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GENERIC PROPERTIES OF NORMAL MAPPINGS

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Abstract. The notion of a normal mapping with respect to a continuous convex function was first introduced by Gabour, Reich and Zaslavski in 2001 for bounded, closed and convex sets. This notion has turned out to be useful in solving minimization problems. In this paper we consider further properties of normal mappings under various assumptions for arbitrary nonempty, closed and convex sets, as well as introduce a more general notion of weak normality, and investigate the properties of normal and weakly normal sequences of mappings. We also present some applications to the minimization of convex functions.

1. Introduction and background

Suppose that $(X, \|\cdot\|)$ is a normed space with norm $\|\cdot\|$, $K \subset X$ is a nonempty, closed and convex subset of X, and that $f : K \to \mathbb{R}$ is a convex function which is bounded from below and uniformly continuous on K. Set

$$\inf f := \inf \{f(x) : x \in K\}.$$

Denote by $\mathfrak{A}$ the set of all bounded self-mappings $A : K \to K$ such that

$$f(Ax) \le f(x) \quad \text{for each } x \in K \tag{1.1}$$

and by $\mathfrak{A}_c$ the set of all continuous mappings $A \in \mathfrak{A}$. For the set $\mathfrak{A}$ define a metric $\rho : \mathfrak{A} \times \mathfrak{A} \to \mathbb{R}$ by

$$\rho(A, B) := \sup \{\|Ax - Bx\| : x \in K\}, \ A, B \in \mathfrak{A}.$$

Clearly, the metric space $\mathfrak{A}$ is complete if $(X, \|\cdot\|)$ is a Banach space, and the metric space $\mathfrak{A}_c$ is a closed subset of $\mathfrak{A}$. Denote by $\mathfrak{M}$ the set of all sequences of elements in $\mathfrak{A}$ and by $\mathfrak{M}_c$ the set of all sequences of elements in $\mathfrak{A}_c$. For the set $\mathfrak{M}$ we consider the following two uniformities and the topologies induced by them. The first uniformity is determined by the following basis:

$$E_1(N, \varepsilon) = \{(\{A_n\}_{n=1}^{\infty}, \{B_n\}_{n=1}^{\infty}) \in \mathfrak{M} \times \mathfrak{M} : \rho(A_n, B_n) < \varepsilon, \ n = 1, \dots, N\},$$

where $N = 1, 2 \dots$ and $\varepsilon > 0$. This uniformity induces a uniform topology on $\mathfrak{M}$, which we denote by τ_1 and call the *weak topology.*

The second uniformity is determined by the following basis:

$$E_2(\varepsilon) = \{(\{A_n\}_{n=1}^{\infty}, \{B_n\}_{n=1}^{\infty}) \in \mathfrak{M} \times \mathfrak{M} : \rho(A_n, B_n) < \varepsilon, \ n = 1, 2, \dots\},$$

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where $\varepsilon > 0$. This uniformity induces a uniform topology on $\mathfrak{M}$, which we denote by τ_2 and call the *strong topology.* It is clear that τ_2 is indeed stronger than τ_1.

It is not difficult to see that the uniform spaces $(\mathfrak{M}, \tau_1)$ and $(\mathfrak{M}, \tau_2)$ are metrizable (by metrics ρ_1 and ρ_2, respectively) and complete if $(X, \|\cdot\|)$ is a Banach space.

Clearly, $\mathfrak{M}_c$ is a closed subset of $\mathfrak{M}$ with respect to the weak topology (and therefore with respect to the strong topology) and hence complete with respect to both the strong and weak topologies. Denote by $\mathfrak{M}_b$ the set of all bounded sequences of elements in $\mathfrak{A}$ and by $\mathfrak{M}_{bc}$ the set of all bounded sequences of elements in $\mathfrak{A}_c$. It can easily be verified that $\mathfrak{M}_b$ and $\mathfrak{M}_{bc}$ are closed subsets of $\mathfrak{M}$ with respect to the strong topology. Evidently, the relative strong topology on $\mathfrak{M}_b$ is determined by the metric $d : \mathfrak{M}_b \times \mathfrak{M}_b \to \mathbb{R}$ defined by

$$d\left(\{A_n\}_{n=1}^{\infty}, \{B_n\}_{n=1}^{\infty}\right) := \sup\left\{\rho\left(A_n, B_n\right)\right\}_{n=1}^{\infty} \quad \{A_n\}_{n=1}^{\infty}, \{B_n\}_{n=1}^{\infty} \in \mathfrak{M}_b.$$

Definition 1.1. A mapping $A : K \to K$ is called *normal* with respect to f if given $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that for each $x \in K$ satisfying $f(x) \geq \inf(f) + \varepsilon$, the inequality

$$f(Ax) < f(x) - \delta(\varepsilon)$$

is true. A sequence $\{A_n\}_{n=1}^{\infty}$ of operators $A_n : K \to K$ is called *normal* with respect to f if given $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that for each $x \in K$ satisfying $f(x) \geq \inf(f) + \varepsilon$ and each integer $n = 1, 2, \ldots,$ the inequality

$$f(A_n x) < f(x) - \delta(\varepsilon)$$

holds.

Example 1.2. Let $X = \mathbb{R}$ and $K = [0, \infty)$. Define $A : K \to K$ by $Ax := 2^{-1}|\sin x|$ for each $x \in K$. Let $f : K \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} x^2 & x \leq 1 \\ 2x - 1 & x > 1 \end{cases}$$

for each $x \in K$. Clearly, $A \in \mathfrak{A}_c$, that is, $\mathfrak{A}_c \subset \mathfrak{A} \neq \emptyset$ and therefore $\mathfrak{M}_c \subset \mathfrak{M} \neq \emptyset$. Let $\varepsilon > 0$ be given and assume $x \in K$ satisfies $f(x) \geq \varepsilon$. Choose $\delta(\varepsilon) := 3 \cdot 8^{-1}\varepsilon$. Then

$$f(Ax) = 4^{-1}\sin^2 x < f(x) - \delta(\varepsilon).$$

We conclude that A is normal with respect to f.

It was shown in [1] that if K is a bounded, closed and convex set in $(X, \|\cdot\|)$, where $(X, \|\cdot\|)$ is a Banach space, then a generic element taken from the spaces $\mathfrak{A}$, $\mathfrak{A}_c$, $\mathfrak{M}$ and $\mathfrak{M}_c$ is normal with respect to f, and that the sequence of values of the function f along any trajectory of such an element tends to the infimum of f on K. These results demonstrate the importance of normal mappings for convex minimization problems. We present analogous results, where the set K is a general nonempty, closed and convex set, which is not necessarily bounded. To this end, we introduce the following weaker notion of normality.

