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POSSIBLE CAUSAL RELATIONSHIP EXTENSIONS AND PROPERTIES OF RCI

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Abstract. In this paper, the causality relationship between pairs of events in Minkowski space-time is extended to the General Relativity in the context of the tangent spaces of pseudo-Riemannian manifolds. Here the space-time interval between events is inferred from the simmetric bilinear forms induced from linear self-adjoint operators. Strong causal isomorphisms are introduced and studied here. Underline that by expansivity of the symmetric bilinear form in the range of an of strong casual isomorphism f with respect to the symmetric bilinear form in the domain of f, it follows the nonexpansivity of f. At least, we show, with an example, that in the setting of General Relativity the causal automorphisms are structurally different from the case of Special Relativity Theory.

1. Introduction

Following E. M. Howard [7],

"Roughly speaking, we define causality as the relation between two events correlated in a regular pattern or between a cause an an effect. All physical theories assume causa-tion as an inherent fundamental assumption. In relativity,an event can influence another event only if there is a causal (timelike or null) curve connecting the two space-timepoints".

However, there is currently no single precise definition of the causal structure of space-time.
For example, in addition to the usual causality, which we will see shortly, we have K-causality (introduced by Sorkin and Woolgar in [11] and also the stable causality [5], both studied extensively by Minguzzi [10]. A celebrated interesting survey containing causality relationships is given in [3], see also [12] [6] : here we approach the issue from a more mathematical than physical point of view and try to introduce causal relationships induced by bounded self-adjoint operators in spaces with inneer product. Our interest will be devoted above all to the study of causal isomorphisms Let's start with the usual causality relationship in special relativity.

In the context of Special Relativity Theory, in Minkowski space-time, given two events, represented by the 4-vectors $X = (x_0, x_1, x_2, x_3)$ and $Y = (y_0, y_1, y_2, y_3)$we

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say that X is causally related to Y if and only if it holds

$$\Delta S^2 = (x_0 - y_0)^2 - \sum_{i=1}^{3}(x_i - y_i)^2 \geq 0$$

Here the x_0 and y_0 components of the two 4-vectors have the meaning of temporal coordinates, i.e. $x_0 = ct_X$ and $y_0 = ct_Y$, where c is the speed of light in the vacuum and t_X, t_Y are the moments of time in which the X and Y events, respectively, take place. The components x_1, x_2, x_3 and y_1, y_2, y_3 are instead the spatial coordinates of the points where the two events take place compared to an orthogonal Cartesian reference frame. The physical interpretation of the relationship introduced is as follows:
X and Y events are causally related (we will write $X \subset Y$) if and only if the space $c|t_X - t_Y|$ that a ray of light travels through in the time $\Delta t = |t_X - t_Y|$ is greater or equal than the spatial distance $S = \left(\sum_{i=1}^{3}(x_i - y_i)^2\right)^{\frac{1}{2}}$ between the points at which the events take place. We observe that this relation can be expressed in the following way:
let $T_M : \mathbb{R}^4 \to \mathbb{R}^4$ be the self-adjoint bijective operator represented in the canonic base of $\mathbb{R}^4$ by the diagonal matrix $diag(1; -1; -1; -1)$ of order 4; it is

$$(\Delta S)^2 = \langle X - Y, T_M X - T_M Y\rangle \geq 0$$

where with the parentheses $\langle \cdot, \cdot \rangle$ we represented the usual scalar product of $\mathbb{R}^4$.
So we can write $X \subset Y$ if and only if $\langle X - Y, T_M X - T_M Y\rangle \geq 0$.
A bijective map $f : \mathbb{R}^4 \to \mathbb{R}^4$ that preserves, with its inverse, the causal relationship, will be called causal automorphism. Therefore a causal automorphism, in the context of Special Relativity Theory, is a bijective map f such that

$$\forall X, Y \in \mathbb{R}^4, \qquad X \subset Y \iff f(X) \subset f(Y)$$

i.e.

$$\langle X - Y, T_M X - T_M Y\rangle \geq 0 \iff \langle f(X) - f(Y), T_M(f(X) - f(Y))\rangle \geq 0$$

An important result about causal automorphisms was obtained by mathematicians Alexandrov and Zeemann who proved in [13] and [1] the following statement, known as Alexandrov-Zeemann theorem:

Theorem 1.1. *All the causal automorphisms of Minkowski's space-time towards itself are the elements of the group generated by Lorentz's transformations, homotheties and translations.*

This group of transformations is nothing else than the Poincaré group $P(\mathbb{R}^4)$; a mapping f in $P(\mathbb{R}^4)$ has the shape $f(X) = \lambda L(X) + v_0$ where L is a Lorentz transformation, λ a real number and v_0 a constant vector of linear space $\mathbb{R}^4$.

