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SPLITTING ALGORITHMS OF COMMON SOLUTIONS BETWEEN EQUILIBRIUM AND INCLUSION PROBLEMS ON HADAMARD MANIFOLDS

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Abstract. The aim of this article is to introduce an iterative algorithm for finding a common solution from the set of an equilibrium point for a bifunction and the set of a singularity of an inclusion problem on a Hadamard manifold. We also discuss some particular cases of the problem by the proposed algorithm. The convergence of a sequence generated by the proposed algorithm is proved under appropriate assumptions. Moreover, we apply our results for solving minimization problems and minimax problems.

1. Introduction

Equilibrium problem (EP) was firstly introduced by Fan [11] and extensively developed later by Blum and Oettli [3]. Let H be a real Hilbert space, K a nonempty closed convex subset of H and $F : K \times K \longrightarrow \mathbb{R}$ a bifunction satisfying $F(x, x) = 0$, for all $x \in K$. The equilibrium problem for a bifunction F is to find $x \in K$ such that

$$F(x, y) \geq 0, \quad \forall y \in K. \tag{1.1}$$

Herein, F is said to be the *equilibrium bifunction*. The theory of equilibrium problems plays a vital role in nonlinear problems, e.g., variational inequalities, optimization problems, Nash equilibrium problems, complementarity problems and so on, (see, for example [14, 17, 20, 27] and the references therein).

In 1976, Rockafellar [22] considered the following inclusion problem:

$$\text{find } x \in K \text{ such that } 0 \in A(x), \tag{1.2}$$

where $A : K \longrightarrow 2^H$ is a maximal monotone operator. The classical method for solving inclusion problem (1.2) is the proximal point method. The proximal method was firstly introduced by Martinet [19] for convex minimization and further generalized by Rockafellar [22]. Many problems in nonlinear analysis, optimization problem, convex programming problem, variational inequality problem, PDEs, economics are reduced to finding a singularity of the problem (1.2), see for example [6–8, 13, 18] and the references therein.

During the last decade, many issues in nonlinear analysis such as fixed point theory, convex analysis, variational inequality, equilibrium theory, and optimization

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theory have, been magnified from linear setting, namely, Banach spaces or Hilbert spaces, etc., to common solution because the problems cannot be posted in the linear space and require a manifold structure (not necessary with linear structure). The main advantages of these extensions are that non-convex problems in the general sense are transformed into convex problems, and constraint problems also transform into unconstraint problems. Eigenvalue optimization problems [24] and geometric models for the human spine [1] are typical examples of the situation. Therefore, many authors have focused on extension and development of nonlinear problems techniques on the Riemannian manifold, see for examples [9, 12, 16, 25] and the reference therein.

In 2012, Calao et al. [9] studied the equilibrium problems on a Hadamard manifold. Let M be an Hadamard manifold, TM the tangent bundle of M, K a nonempty closed geodesic convex subset of M, and $F : K \times K \longrightarrow \mathbb{R}$ a bifunction satisfying $F(x,x) = 0$, for all $x \in K$. Then, the equilibrium problem on the Hadamard manifold is to find $x \in K$ such that

$$F(x,y) \geq 0, \quad \forall y \in K. \tag{1.3}$$

We denote by $EP(F)$ the set of equilibrium points of the equilibrium problem (1.3). They [9] studied the existence of an equilibrium point for a bifunction under suitable conditions and applied their results to solve mixed variational inequality problems, fixed point problems and Nash equilibrium problems in Hadamard manifolds. The authors also introduced Picard iterative method to approximate solutions of the problem (1.3). However, Wang et al. [28] found some gaps in the existence proof of the mixed variational inequalities and the domain of the resolvent for the equilibrium problems in [9].

The inclusion problem (1.2) is generalized by Li et al. [15] in Hadamard manifolds, and it reads as follows:

$$\text{find } x \in K \text{ such that } \mathbf{0} \in A(x), \tag{1.4}$$

where $A : K \longrightarrow 2^{TM}$ is a multivalued vector field on Hadamard manifolds and $\mathbf{0}$ denotes the zero section of TM. We denote by $A^{-1}(\mathbf{0})$ the set of singularities of the inclusion problem (1.4). The authors also extended the general proximal point method from Euclidean spaces to Hadamard manifolds for solving the inclusion problem (1.4).