Definition 1.3. A sequence $\{A_n\}_{n=1}^{\infty}$ of operators $A_n : K \to K$ is called *weakly normal* with respect to f if given $\varepsilon > 0$, there exists a sequence $\{\delta_n\}_{n=1}^{\infty}$ of positive numbers such that $\limsup_{n\to\infty} n\delta_n = \infty$, and for each positive integer n, each $x \in K$ satisfying $f(x) \geq \inf(f) + \varepsilon$ and each integer $k = 1, 2, \ldots, n$, the inequality

$$f(A_k x) < f(x) - \delta_n$$

holds.

Remark 1.4. It is not difficult to see that for each $\alpha \in (0,1)$ and each $\{A_n\}_{n=1}^{\infty}, \{B_n\}_{n=1}^{\infty} \in \mathfrak{M}$, their convex combination, $\alpha\{A_n\}_{n=1}^{\infty} + (1-\alpha)\{B_n\}_{n=1}^{\infty}$, is also an element of $\mathfrak{M}$ and if one of them is normal, then the sequence $\alpha\{A_n\}_{n=1}^{\infty} + (1-\alpha)\{B_n\}_{n=1}^{\infty}$ is also normal. Evidently, each normal sequence of mappings is, in particular, weakly normal, but not *vice versa*, as is shown in the following example.

Example 1.5. Let $X = \mathbb{R}$ and $K = (-\infty, 1]$. Let $g : K \to \mathbb{R}$ be defined by

$$g(x) := \begin{cases} x & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

for each $x \in K$. For each positive integer n, define $A_n : K \to K$ by

$$A_n x := \left(1 - n^{-2^{-1}}\right)^{2^{-1}} g(x)$$

for each $x \in K$. Let $f : K \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

for each $x \in K$. Clearly, f is convex. Let $\varepsilon > 0$ be arbitrary. For each positive integer n, set $\delta_n := n^{-2^{-1}}\varepsilon$. Then $\inf(f) = 0$ and for each $x \in K$ such that $f(x) \geq \varepsilon$ and each $k = 1, 2, \ldots, n$, we have

$$f(A_k x) = \left(1 - k^{-2^{-1}}\right) f(x) \leq f(x) - k^{-2^{-1}}\varepsilon = f(x) - \delta_k \leq f(x) - \delta_n.$$

Clearly, $\lim_{n\to\infty} n\delta_n = \infty$. Therefore the sequence $\{A_n\}_{n=1}^{\infty}$ is weakly normal with respect to f, but it is not normal with respect to f because $\lim_{n\to\infty} f(A_n x) = f(x)$ for each $x \in K$ such that $f(x) \geq \varepsilon$. As a matter of fact, we also have $\{A_n\}_{n=1}^{\infty} \in \mathfrak{M}_c$, that is, $\mathfrak{M}_c \subset \mathfrak{M} \neq \emptyset$.

In the sequel we assume that the function f is clearly understood and therefore use the notions of normality and weak normality without referring explicitly to f. We also assume that $(X, \|\cdot\|)$ is a Banach space.

The rest of the paper is organized as follows. In Section 2 we state our main theorems. Several auxiliary results are presented in Section 3. Section 4 is devoted to results concerning the existence of residual sets of normal mappings, normal sequences of mappings and of weakly normal sequences of mappings. In Section 5 we provide some applications of the concepts of normality and weak normality to solving certain minimization problems. Finally, the proofs of our main theorems, which are stated in Section 2, are provided in Section 6.

2. Statements of the main results

In this section we state our three main theorems. We establish them in the last section of our paper.

Theorem 2.1. *There exist sets* $\mathcal{F} \subset \mathfrak{M}$, $\mathcal{F}_b \subset \mathcal{F} \cap \mathfrak{M}_b$, $\mathcal{F}_c \subset \mathcal{F} \cap \mathfrak{M}_c$ *and* $\mathcal{F}_{bc} \subset \mathcal{F}_b \cap \mathfrak{M}_c$ *of weakly normal sequences of mappings which are countable intersections of open (in the relative weak topology) and dense (respectively, in the weak topology, in the relative strong topology, in the relative weak topology and in the relative strong topology) sets in, respectively,* $\mathfrak{M}$, $\mathfrak{M}_b$, $\mathfrak{M}_c$ *and* $\mathfrak{M}_{bc}$ *such that for each* $\{A_n\}_{n=1}^{\infty} \in \mathcal{F}$, *the following assertion holds:*

For each $\varepsilon > 0$ *and each* $B_0 \in \mathfrak{A}$, *there exists a neighborhood* U *(in the weak topology) of* $\{A_n\}_{n=1}^{\infty}$ *and a positive integer* N *satisfying*

$$f\left(B_N \dots B_1 B_0 x\right) < \inf\left(f\right) + \varepsilon$$

for each $\{B_n\}_{n=1}^{\infty} \in U$ *and each* $x \in K$.

Theorem 2.2. *There exist a set* $\mathcal{F} \subset \mathfrak{A}$ *of normal mappings, which is a countable intersection of open and dense sets in* $\mathfrak{A}$, *and a set* $\mathcal{F}_c \subset \mathcal{F} \cap \mathfrak{A}_c$ *of normal mappings, which is a countable intersection of open and dense sets in* $\mathfrak{A}_c$, *such that for each* $A \in \mathcal{F}$, *the following assertion holds:*

For each $\varepsilon > 0$, *there exists a neighborhood* U *of* A *in* $\mathfrak{A}$ *such that for each* $B_0 \in \mathfrak{A}$, *there is a positive integer* N *satisfying*

$$f\left(B^N B_0 x\right) < \inf\left(f\right) + \varepsilon$$

for each $B \in U$ *and each* $x \in K$. *In particular, for each* $B \in U$, *there is a positive integer* N *such that we have*

$$f\left(B^N x\right) < \inf\left(f\right) + \varepsilon$$

for each $x \in K$.