Remark 1.2. The causality relationship is reflexive and symmetric but not transitive, as is shown by the following example.

Example 1.3. In space $\mathbb{R}^3$ let be $P = (x_1, x_2, x_3)$ a point from which two rays of light are emitted, at time t, in opposite directions along a line that passes through P_X; besides let be $P_Y = (y_1, y_2, y_3)$ and $P_Z = (z_1, z_2, z_3)$ two points placed along that line, from opposite parts of P and equidistant from it. We have $Y = (ct', y_1, y_2, y_3)$ and $Z = (ct', z_1, z_2, z_3)$, where t' is the time when the two rays of light arrive at the P_Y and P_Z points. We have $Y \subset X$ and $X \subset Z$, being

$$c^2(t-t')^2 - \sum_{i=1}^{3}(x_i - y_i)^2 = 0$$

and

$$c^2(t-t')^2 - \sum_{i=1}^{3}(z_i - x_i)^2 = 0$$

but it is not $Y \subset Z$ being

$$c^2(t'-t')^2 - \sum_{i=1}^{3}(y_i - z_i)^2 = -\sum_{i=1}^{3}(y_i - z_i)^2 < 0$$

Our aim (in Section 2 and 3) will be now to extend, in a purely formal way, the causality relationship from the Minkowski's space-time, to a more general context, in order to achieve the coherent definition of a causal relationship in the context of the General Relativity Theory. Then sets of strong causal isomorphisms and causal isomorphisms will be introduced highlighting some of their notable formal properties.

2. Extension of the causality relationship to pairs of points of a separable inner product space

Consider here a separable real field linear space E with inner product $\langle \cdot, \cdot \rangle$ definite positive; then let $T : E \to E$ be a self-adjoint bijective linear operator. As well known the form $\tau(x, y) := \langle x, Ty \rangle$ is a bilinear form (generally degenerate and not definite positive). Relations of causality induced by self-adjoint bijective operators, formally identical to that introduced in Minkowski's space-time can be defined in these spaces, i.e.

$$x \subset_T y \iff \langle x - y, T(x - y) \rangle \geq 0 \quad \forall x, y \in E.$$

Of course, according to Example 1.3, these binary relationships are reflexive and symmetric, but generally not transitive.

In this setting, analogously to the case of Special Relativity Theory, we can introduce the notion of causal isomorphism.

Definition 2.1. Let $(E_1, \langle \cdot, \cdot \rangle_1, \subset_T)$ be an inner product space E_1 endowed with the causality relationship $\subset_T$ induced by self-adjoint bijective operator T. Let $(E_2, \langle \cdot, \cdot \rangle_2, \subset_S)$ be an inner product space E_2 endowed with the causality relationship $\subset_S$ induced by self-adjoint bijective operator S. A bijective mapping $f : E_1 \to E_2$ is called an causal isomorphism if the images of vectors causally related they are also casually related

$$x \subset_T y \iff f(x) \subset_S f(y).$$

That is, equivalently

$$\langle x-y, T(x-y)\rangle_1 \geq 0 \iff \langle f(x)-f(y), S(f(x)-f(y))\rangle_2 \geq 0$$

Note that, as in Special Relativity Theory, f can be not linear. However this can not happen if f is a **strong** causal isomorphism in the sense of the following definition:

Definition 2.2. Let $(E_1, \langle\cdot,\cdot\rangle_1, \subset_T)$ be an inner product space E_1 endowed with the causality relationship $\subset_T$ induced by self-adjoint bijective operator T. Let $(E_2, \langle\cdot,\cdot\rangle_2, \subset_S)$ be an inner product space E_2 endowed with the causality relationship $\subset_S$ induced by self-adjoint bijective operator S. A bijective mapping $f : E_1 \to E_2$ is called a strong causal isomorphism if

$$\langle x, T(y)\rangle_1 = \langle f(x), S(f(y))\rangle_2, \qquad \forall x, y \in E_1$$

Of course a strong casual isomorphism is a casual isomorphism. But there is more. Indeed, we show the following results:

Theorem 2.3. *Let $(E_1, \langle\cdot,\cdot\rangle_1)$, $(E_2, \langle\cdot,\cdot\rangle_1)$ be two separable inner product spaces on the complex number field $\mathbb{C}$ having the same dimension, finite or countable. Let $G_1 : E_1 \to E_1$, G :$_2$: $E_2 \to E_2$ be two bijective linear operators. Moreover, let $f : E_1 \to E_2$ be a bijective map such that $\forall x, y, \in E_1$, it results*

$$\langle x, G_1 y\rangle_1 = \langle f(x), G_2(f(y))\rangle_2.$$

Then f is linear. It follows that any strong causal isomorphism is a linear operator.