Motivated by above results, we introduce iterative algorithm for finding a common solution of the equilibrium problem (1.3) and the inclusion problem (1.4) on Hadamard manifolds. Our proposed algorithm can be regraded as the double-backward method for the two underlying problems.

The rest of this paper is organized in the following: In Section 2, we give some basic concepts and fundamental results of Riemannian manifolds as well as some useful results. In Section 3, we introduce the problem of finding $x \in EP(F) \cap A^{-1}(\mathbf{0})$, which is a common solution of the sets of equilibrium points and singularlity of an inclusion problem. We propose an iterative algorithm for finding a common solution of the proposed problem, and establish convergence results of a sequence generated by the proposed algorithm converges to a solution of the proposed problem on Hadamard manifolds. In the last section, we devote our results to minimization problems and minimax problems on Hadamard manifolds.

2. Preliminaries

In this section, we recall some fundamental definitions, properties, useful results, and notations of Riemannian geometry. Readers refer to some textbooks [5, 23, 26] for more details.

Let M be a connected finite-dimensional manifold. For $p \in M$, we denote T_pM the *tangent space* of M at p which is a vector space of the same dimension as M, and by $TM = \bigcup_{p\in M} T_pM$ the *tangent bundle* of M. We always suppose that M can be endowed with a Riemannian metric $\langle\cdot,\cdot\rangle_p$, with corresponding norm denoted by $\|\cdot\|_p$, to become a *Riemannian manifold*. The angle $\angle_p(u,v)$ between $u, v \in T_pM$ $(u, v \neq \mathbf{0})$ is set by $\cos\angle_p(u,v) = \frac{\langle u,v\rangle_p}{\|u\|\|v\|}$. If there is no confusion, we denote $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_p$, $\|\cdot\| := \|\cdot\|_p$ and $\angle(u,v) := \angle_p(u,v)$. Let $\gamma : [a,b] \longrightarrow M$ be a piecewise smooth curve joining $\gamma(a) = p$ to $\gamma(b) = q$, we define the length of the curve γ by using the metric as

$$L(\gamma) = \int_a^b \|\gamma^{'}(t)\|dt.$$

Minimizing the length function over the set of all such curves, we obtain a Riemannian distance $d(p,q)$ which induces the original topology on M.

Let ∇ be a Levi-Civita connection associated to $(M, \langle\cdot,\cdot\rangle)$. Given a smooth curve γ, a smooth vector field X along γ is said to be *parallel* if $\nabla_{\gamma^{'}}X = \mathbf{0}$. If $\gamma^{'}$ itself is parallel, we say that γ is a *geodesic*, and in this case $\|\gamma^{'}\|$ is a constant. When $\|\gamma^{'}\| = 1$, γ is said to be *normalized*. A geodesic joining p to q in M is said to be a *minimal geodesic* if its length equals to $d(p,q)$.

A Riemannian manifold is complete if for any $p \in M$ all geodesic emanating from p are defined for all $t \in \mathbb{R}$. From the Hopf-Rinow theorem we know that if M is complete then any pair of points in M can be joined by a minimal geodesic. Moreover, (M,d) is a complete metric space and every bounded closed subset is compact.

Let M be a complete Riemannian manifold and $p \in M$. The exponential map $\exp_p : T_pM \longrightarrow M$ is defined as $\exp_p v = \gamma_v(1,p)$, where $\gamma(\cdot) = \gamma_v(\cdot,p)$ is the geodesic starting at p with velocity v (i.e., $\gamma_v(0,p) = p$ and $\gamma_v^{'}(0,p) = v$). Then, for any value of t, we have $\exp_p tv = \gamma_v(t,p)$ and $\exp_p \mathbf{0} = \gamma_v(0,p) = p$. Note that the exponential $\exp_p$ is differentiable on T_pM for all $p \in M$. It is well-known that the derivative $D\exp_p(\mathbf{0})$ of $\exp_p(\mathbf{0})$ is equal to the identity vector of T_pM. Therefore, by the inverse mapping theorem, there exists an inverse exponential map $\exp_p^{-1} : M \longrightarrow T_pM$. Moreover, for any $p, q \in M$, we have $d(p,q) = \|\exp_p^{-1} q\|$.