Theorem 2.3. *There exist sets* $\mathcal{F}_b \subset \mathfrak{M}_b$ *and* $\mathcal{F}_{bc} \subset \mathcal{F}_b \cap \mathfrak{M}_c$ *of normal sequences of mappings, which are countable intersections of open (in the relative strong topology) and dense (in the relative strong topology) sets in, respectively,* $\mathfrak{M}_b$ *and* $\mathfrak{M}_{bc}$, *such that for each* $\{A_n\}_{n=1}^{\infty} \in \mathcal{F}$, *the following assertion holds:*

For each $\varepsilon > 0$, *there exists a neighborhood* U *(in the strong topology) of* $\{A_n\}_{n=1}^{\infty}$ *such that for each* $B_0 \in \mathfrak{A}$ *there is a positive integer* N *satisfying*

$$f\left(B_{r(N)} \dots B_{r(1)} B_0 x\right) < \inf\left(f\right) + \varepsilon$$

for each $\{B_n\}_{n=1}^{\infty} \in U$, *each mapping* $r : \{1, 2, \dots\} \to \{1, 2, \dots\}$ *and each* $x \in K$. *In particular, for each* $\{B_n\}_{n=1}^{\infty} \in U$ *and each mapping* $r : \{1, 2, \dots\} \to \{1, 2, \dots\}$, *there is a positive integer* N *such that*

$$f\left(B_{r(N)} \dots B_{r(1)} x\right) < \inf\left(f\right) + \varepsilon$$

for each $x \in K$.

These theorems generalize the corresponding results in [1] and [4].

3. Auxiliary results

We first prove that there exists a continuous and normal operator $A_* : K \to K$. Our proof follows in the footsteps of the proof of Proposition 2.1 of [1]. In this connection, see also [4].

Proposition 3.1. *There exists an operator $A_* : K \to K$ which is continuous, normal and satisfies* (1.1)*.*

Proof. Without loss of generality we may assume that f does not attain its minimum on K. Define a set-valued map $a : K \to 2^K$ as follows: for each $x \in K$, denote by $a(x)$ the closure (in the relative topology induced by the norm $\|\cdot\|$) of the set

$$\left\{y \in K : f(y) < 2^{-1}(f(x) + \inf(f))\right\}.$$

It is clear that for each $x \in K$, the set $a(x)$ is nonempty, closed and convex. We claim that a is lower semi-continuous. Let U be an arbitrary open set in K. We have to show that the set $V = \{x \in K : a(x) \cap U \neq \emptyset\}$ is open. To this end, let $x_0 \in V$. Then there exists a point $y_0 \in a(x_0) \cap U$. By definition of $a(x_0)$, there also exists a point $y_1 \in U$ such that

$$f(y_1) < 2^{-1}(f(x_0) + \inf(f)).$$

Since the function f is continuous, there is a number $\delta > 0$ such that for each $x \in K$ satisfying $\|x - x_0\| < \delta$, we have

$$f(y_1) < 2^{-1}(f(x) + \inf(f)).$$

Hence $y_1 \in a(x) \cap U$ for each $x \in K$ satisfying $\|x - x_0\| < \delta$, and therefore x_0 is an interior point of V. Thus V is indeed open and therefore a is lower semi-continuous, as claimed. By Michael's selection theorem, there exists a continuous mapping $A_* : K \to K$ such that $A_*x \in a(x)$ for each $x \in K$. It follows from the definition of a that for each point $x \in K$, we have

$$f(A_*x) \leq 2^{-1}(f(x) + \inf(f)).$$

Given $\varepsilon > 0$, choose $\delta(\varepsilon) = 4^{-1}\varepsilon$. Then for each $x \in K$ satisfying $f(x) \geq \inf(f) + \varepsilon$, we have $f(A_*x) < f(x) - \delta(\varepsilon)$. Hence A_* is normal and satisfies (1.1), as asserted. □

Lemma 3.2. *Let $\{A_n\}_{n=1}^{\infty} \in \mathfrak{M}$ be normal and let $\varepsilon > 0$ be given. Then there exist a number $\delta > 0$ and a neighborhood U of $\{A_n\}_{n=1}^{\infty}$ in $\mathfrak{M}$ with the strong topology such that for each $\{B_n\}_{n=1}^{\infty} \in U$ and each $x \in K$ satisfying $f(x) \geq \inf(f) + \varepsilon$, we have $f(B_nx) < f(x) - \delta$ for each $n = 1, 2, \ldots$.*

Proof. Since $\{A_n\}_{n=1}^{\infty}$ is normal, there is $\delta_0 > 0$ such that for each $n = 1, 2, \ldots$ and each $x \in K$ satisfying $f(x) \geq \inf(f) + \varepsilon$, we have

$$f(A_nx) < f(x) - \delta_0. \tag{3.1}$$

Since f is uniformly continuous, there is $\delta \in \left(0, 2^{-1}\delta_0\right)$ such that $|f(y) - f(z)| < 2^{-1}\delta_0$ for each $y, z \in K$ satisfying $\|y - z\| < \delta$. Set

$$U := \left\{\{B_n\}_{n=1}^{\infty} : \left(\{B_n\}_{n=1}^{\infty}, \{A_n\}_{n=1}^{\infty}\right) \in E_2(\delta)\right\}.$$

It is clear that U is a neighborhood of $\{A_n\}_{n=1}^{\infty}$ in $\mathfrak{M}$ with the strong topology. Assume that $\{B_n\}_{n=1}^{\infty} \in U$ and that $x \in K$ satisfies $f(x) \geq \inf(f) + \varepsilon$. Then by (3.1) we have

$$f(A_n x) < f(x) - \delta_0 \tag{3.2}$$

for each $n = 1, 2, \ldots$. The definitions of δ and U imply that $\|A_n x - B_n x\| < \delta$ and $|f(A_n x) - f(B_n x)| < 2^{-1}\delta_0$ for each $n = 1, 2, \ldots$. When combined with (3.2), this implies that

$$f(B_n x) < f(x) + 2^{-1}\delta_0 - \delta_0 < f(x) - \delta$$

for each $n = 1, 2, \ldots$, as asserted. □

Lemma 3.3. *Let $\{A_n\}_{n=1}^{\infty} \in \mathfrak{M}$ be weakly normal and let $\varepsilon > 0$ be given. Then there exist a sequence of positive numbers $\{\delta_N\}_{N=1}^{\infty}$ and a sequence $\{U_N\}_{N=1}^{\infty}$ of neighborhoods of $\{A_n\}_{n=1}^{\infty}$ in $\mathfrak{M}$ with the weak topology such that $\limsup_{N\to\infty} \delta_N N = \infty$ and for each positive integer N, the following assertion holds:*

For each $\{B_n\}_{n=1}^{\infty} \in U_N$ and each $x \in K$ satisfying $f(x) \geq \inf(f) + \varepsilon$, we have $f(B_n x) < f(x) - \delta_N$ for each $n = 1, 2, \ldots, N$.