Proof. For all $y \in E_2$, let $v \in E_1$ and $w \in E_2$ be such that $G_2 w = y$ and $f(v) = w$. In the following, for convenience of notation we will not specify whether the vectors we consider are in E_1 or E_2. And we do not want even specify if wea are considering $\langle\cdot,\cdot\rangle_1$ or $\langle\cdot,\cdot\rangle_2$. All of this is very clear.

$$\begin{aligned}\langle f(\alpha x), y\rangle &= \langle f(\alpha x), G_2 w\rangle = \langle f(\alpha x), G_2 f(v)\rangle \geq \langle \alpha x, G_1 v\rangle \\ &= \alpha\langle x, G_1 v\rangle = \alpha\langle f(x), G_2 f(v)\rangle = \alpha\langle f(x), G_2 w\rangle = \alpha\langle f(x), y\rangle\end{aligned}$$

$>$From which $f(\alpha x) = \alpha f(x)$ for all scalar α and for all vectors x. Besides,

$$\begin{aligned}\langle f(x_1+x_2), y\rangle &= \langle f(x_1+x_2), G_2 f(v)\rangle = \langle x_1+x_2, G_1 v\rangle \\ &= \langle f(x_1), G_2 f(v)\rangle + \langle f(x_2), G_2 f(v)\rangle = \langle f(x_1)+f(x_2), y\rangle\end{aligned}$$

that gives $f(x_1+x_2) = f(x_1) + f(x_2)$. $\square$

But there is even more. We can show that f is also continuous under the assumptions of continuity of G_1 and G_2. For this, we use the following celebrate result that a linear mapping between normed spaces is continuous if and only if it is weakly continuous. Following [4],

The weak continuity of a bounded linear operator was as first noticed by Banach in his 1922 masterpiece: the converse was proved by Dunford.

Theorem 2.4. *A linear mapping $F : N \to M$ between the normed linear spaces N and M is norm-to-norm continuous if and only if F is weak-to-weak continuous.*

Proof. A Proof complete can be found in [4], page 12. $\square$

So now we have the tools to prove the result announced above:

Theorem 2.5. *Let $(E_1, \langle\cdot,\cdot\rangle_1)$, $(E_2, \langle\cdot,\cdot\rangle_1)$ be two separable inner product spaces on the complex number field $\mathbb{C}$ having the same dimension, finite or countable. Let $G_1 : E_1 \to E_1$, $G_2 : E_2 \to E_2$ be two bijective linear operators. Moreover, let $f : E_1 \to E_2$ be a bijective map such that $\forall x, y, \in E_1$, it results*

$$\langle x, G_1 y\rangle_1 = \langle f(x), G_2(f(y))\rangle_2. \tag{2.1}$$

If $G_1$and G_2 are continuous, then the map f is continuous with the bound

$$\|f\| \leq \sqrt{\frac{\|G_1\|}{\|G_2\|}}$$

It follows that any strong causal isomorphism is a linear bounded operator.

Proof. Step 1. Thanks to Theorem 2.5, to show the continuity of f it is equivalent to show its weak continuity.
Let thus $\langle x_n, y\rangle \to 0$, for all $y \in E_1$. We need to show that $\langle f(x_n), z\rangle \to 0$ for all $z \in E_2$.
Indeed, any $z \in E_2$ can be written as $z = G_2 f(y)$, $w \in E_1$, and so

$$\langle f(x_n), z)\rangle = \langle f(x_n), G_2 f(y)\rangle = \text{by (2.1)} = \langle x_n, G_1 y)\rangle \to 0.$$

Thanks the bijectivity of G_1 the thesis follows.
Step 2. Show now the bound of $\|f\|$. Indeed, we can write the equality $\|f(x)\|^2 = \langle f(x), G_2(G_2^{-1} f(x))\rangle_2$ as $\|f(x)\|^2 = \langle f(x), G_2(f(y))\rangle_2$, where $y \in E_1$ is such that $f(y) = G_2^{-1} f(x)$ i.e.