A complete simply connected Riemannian manifold of non-positive sectional curvature is said to be an *Hadamard manifold.* Throughout the remainder of the paper, we always assume that M is a finite-dimensional Hadamard manifold. The following proposition is well-known and will be useful.

Proposition 2.1 ([23]). *Let $p \in M$. The exponential map $\exp_p : T_pM \longrightarrow M$ is a diffeomorphism, and for any two points $p, q \in M$ there exists a unique normalized geodesic joining p to q, which is can be expressed by the formula*

$$\gamma(t) = \exp_p t \exp_p^{-1} q, \quad \forall t \in [0, 1].$$

This proposition yields that M is diffeomorphic to the Euclidean space $\mathbb{R}^n$. Then, M has same topology and differential structure as $\mathbb{R}^n$. Moreover, Hadamard manifolds and Euclidean spaces have some similar geometrical properties. One of the most important properties is illustrated in the following propositions.

A geodesic triangle $\triangle(p_1, p_2, p_3)$ of a Riemannian manifold M is a set consisting of three points p_1, p_2 and p_3, and three minimal geodesics γ_i joining p_i to p_{i+1} where $i = 1, 2, 3 \pmod 3$.

Proposition 2.2 ([23]). *Let $\triangle(p_1, p_2, p_3)$ be a geodesic triangle in M. For each $i = 1, 2, 3 \pmod 3$, given $\gamma_i : [0, l_i] \longrightarrow M$ the geodesic joining p_i to p_{i+1} and set $l_i := L(\gamma_i)$, $\alpha_i := \angle(\gamma_i^{'}(0), -\gamma_{i-1}^{'}(l_{i-1}))$. Then,*

$$\alpha_1 + \alpha_2 + \alpha_3 \leq \pi; \tag{2.1}$$

$$l_i^2 + l_{i+1}^2 - 2 l_i l_{i+1} \cos \alpha_{i+1} \leq l_{i-1}^2. \tag{2.2}$$

In the terms of the distance and the exponential map, the inequality (2.2) *can be rewritten as*

$$d^2(p_i, p_{i+1}) + d^2(p_{i+1}, p_{i+2}) - 2\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2} \rangle \leq d^2(p_{i-1}, p_i), \tag{2.3}$$

where $\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2} \rangle = d(p_i, p_{i+1}) d(p_{i+1}, p_{i+2}) \cos \alpha_{i+1}$.

The following relation between geodesic triangles in Riemannian manifolds and triangles in $\mathbb{R}^2$ can be referred to [4].

Lemma 2.3 ([4]). *Let $\triangle(p_1, p_2, p_3)$ be a geodesic triangle in M. Then, there exists a triangle $\triangle(\overline{p_1}, \overline{p_2}, \overline{p_3})$ for $\triangle(p_1, p_2, p_3)$ such that $d(p_i, p_{i+1}) = \|\overline{p_i} - \overline{p_{i+1}}\|$, indices taken modulo 3; it is unique up to an isometry of $\mathbb{R}^2$.*