Proof. Since $\{A_n\}_{n=1}^{\infty}$ is weakly normal, there is a sequence $\{\delta'_N\}_{N=1}^{\infty}$ of positive numbers such that $\limsup_{N\to\infty} \delta'_N N = \infty$ and for each $x \in K$ satisfying $f(x) \geq \inf(f) + \varepsilon$, we have

$$f(A_n x) < f(x) - \delta'_N \tag{3.3}$$

for all $N = 1, 2 \ldots$ and each $n = 1, 2, \ldots, N$.
Let N be a positive integer. Set $\delta_N := 2^{-1}\delta'_N$. Since f is uniformly continuous, there is a number $\delta''_N > 0$ such that $|f(y) - f(z)| < \delta_N$ for each $y, z \in K$ satisfying $\|y - z\| < \delta''_N$. Set

$$U_N := \left\{\{B_n\}_{n=1}^{\infty} : (\{B_n\}_{n=1}^{\infty}, \{A_n\}_{n=1}^{\infty}) \in E_1\left(N, \delta''_N\right)\right\}.$$

Clearly, U_N is a neighborhood of $\{A_n\}_{n=1}^{\infty}$ in $\mathfrak{M}$ with the weak topology. Assume that $\{B_n\}_{n=1}^{\infty} \in U_N$ and that $x \in K$ satisfies $f(x) \geq \inf(f) + \varepsilon$. Then by (3.3) we have

$$f(A_n x) < f(x) - \delta'_N \tag{3.4}$$

for each $n = 1, 2, \ldots, N$. The definitions of δ''_N and U_N imply that $\|A_n x - B_n x\| < \delta''_N$ and $|f(A_n x) - f(B_n x)| < \delta_N$ for each $n = 1, 2, \ldots, N$. When combined with (3.4), this implies that

$$f(B_n x) < f(x) + \delta_N - \delta'_N = f(x) - \delta_N$$

for each $n = 1, 2, \ldots, N$. In this way we have constructed two sequences $\{\delta_N\}_{N=1}^{\infty}$ and $\{U_N\}_{N=1}^{\infty}$. Using the weak normality of $\{A_n\}_{n=1}^{\infty}$ and the definition of $\{\delta_N\}_{N=1}^{\infty}$, we obtain that $\limsup_{N\to\infty} \delta_N N = \infty$. Hence we see that the sequences $\{\delta_N\}_{N=1}^{\infty}$ and $\{U_N\}_{N=1}^{\infty}$ have all the asserted properties. □

Let A_* be the mapping the existence of which is guaranteed by Proposition 3.1 and let $\{A_n\}_{n=1}^{\infty}$ be an arbitrary sequence in $\mathfrak{M}$. For each $\gamma \in (0, 1)$, we define a sequence of mappings $\{A_n^\gamma\}_{n=1}^\infty$, $A_n^\gamma : K \to K$, by

$$A_n^\gamma := (1-\gamma) A_n + \gamma A_1 A_*, \; n = 1, 2, \ldots . \tag{3.5}$$

By (1.1) and Proposition 3.1, $A_1 A_* \in \mathfrak{A}$ and $A_1 A_*$ is normal. By Proposition 3.1 and Remark 1.4, the sequence $\{A_n^\gamma\}_{n=1}^\infty \in \mathfrak{M}$. For each $\gamma \in (0,1)$ and for each $N = 1, 2 \ldots$, we have

$$(\forall n \in \{1, \ldots, N\})\, \rho\left(A_n^\gamma, A_n\right) \le 2\gamma \max \left\{ \sup_{x \in K} \|A_k x\| \right\}_{k=1}^N . \tag{3.6}$$

If, in addition, $\{A_n\}_{n=1}^\infty \in \mathfrak{M}_b$, then we also have

$$(\forall n \in \{1, 2 \ldots\})\, \rho\left(A_n^\gamma, A_n\right) \le 2\gamma \sup \left\{ \sup_{x \in K} \|A_k x\| \right\}_{k=1}^\infty , \tag{3.7}$$

where $\sup \{\sup_{x \in K} \|A_k x\|\}_{k=1}^\infty < \infty$. For an arbitrary operator $A \in \mathfrak{A}$, we define

$$A_\gamma := (1-\gamma) A + \gamma A A_*.$$

Evidently,

$$\rho\left(A_\gamma, A\right) \le 2\gamma \sup_{x \in K} \|Ax\| . \tag{3.8}$$

Lemma 3.4. *For each $\varepsilon > 0$, there exists a positive number δ such that for each $\{A_n\}_{n=1}^\infty \in \mathfrak{M}$ and each $\gamma \in (0,1)$, there is a sequence $\{U_N\}_{N=1}^\infty$ of neighborhoods of $\{A_n^\gamma\}_{n=1}^\infty$ in $\mathfrak{M}$ with the weak topology such that the following assertion holds for each positive integer N:*

For each $\{B_n\}_{n=1}^\infty \in U_N$ and each $x \in K$ such that $f(x) \ge \inf(f) + \varepsilon$, we have $f(B_n x) < f(x) - \gamma\delta$ for each $n = 1, 2, \ldots, N$.

Proof. Let $\varepsilon > 0$. Since A_* is normal, there is a positive number δ' such that for each $x \in K$ satisfying $f(x) \ge \inf(f) + \varepsilon$, we have

$$f(A_* x) < f(x) - \delta'.$$

Set $\delta := 2^{-1}\delta'$. Let $\{A_n\}_{n=1}^\infty \in \mathfrak{M}$ and $\gamma \in (0,1)$. By the convexity of f, we have

$$f\left(A_n^\gamma x\right) < f(x) - \gamma\delta' \tag{3.9}$$

for each $n = 1, 2 \ldots$. Let N be a positive integer. Since f is uniformly continuous, there is a number $\delta'' > 0$ such that $|f(y) - f(z)| < \gamma\delta$ for each $y, z \in K$ satisfying $\|y - z\| < \delta''$. Set

$$U_N := \left\{ \{B_n\}_{n=1}^\infty \in \mathfrak{M} : \left(\{A_n^\gamma\}_{n=1}^\infty, \{B_n\}_{n=1}^\infty\right) \in E_1\left(N, \delta''\right) \right\}.$$