$$y = f^{-1} G_2^{-1} f(x) \tag{2.2}$$

So we obtain

$$\begin{aligned} \|f(x)\|^2 &= \langle f(x), G_2 f(y)\rangle_2 = \text{ (since } f \text{ is a strong causal isomorphism)} \\ &= \langle x, G_1 y\rangle \leq \|x\|\|G_1 y\| = \text{ (by (2.2))} \\ &= \|x\|\|G_1 f^{-1} G_2^{-1} f(x)\| \leq \|x\|\|G_1 f^{-1} G_2^{-1}\|\|f(x)\| \end{aligned}$$

that yields $\|f(x)\| \leq \|x\|\|G_1 f^{-1} G_2^{-1}\|$ hence f is bounded and

$$\|f\| \leq \|G_1 f^{-1} G_2^{-1}\| \leq \|G_1\|\|f^{-1}\|\|G_2^{-1}\| \leq \frac{\|G_1\|}{\|f\|\|G_2\|}.$$

And so, finally

$$\|f\| \leq \sqrt{\frac{\|G_1\|}{\|G_2\|}}$$

$\square$

Corollary 2.6. *Under the assumptions of Theorem 2.5 and if moreover G_2 is expansive with respect to G_1 (i.e. if $\|G_2\| \geq \|G_1\|$) then f is nonexpansive.*

3. Extension of the causality relationship to pairs of points of a Pseudo-Riemannian manifold.

Now we want to extend the causal relationship in the context of the General Relativity Theory.
In this setting, the linear spaces introduced can be interpreted as tangent spaces of a 4-dimensional pseudo-Riemannian manifold in two particular points of it. In this manifold, fixed a point $P(\bar{X})$ and an open set $\mathcal{T}(P)$, if $P_1(X)$ and $P_2(X)$ are in $\mathcal{T}(P)$, (with $\bar{X}, X, Y$ we denoted local coordinates of P, P_1, P_2), we will say that the event X is causally related to event Y if and only if $\int_\gamma G_{i,j} dx^i dx^j \geq 0$; in this last formula $G_{i,j}$ is the metric tensor defined in the open set $\mathcal{T}(P)$ and γ is the geodesic that has as extremes P_1 and P_2.
$(dS)^2 = G_{i,j} dx^i dx^j$ is the square of the space time interval that separates two indefinitely neighgboring events, expressed in differential form [8],[2],[9]. So that if P' is a point distinct from P and $\mathcal{T}(P')$ is its open set, let

$$f : \mathcal{T}(P) \to \mathcal{T}(P')$$

be a one-to-one map that preserves, with its inverse, the causal relationship introduced above, that is, an application such that $\forall x, y \in \mathcal{T}(P)$ $X \subset Y \iff f(X) \subset f(Y)$. Then the map f is called *causal isomorphism.*
In our approximation we consider, in place of open sets $\mathcal{T}(P), \mathcal{T}(P')$ of the manifold, the tangent spaces in the points P and P' with associated inner products induced by operators T and S, whose matrices elements are coincident with the values of components of metric tensor $G_{i,j}$ in points P and P' respectively [2, 9].
Thus, in place of the causal relationships expressed above in the integral form, we consider those defined from the bilinear forms induced by the operators $T$and S. This is a reasonable approximation that will allow us to draw formal properties of considerable interest of the corresponding sets of causal isomorphisms. Let $(E_1, \langle\cdot,\cdot\rangle_1, T)$ and $(E_2, \langle\cdot,\cdot\rangle_2, S)$ be real linear separable spaces with associated bilinear forms induced by linear self-adjoint bijective operators T and S.
Let $\mathcal{L}(E_1, E_2)$ be the set of strong isomorphisms from E_1 to E_2.
We know that any $L \in \mathcal{L}(E_1, E_2)$ is linear (Theorem 2.3) and also continuous if T and S are (Theorem 3.1). Show now that $\mathcal{L}(E_1, E_2)$ is closed with respect to the invertibility.

Theorem 3.1. *$L \in \mathcal{L}(E_1, E_2)$ implies that $L^{-1} \in \mathcal{L}(E_2, E_1)$*

Proof. $\langle L^{-1}v, TL^{-1}w\rangle = \langle LL^{-1}v, SLL^{-1}w\rangle = \langle v, Sw\rangle$ □

Let then $\mathcal{K}(E_1, E_2)$ be the set of causal isomorphisms from E_1 to E_2.