The triangle $\triangle(\overline{p_1}, \overline{p_2}, \overline{p_3})$ in Lemma 2.3 is said to be a *comparison triangle* for $\triangle(p_1, p_2, p_3)$. The geodesic side from x to y will be denoted $[x, y]$. A point $\overline{x} \in [\overline{p_1}, \overline{p_2}]$ is said to be a *comparison point* for $x \in [p_1, p_2]$ if $\|\overline{x} - \overline{p_1}\| = d(x, p_1)$. The interior angle of $\triangle(\overline{p_1}, \overline{p_2}, \overline{p_3})$ at $\overline{p_1}$ is said to be the *comparison angle* between $\overline{p_2}$ and $\overline{p_3}$ at $\overline{p_1}$ and is denoted $\angle_{\overline{p_1}}(\overline{p_2}, \overline{p_3})$. With all notation as in the statement of Proposition 2.2, according to the law of cosine, (2.2) is valid if and only if

$$\langle \overline{p_2} - \overline{p_1}, \overline{p_3} - \overline{p_1} \rangle_{\mathbb{R}^2} \leq \langle \exp_{p_1}^{-1} p_2, \exp_{p_1}^{-1} p_3 \rangle \tag{2.4}$$

or,

$$\alpha_1 \leq \angle_{\overline{p_1}}(\overline{p_2}, \overline{p_3})$$

or, equivalently, $\triangle(p_1, p_2, p_3)$ satisfies the CAT(0) inequality and that is, given a comparison triangle $\overline{\triangle} \subset \mathbb{R}^2$ for $\triangle(p_1, p_2, p_3)$ for all $x, y \in \triangle$,

$$d(x, y) \leq \|\overline{x} - \overline{y}\|, \tag{2.5}$$

where $\overline{x}, \overline{y} \in \overline{\triangle}$ are the respective comparison points of x, y.

A subset K is called *geodesic convex* if for every two points p and q in K, the geodesic joining p to q is contained in K, that is, if $\gamma : [a, b] \longrightarrow M$ is a geodesic such that $p = \gamma(a)$ and $q = \gamma(b)$, then $\gamma((1-t)a + tb) \in K$ for all $t \in [0, 1]$.

A real function $f : M \longrightarrow \mathbb{R}$ is called *geodesic convex* if for any geodesic γ in M, the composition function $f \circ \gamma : [a, b] \longrightarrow \mathbb{R}$ is convex, that is,

$$(f \circ \gamma)(ta + (1-t)b) \leq t(f \circ \gamma)(a) + (1-t)(f \circ \gamma)(b)$$

where $a, b \in \mathbb{R}$, and $t \in [0, 1]$.

Proposition 2.4 ([23])**.** *Let $d : M \times M \longrightarrow \mathbb{R}$ be the distance function. Then, $d(\cdot, \cdot)$ is a geodesic convex function with respect to the product Riemannian metric, that is, for any pair of geodesics $\gamma_1 : [0, 1] \longrightarrow M$ and $\gamma_2 : [0, 1] \longrightarrow M$ the following inequality holds for all $t \in [0, 1]$*

$$d(\gamma_1(t), \gamma_2(t)) \leq (1-t)d(\gamma_1(0), \gamma_2(0)) + td(\gamma_1(1), \gamma_2(1)).$$

In particular, for each $y \in M$, the function $d(\cdot, y) : M \longrightarrow \mathbb{R}$ is a geodesic convex function.

The following notion and lemma are crucial in establishing our main convergence results.

Definition 2.5. [12] Let K be a nonempty subset of M and $\{x_n\}$ be a sequence in M. Then, $\{x_n\}$ is said to be *Fejér convergent* with respect to K if for all $p \in K$ and $n \in \mathbb{N}$,

$$d(x_{n+1}, p) \leq d(x_n, p).$$

Lemma 2.6 ([12])**.** *Let K be a nonempty subset of M and $\{x_n\}$ be a sequence in M such that $\{x_n\}$ be a Fejér convergent with respect to K. Then, the following hold:*

(i) *For every $p \in K$, $d(x_n, p)$ converges;*
(ii) *$\{x_n\}$ is bounded;*
(iii) *Assume that every cluster point of $\{x_n\}$ belongs to K. Then, $\{x_n\}$ converges to a point in K.*

Recall that for all $x, y \in \mathbb{R}^2$,

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2, \quad \forall t \in [0, 1].$$

Next, let us present the concepts of the monotonicity of vector fields.