Clearly, U_N is a neighborhood of $\{A_n^\gamma\}_{n=1}^\infty$ in $\mathfrak{M}$ with the weak topology. Assume that $\{B_n\}_{n=1}^\infty \in U_N$ and that $x \in K$ satisfies $f(x) \ge \inf(f) + \varepsilon$. The definitions of δ'' and U_N imply that $\|A_n x - B_n x\| < \delta''$ and $|f(A_n^\gamma x) - f(B_n x)| < \gamma\delta$ for each $n = 1, 2, \ldots, N$. When combined with (3.9), this implies that

$$f(B_n x) < f(x) + \gamma\delta - \gamma\delta' = f(x) - \gamma\delta$$

for each $n = 1, 2, \ldots, N$. In this way we have found a number δ and constructed a sequence $\{U_N\}_{N=1}^{\infty}$ which have all the asserted properties. □

4. Residual sets of normal mappings, normal sequences of mappings and of weakly normal sequences of mappings

Recall that a subset Z of a topological space Y is called *residual* if it contains a countable intersection of open and dense subsets of Y. In the case where the space Y is completely pseudo-metrizable, the Baire category theorem guarantees that Z is also a dense subset of Y. In this section we prove that there exist residual sets of normal mappings, normal sequences of mappings and weakly normal sequences of mappings.

Theorem 4.1. *There exist sets $\mathcal{F} \subset \mathfrak{M}$, $\mathcal{F}_b \subset \mathcal{F} \cap \mathfrak{M}_b$, $\mathcal{F}_c \subset \mathcal{F} \cap \mathfrak{M}_c$ and $\mathcal{F}_{bc} \subset \mathcal{F}_b \cap \mathfrak{M}_c$, which are countable intersections of open (in the relative weak topology) and dense (respectively, in the weak topology, in the relative strong topology, in the relative weak topology and in the relative strong topology) sets in, respectively, $\mathfrak{M}$, $\mathfrak{M}_b$, $\mathfrak{M}_c$ and $\mathfrak{M}_{bc}$ such that each sequence $\{B_n\}_{n=1}^{\infty} \in \mathcal{F}$ is weakly normal.*

Proof. Define

$$T := \left\{ \gamma \in (0,1) : \gamma = N^{-2^{-1}} \text{ for some positive integer } N \right\}$$

and for each positive integer N, define

$$T_N := \left\{ \gamma \in T : \gamma < N^{-2^{-1}} \right\}.$$

By (3.6) and Proposition 3.1, for each $N = 1, 2 \ldots$, the set

$$\boldsymbol{A^N} = \{\{A_n^{\gamma}\}_{n=1}^{\infty} : \{A_n\}_{n=1}^{\infty} \in \mathfrak{M},\ \gamma \in T_N\}$$

is dense in $\mathfrak{M}$ with the weak topology and the set

$$\boldsymbol{A}_c^N = \{\{A_n^{\gamma}\}_{n=1}^{\infty} : \{A_n\}_{n=1}^{\infty} \in \mathfrak{M}_c,\ \gamma \in T_N\}$$

is dense in $\mathfrak{M}_c$ with the relative weak topology. By (3.7) and Proposition 3.1, for each $N = 1, 2 \ldots$, the set

$$\boldsymbol{A}_b^N = \{\{A_n^{\gamma}\}_{n=1}^{\infty} : \{A_n\}_{n=1}^{\infty} \in \mathfrak{M}_b,\ \gamma \in T_N\}$$

is dense in $\mathfrak{M}_b$ with the relative strong topology, and the set

$$\boldsymbol{A}_{bc}^N = \{\{A_n^{\gamma}\}_{n=1}^{\infty} : \{A_n\}_{n=1}^{\infty} \in \mathfrak{M}_{bc},\ \gamma \in T_N\}$$

is dense in $\mathfrak{M}_{bc}$ with the relative strong topology.

Let q be an arbitrary positive integer. By Lemma 3.4, there exists a number $\delta(q) > 0$ such that for each $(\{A_n\}_{n=1}^{\infty}, \gamma) \in \mathfrak{M} \times T$, there is a sequence $\{U_N(\{A_n\}_{n=1}^{\infty}, \gamma)(q)\}_{N=1}^{\infty}$ of open neighborhoods of $\{A_n^{\gamma}\}_{n=1}^{\infty}$ with the weak topology such that the following assertion holds for each positive integer N:

For each $\{B_n\}_{n=1}^{\infty} \in U_N(\{A_n\}_{n=1}^{\infty}, \gamma)(q)$ and each $x \in K$ satisfying $f(x) \geq \inf(f) + 2^{-q}$, we have

$$f(B_n x) < f(x) - \gamma\delta(q) \tag{4.1}$$

for each $n = 1, 2 \ldots, N$.

For each positive integers q and N, let

$$\mathcal{D}_{q,N} = \cup_{\left(\{A_n\}_{n=1}^{\infty}, \gamma\right) \in \mathfrak{M} \times T_N} \left(U_{\gamma^{-2}}\left(\{A_n\}_{n=1}^{\infty}, \gamma\right)(q)\right),$$

$$\mathcal{D}_{q,N}^{b} = \cup_{\left(\{A_n\}_{n=1}^{\infty}, \gamma\right) \in \mathfrak{M}_b \times T_N} U_{\gamma^{-2}}\left(\{A_n\}_{n=1}^{\infty}, \gamma\right)(q) \cap \mathfrak{M}_b,$$

$$\mathcal{D}_{q,N}^{c} = \cup_{\left(\{A_n\}_{n=1}^{\infty}, \gamma\right) \in \mathfrak{M}_c \times T_N} U_{\gamma^{-2}}\left(\{A_n\}_{n=1}^{\infty}, \gamma\right)(q) \cap \mathfrak{M}_c,$$

and

$$\mathcal{D}_{q,N}^{bc} = \cup_{\left(\{A_n\}_{n=1}^{\infty}, \gamma\right) \in \mathfrak{M}_{bc} \times T_N} U_{\gamma^{-2}}\left(\{A_n\}_{n=1}^{\infty}, \gamma\right)(q) \cap \mathfrak{M}_{bc}.$$