Of course $\mathcal{L}(E_1, E_2) \subseteq \mathcal{K}(E_1, E_2)$ and also is closed with respect to the invertibility.
Finally we want to see that $\mathcal{K}(E_1, E_2)$ is a larger set than the analogous set of causal isomorphisms in the Special Relativity Theory, that is the Poincaré group $P(\mathbb{R}^4)$, i.e

the set of mappings f in $P(\mathbb{R}^4)$ has the shape$f(X) = \lambda L(X) + v_0$. For this purpose, define $P(E_1, E_2) := \{f : E_1 \to E_2 | f(x) = \lambda Lx + v_0, \lambda \neq 0, L \in \mathcal{L}(E_1, E_2)\}$.

The first result is expected.

Theorem 3.2. *$P(E_1, E_2) \subseteq \mathcal{K}(E_1, E_2)$.*

Proof. $f \in P(E_1, E_2)$ implies that $f(x) = \lambda Lx + v_0$. See that f is a causal isomorphism, that is

$$x \subset_T y \iff f(x) \subset_S f(y)$$

i.e.

$$\langle x - y, T(x - y)\rangle_1 \geq 0 \iff \langle f(x) - f(y), S(f(x) - f(y))\rangle_2 \geq 0.$$

Suppose thus $\langle x - y, T(x - y)\rangle_1 \geq 0$.
Then

$$\begin{aligned} \langle (x) - f(y), S(f(x) - f(y))\rangle_2 &= \langle \lambda Lx - \lambda Ly, S(\lambda Lx - \lambda Ly)\rangle \\ &= \lambda^2 \langle Lx - Ly, S(Lx - Ly)\rangle \\ &= \lambda^2 \langle x - y, T(x - y)\rangle_1 \geq 0. \end{aligned}$$

□

We see now that also $P(E_1, E_2)$ is closed with respect to the invertibility.

Theorem 3.3. *$f \in P(E_1, E_2)$ implies that $f^{-1} \in P(E_2, E_1)$.*

Proof. We show that if $f(x) = v$, then it results

$$f^{-1} = \frac{1}{\lambda} L^{-1} - \frac{1}{\lambda} L^{-1} v_0$$

Indeed, this follows by the two formulas

$$\left(f \circ \left(\frac{1}{\lambda} L^{-1} - \frac{1}{\lambda} L^{-1} v_0\right)\right)(v) = \lambda L \left(\frac{1}{\lambda} L^{-1} v - \frac{1}{\lambda} L^{-1} v_0\right) + v_0 = v$$

and

$$\begin{aligned} \left(\left(\frac{1}{\lambda} L^{-1} - \frac{1}{\lambda} L^{-1} v_0\right) \circ f\right)(x) &= \left(\frac{1}{\lambda} L^{-1} - \frac{1}{\lambda} L^{-1} v_0\right)(\lambda Lx + v_0) \\ &= \frac{1}{\lambda} L^{-1}(\lambda Lx + v_0) - \frac{1}{\lambda} L^{-1} v_0 = x \end{aligned}$$

□

Corollary 3.4. *If $E_1 = E_2 = E$, then $\mathcal{L}(E) \subseteq P(E) \subseteq \mathcal{K}(E)$ are non abelian transformations groups respect to the composition.*

Finally, we underline another difference between the Special and General Relativity Theory. We know that in the Special Relativity Theory it results $P(E_1, E_2) = \mathcal{K}(E_1, E_2)$.
The last our result shows that this does not hold in the General Relativity Theory.

Theorem 3.5. . *In general $P(E_1, E_2) \neq \mathcal{K}(E_1, E_2)$.*

Proof. It is enough to give a counterexample. Indeed, let be $E_1 = E_2 = \mathbb{R}$. Fix a positive number a and take the linear, bijective, self-adjoint operator T defined by $Tx := ax$. The relationschip causality induced is given by

$$x \subset_T y \iff (x-y)a(x-y) = a(x-y)^2 \geq 0 \qquad \forall x, y \in \mathbb{R}.$$

So in this context a causal isomorphism is any bijective map from $\mathbb{R}$ to $\mathbb{R}$. For example $f(x) := x^3$ is such a map, but it has not the form $f(x) = \lambda x = b$. □

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