Clearly, the sets $\mathcal{D}_{q,N}$, $\mathcal{D}_{q,N}^{b}$, $\mathcal{D}_{q,N}^{c}$ and $\mathcal{D}_{q,N}^{bc}$ are open (in the relative weak topology) and dense (respectively, in the weak topology, in the relative strong topology, in the relative weak topology and in the relative strong topology) sets in, respectively, $\mathfrak{M}$, $\mathfrak{M}_b$, $\mathfrak{M}_c$ and $\mathfrak{M}_{bc}$ for each pair of positive integers q and N, because these sets contain, respectively, $\boldsymbol{A}^N$, $\boldsymbol{A}_b^N$, $\boldsymbol{A}_c^N$ and $\boldsymbol{A}_{bc}^N$. Define $\mathcal{F} = \cap_{q=1}^{\infty} \cap_{N=1}^{\infty} \mathcal{D}_{q,N}$, $\mathcal{F}_b = \cap_{q=1}^{\infty} \cap_{N=1}^{\infty} \mathcal{D}_{q,N}^{b}$, $\mathcal{F}_c = \cap_{q=1}^{\infty} \cap_{N=1}^{\infty} \mathcal{D}_{q,N}^{c}$ and $\mathcal{F}_{bc} = \cap_{q=1}^{\infty} \cap_{N=1}^{\infty} \mathcal{D}_{q,N}^{bc}$. Evidently, $\mathcal{F}$, $\mathcal{F}_b$, $\mathcal{F}_c$ and $\mathcal{F}_{bc}$ are countable intersections of open (in the relative weak topology) and dense (respectively, in the weak topology, in the relative strong topology, in the relative weak topology and in the relative strong topology) sets in, respectively, $\mathfrak{M}$, $\mathfrak{M}_b$, $\mathfrak{M}_c$ and $\mathfrak{M}_{bc}$.

Assume now that $\{B_n\}_{n=1}^{\infty} \in \mathcal{F}$ and let $\varepsilon > 0$ be an arbitrary positive number. Choose a positive integer q_0 such that $2^{-q_0} < \varepsilon$. Then for each positive integer N, there exists a pair $\left(\{A_n\}_{n=1}^{\infty}, \gamma_N\right) \in \mathfrak{M} \times T_N$ such that

$$\{B_n\}_{n=1}^{\infty} \in U_{\gamma_N^{-2}}\left(\{A_n\}_{n=1}^{\infty}, \gamma_N\right)(q_0),$$

and it follows from (4.1) that for each point $x \in K$ satisfying $f(x) \geq \inf(f) + \varepsilon$, we have

$$f(B_n x) < f(x) - \gamma_N \delta(q_0) \tag{4.2}$$

for each $n = 1, 2 \ldots, \gamma_N^{-2}$.

Consider the sequences $\{\gamma_N\}_{N=1}^{\infty}$ and $\left\{\gamma_N^{-2}\right\}_{N=1}^{\infty}$. Since for each positive integer N we have $\gamma_N^{-2} > N$, it is clear that there exists a strictly increasing subsequence $\left\{\gamma_{N_k}^{-2}\right\}_{k=1}^{\infty}$ of $\left\{\gamma_N^{-2}\right\}_{N=1}^{\infty}$. For each positive integer M, set $\delta_M := \gamma_{N_{\min\left\{k:\, \gamma_{N_k}^{-2} \geq M\right\}}}$. Since $\gamma^{-2}_{N_{\min\left\{k:\, \gamma_{N_k}^{-2} \geq M\right\}}} \geq M$, we conclude from (4.2) that for each point $x \in K$ satisfying $f(x) \geq \inf(f) + \varepsilon$, we have

$$f(B_n x) < f(x) - \delta_M \delta(q_0)$$

for each $n = 1, 2 \ldots, M$. Since for each $k = 1, 2 \ldots$, we have

$$\delta_{\gamma_{N_k}^{-2}} = \gamma_{N_{\min\left\{i:\, \gamma_{N_i}^{-2} \geq \gamma_{N_k}^{-2}\right\}}} = \gamma_{N_k},$$

it follows that

$$\lim_{n \to \infty} \gamma_{N_k}^{-2} \delta_{\gamma_{N_k}^{-2}} = \lim_{n \to \infty} \gamma_{N_k}^{-1} = \infty.$$

Hence $\{B_n\}_{n=1}^{\infty}$ is weakly normal. This completes the proof of Theorem 4.1. □

Theorem 4.2. *There exist a set $\mathcal{F} \subset \mathfrak{M}_b$, which is a countable intersection of open and dense sets in $\mathfrak{A}$, and a set $\mathcal{F}_c \subset \mathcal{F} \cap \mathfrak{A}_c$, which is a countable intersection of open and dense sets in $\mathfrak{A}_c$, such that each mapping $B \in \mathcal{F}$ is normal.*

Proof. By (3.8) and Proposition 3.1, the set

$$\boldsymbol{A} = \{A_\gamma : A \in \mathfrak{A},\ \gamma \in (0,1)\}$$

is dense in $\mathfrak{A}$, and the set

$$\boldsymbol{A}_c = \{A_\gamma : A \in \mathfrak{A}_c,\ \gamma \in (0,1)\}$$

is dense in $\mathfrak{A}_c$.

By Remark 1.4, for each $(A,\gamma) \in \mathfrak{A} \times (0,1)$, the mapping A_γ is normal. Assume q is an arbitrary positive integer. By Lemma 3.2, for each $(A,\gamma) \in \mathfrak{A} \times (0,1)$, there exist a number $\delta_q(A,\gamma) > 0$ and an open neighborhood $U_q(A,\gamma)$ of A_γ in $\mathfrak{A}$ such that the following assertion holds:

For each $B \in U_q(A,\gamma)$ and each $x \in K$ satisfying $f(x) \geq \inf(f) + 2^{-q}$, we have

$$f(Bx) < f(x) - \delta_q. \tag{4.3}$$

For each positive integer q, set

$$\mathcal{D}_q = \cup_{(A,\gamma)\in\mathfrak{A}\times(0,1)} U_q(A,\gamma),$$
$$\mathcal{D}_q^c = \cup_{(A,\gamma)\in\mathfrak{A}_c\times(0,1)} U_q(A,\gamma) \cap \mathfrak{A}_c.$$

It is clear that the sets $\mathcal{D}_q$ and $\mathcal{D}_q^c$ are open and dense sets in, respectively, $\mathfrak{A}$ and $\mathfrak{A}_c$ for each $q = 1, 2, \ldots$, since these sets contain, respectively, $\boldsymbol{A}$ and $\boldsymbol{A}_c$. Define $\mathcal{F} = \cap_{q=1}^{\infty} \mathcal{D}_q$ and $\mathcal{F}_c = \cap_{q=1}^{\infty} \mathcal{D}_q^c$. Evidently, $\mathcal{F}$ and $\mathcal{F}_c$ are countable intersections of open and dense sets in, respectively, $\mathfrak{A}$ and $\mathfrak{A}_c$.

Assume now that $B \in \mathcal{F}$. Let $\varepsilon > 0$ be an arbitrary positive number and choose a positive integer q_0 such that $2^{-q_0} < \varepsilon$. There exists a pair $(A,\gamma) \in \mathfrak{A} \times (0,1)$ such that $B \in U_{q_0}(A,\gamma)$. It follows from (4.3) that for each point $x \in K$ satisfying $f(x) \geq \inf(f) + \varepsilon$, we have

$$f(Bx) < \inf(f) - \delta_{q_0}.$$

Hence B is normal. This completes the proof of Theorem 4.2. □

Theorem 4.3. *There exist sets $\mathcal{F}_b \subset \mathcal{F} \cap \mathfrak{M}_b$ and $\mathcal{F}_{bc} \subset \mathcal{F}_b \cap \mathfrak{M}_{bc}$, which are countable intersections of open (in the relative strong topology) and dense (in the relative strong topology) sets in, respectively, $\mathfrak{M}_b$ and $\mathfrak{M}_{bc}$, such that each sequence $\{B_n\}_{n=1}^{\infty} \in \mathcal{F}$ is normal.*

Proof. By (3.7) and Proposition 3.1 the set

$$\boldsymbol{A}_b = \{\{A_n^\gamma\}_{n=1}^{\infty} : \{A_n\}_{n=1}^{\infty} \in \mathfrak{M}_b,\ \gamma \in (0,1)\}$$

is dense in $\mathfrak{M}_b$ with the relative strong topology and the set

$$\boldsymbol{A}_{bc} = \{\{A_n^\gamma\}_{n=1}^{\infty} : \{A_n\}_{n=1}^{\infty} \in \mathfrak{M}_{bc},\ \gamma \in (0,1)\}$$

is dense in $\mathfrak{M}_{bc}$ with the relative strong topology.

By Remark 1.4, for each pair $(\{A_n\}_{n=1}^{\infty}, \gamma) \in \mathfrak{M} \times (0,1)$, the sequence $\{A_n^{\gamma}\}_{n=1}^{\infty}$ is normal. Let q be an arbitrary positive integer. By Lemma 3.2, for each pair $(\{A_n\}_{n=1}^{\infty}, \gamma) \in \mathfrak{M} \times (0,1)$, there exist a number $\delta_q(\{A_n\}_{n=1}^{\infty}, \gamma) > 0$ and an open neighborhood $U_q(\{A_n\}_{n=1}^{\infty}, \gamma)$ of $\{A_n^{\gamma}\}_{n=1}^{\infty}$ in $\mathfrak{M}$ with the strong topology such that the following assertion holds:

For each $\{B_n\}_{n=1}^{\infty} \in U_q(\{A_n\}_{n=1}^{\infty}, \gamma)$ and each $x \in K$ satisfying $f(x) \geq \inf(f) + 2^{-q}$, we have

$$f(B_n x) < f(x) - \delta_q \tag{4.4}$$

for each $n = 1, 2, \ldots$. For each positive integer q, set

$$\mathcal{D}_q^b = \cup_{(\{A_n\}_{n=1}^{\infty}, \gamma) \in \mathfrak{M}_b \times (0,1)} U_q(\{A_n\}_{n=1}^{\infty}, \gamma) \cap \mathfrak{M}_b,$$

$$\mathcal{D}_q^{bc} = \cup_{(\{A_n\}_{n=1}^{\infty}, \gamma) \in \mathfrak{M}_c \times (0,1)} U_q(\{A_n\}_{n=1}^{\infty}, \gamma) \cap \mathfrak{M}_{bc}.$$

Clearly, the sets $\mathcal{D}_q^b$ and $\mathcal{D}_q^{bc}$ are open and dense sets in, respectively, $\mathfrak{M}_b$, $\mathfrak{M}_{bc}$ for each $q = 1, 2 \ldots$, since these sets contain, respectively, $\boldsymbol{A}_b$ and $\boldsymbol{A}_{bc}$. Define $\mathcal{F} = \cap_{q=1}^{\infty} \mathcal{D}_q^b$ and $\mathcal{F}_{bc} = \cap_{q=1}^{\infty} \mathcal{D}_q^{bc}$. Evidently, $\mathcal{F}$ and $\mathcal{F}_c$ are countable intersections of open and dense sets in, respectively, $\mathfrak{M}_b$ and $\mathfrak{M}_{bc}$.

Assume now that $\{B_n\}_{n=1}^{\infty} \in \mathcal{F}$. Let $\varepsilon > 0$ be an arbitrary positive number. Choose a positive integer q_0 such that $2^{-q_0} < \varepsilon$. There exists a pair $(\{A_n\}_{n=1}^{\infty}, \gamma) \in \mathfrak{M}_b \times (0,1)$ such that $\{B_n\}_{n=1}^{\infty} \in U_{q_0}(\{A_n\}_{n=1}^{\infty}, \gamma)$. It follows from (4.4) that for each point $x \in K$ satisfying $f(x) \geq \inf(f) + \varepsilon$, we have

$$f(B_n x) < \inf(f) - \delta_{q_0}$$

for each $n = 1, 2, \ldots$. Hence $\{B_n\}_{n=1}^{\infty}$ is normal. This completes the proof of Theorem 4.3. □

5. Applications of normality and weak normality to the minimization of convex functions

In this section we present several applications of the concepts of normality and weak normality to solving certain minimization problems.

Theorem 5.1. *Let $\{A_n\}_{n=1}^{\infty} \in \mathfrak{M}$ be weakly normal and let $\varepsilon > 0$. Then for each $B_0 \in \mathfrak{A}$, there exist a neighborhood U of $\{A_n\}_{n=1}^{\infty}$ in $\mathfrak{M}$ with the weak topology and a positive integer N such that for each $\{B_n\}_{n=1}^{\infty} \in U$, we have*

$$f(B_N \ldots B_1 B_0 x) < \inf(f) + \varepsilon$$

for each $x \in K$.

Proof. Let $B_0 \in \mathfrak{A}$. Set $d_0 = \sup\{|f(B_0 x)| : x \in K\}$. Evidently, d_0 is finite because f is uniformly continuous. Since $\{A_n\}_{n=1}^{\infty}$ is weakly normal, employing Lemma 3.3, we see that there exist a positive integer N, a positive number $\delta_N > 0$ satisfying $\delta_N N > d_0 - \inf(f)$, and a neighborhood U_N of $\{A_n\}_{n=1}^{\infty}$ in $\mathfrak{M}$ with the weak topology such that the following assertion holds:

For each $\{B_n\}_{n=1}^{\infty} \in U_N$ and each $x \in K$ satisfying $f(x) \geq \inf(f) + \varepsilon$, we have

$$f(B_n x) < f(x) - \delta_N \tag{5.1}$$

for each $n = 1, 2, \dots, N$.

Let $\{B_n\}_{n=1}^{\infty} \in U_N$. We claim that

$$f(B_N \dots B_1 B_0 x) < \inf(f) + \varepsilon \tag{5.2}$$

for each $x \in K$. Suppose to the contrary that this is not true. Then there exists $x \in K$ such that

$$f(B_n \dots B_1 B_0 x) \geq \inf(f) + \varepsilon, \; n = 0, \dots, N.$$

By (5.1) and by induction it follows that for each $n = 1, \dots, N$,

$$f(B_n \dots B_1 B_0 x) < f(B_0 x) - n\delta_N.$$

This implies that

$$f(B_N \dots B_1 B_0 x) < f(B_0 x) - N\delta_N < d_0 - (d_0 - \inf(f)) = \inf(f),$$

a contradiction. Therefore, (5.2) is, in fact, valid and Theorem 5.1 is proved. □

Theorem 5.2. *Let $A \in \mathfrak{A}$ be normal and let $\varepsilon > 0$. Then there exists a neighborhood U of A in $\mathfrak{A}$ such that for each $B_0 \in \mathfrak{A}$, the following assertion holds:*

There is a positive integer N such that for each $B \in U$, we have

$$f\left(B^N B_0 x\right) < \inf(f) + \varepsilon$$

for each $x \in K$. In particular, for each $B \in U$, there is a positive integer N such that we have

$$f\left(B^N x\right) < \inf(f) + \varepsilon$$

for each $x \in K$.

Proof. By Lemma 3.2, there exist a neighborhood U of A in $\mathfrak{A}$ and a number $\delta > 0$ such that the following property holds:

For each $B \in U$ and each $x \in K$ satisfying $f(x) \geq \inf(f) + \varepsilon$, we have

$$f(Bx) < f(x) - \delta \tag{5.3}$$

Let $B_0 \in \mathfrak{A}$. Choose a positive integer N such that

$$\delta N > d_0 - \inf(f),$$

where $d_0 = \sup\{|f(B_0 x)| : x \in K\}$. It is clear that d_0 is finite because f is uniformly continuous. Assume that $B \in U$. We claim that

$$f\left(B^N B_0 x\right) < \inf(f) + \varepsilon \tag{5.4}$$

for each $x \in K$. Suppose to the contrary that this is not true. Then there exists $x \in K$ such that

$$f(B^n B_0 x) \geq \inf(f) + \varepsilon, \; n = 0, \dots, N.$$

By (5.3) and by induction it follows that for each $n = 1, \dots, N$,

$$f(B^n B_0 x) < f(B_0 x) - n\delta.$$

This implies that

$$f\left(B^N B_0 x\right) < f(B_0 x) - N\delta < d_0 - (d_0 - \inf(f)) = \inf(f),$$

a contradiction. Therefore, (5.4) is indeed valid, as claimed, and Theorem 5.2 is proved. □

Theorem 5.3. *Let $\{A_n\}_{n=1}^{\infty} \in \mathfrak{M}$ be normal and let $\varepsilon > 0$. Then there exists a neighborhood U of $\{A_n\}_{n=1}^{\infty}$ in $\mathfrak{M}$ with the strong topology such that for each $B_0 \in \mathfrak{A}$, the following assertion holds:*

There is a positive integer N such that for each $\{B_n\}_{n=1}^{\infty} \in U$ and each mapping $r : \{1, 2, \dots\} \to \{1, 2, \dots\}$, we have

$$f\left(B_{r(N)} \dots B_{r(1)} B_0 x\right) < \inf (f) + \varepsilon$$

for each $x \in K$. In particular, for each $\{B_n\}_{n=1}^{\infty} \in U$ and each mapping $r : \{1, 2, \dots\} \to \{1, 2, \dots\}$, there is a positive integer N such that

$$f\left(B_{r(N)} \dots B_{r(1)} x\right) < \inf (f) + \varepsilon$$

for each $x \in K$.

Proof. By Lemma 3.2, there exist a neighborhood U of $\{A_n\}_{n=1}^{\infty}$ in $\mathfrak{M}$ with the strong topology and a number $\delta > 0$ such that the following property holds:

For each $\{B_n\}_{n=1}^{\infty} \in U$ and each point $x \in K$ satisfying $f(x) \geq \inf (f) + \varepsilon$, we have

$$f(B_n x) < f(x) - \delta \tag{5.5}$$

for each $n = 1, 2, \dots$.

Let $B_0 \in \mathfrak{A}$. Choose a positive integer N such that

$$\delta N > d_0 - \inf (f),$$

where $d_0 = \sup \{|f(B_0 x)| : x \in K\}$. Clearly, d_0 is finite because f is uniformly continuous. Now assume that $\{B_n\}_{n=1}^{\infty} \in U$ and $r : \{1, 2, \dots\} \to \{1, 2, \dots\}$. We claim that

$$f\left(B_{r(N)} \dots B_{r(1)} B_0 x\right) < \inf (f) + \varepsilon \tag{5.6}$$

for each $x \in K$. Suppose to the contrary that this is not true. Then there exists $x \in K$ such that

$$f\left(B_{r(n)} \dots B_{r(1)} B_0 x\right) \geq \inf (f) + \varepsilon, \ n = 0, \dots, N.$$

Using (5.5) and induction, we see that for each $n = 1, \dots, N$,

$$f\left(B_{r(n)} \dots B_{r(1)} B_0 x\right) < f(B_0 x) - n\delta.$$

This implies that

$$f\left(B_{r(N)} \dots B_{r(1)} B_0 x\right) < f(B_0 x) - N\delta < d_0 - (d_0 - \inf (f)) = \inf (f),$$

a contradiction. Therefore, (5.6) is indeed valid, as claimed, and Theorem 5.3 is established. □

6. Proofs of the main results

Theorem 2.1 is a direct consequence of Theorems 4.1 and 5.1. Theorem 2.2 is a direct consequence of Theorems 4.2 and 5.2. Theorem 2.3 is a direct consequence of Theorems 4.3 and 5.3.

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