Given K be a nonempty subset of M. Let $\mathfrak{X}(K)$ denote to the set of all multi-valued vector fields $A : K \longrightarrow 2^{TM}$ such that $A(x) \subseteq T_x M$ for each $x \in K$, and denote $D(A)$ the domain of A defined by $D(A) = \{x \in K : A(x) \neq \emptyset\}$.

Definition 2.7 ([10])**.** A vector field $A \in \mathfrak{X}(K)$ is said to be

(i) *monotone* if for all $x, y \in D(A)$

$$\langle u, \exp_x^{-1} y\rangle \leq \langle v, -\exp_y^{-1} x\rangle, \quad \forall u \in A(x) \text{ and } \forall v \in A(y);$$

(ii) *maximal monotone* if it is monotone and for all $x \in K$ and $u \in T_xK$, the condition

$$\langle u, \exp_x^{-1} y\rangle \leq \langle v, -\exp_y^{-1} x\rangle, \quad \forall y \in D(A) \text{ and } \forall v \in A(y),$$

implies that $u \in A(x)$.

The concept of Kuratowski semicontinuity on Hadamard manifolds was introduced by Li et al. [15].

Definition 2.8 ([15])**.** Let a vector field $A \in \mathfrak{X}(K)$ and $x_0 \in K$. Then A is said to be *upper Kuratowski semicontinuous at* x_0 if for any sequences $\{x_n\} \subseteq K$ and $\{v_n\} \subset TM$ with each $v_n \in A(x_n)$, the relations $\lim_{n\longrightarrow\infty} x_n = x_0$ and $\lim_{n\longrightarrow\infty} v_n = v_0$ imply that $v_0 \in A(x_0)$. Moreover, A is said to be *upper Kuratowski semicontinuous on* K if it is upper Kuratowski semicontinuous for each $x \in K$.

The definition of the resolvent of a multivalued vector field and firmly nonexpansive mappings on Hadamard manifolds was introduced by Li et al. [16].

Definition 2.9 ([16])**.** Let a vector field $A \in \mathfrak{X}(K)$ and $\lambda \in (0, \infty)$. The λ-*resolvent* of A is a multivalued map $J_\lambda^A : K \longrightarrow 2^K$ defined by

$$J_\lambda^A(x) := \{z \in K : x \in \exp_z \lambda A(z)\}, \quad \forall x \in K.$$

Remark 2.10 ([16])**.** Let $\lambda > 0$. By the definition of the resolvent of a vector field, then the range of the resolvent J_λ^A is contained the domain of A and $\mathrm{Fix}(J_\lambda^A) = A^{-1}(\mathbf{0})$.

Definition 2.11 ([16])**.** Let K be a nonempty subset of M and $T : K \longrightarrow M$ be a mapping. Then T is called *firmly nonexpansive* if for all $x, y \in K$, the function $\Phi : [0, 1] \longrightarrow [0, \infty)$ defined by

$$\Phi(t) := d(\exp_x t \exp_x^{-1} Tx, \exp_y t \exp_y^{-1} Ty), \quad \forall t \in [0, 1],$$

is nonincreasing.

A mapping $T : K \longrightarrow K$ is called *nonexpansive* if $d(T(x), T(y)) \leq d(x, y)$, for all $x, y \in K$, where $d(x, y)$ is a Riemannian distance. It turns out that the monotonicity and nonexpansivity are closely related.

Theorem 2.12 ([16])**.** *Let a vector field $A \in \mathfrak{X}(K)$. Then, for any $\lambda > 0$, the vector field A is monotone if and only if J_λ^A is single-valued and firmly nonexpansive.*

Proposition 2.13 ([16])**.** *Let K be a nonempty subset of M and $T : K \longrightarrow M$ be a firmly nonexpansive mapping. Then*

$$\langle \exp_{Ty}^{-1} x, \exp_{Ty}^{-1} y\rangle \leq 0$$

holds for any $x \in \mathrm{Fix}(T)$ and for all $y \in K$.

The following lemma which is useful in establishing our main result.

Lemma 2.14 ([2]). *Let K be a nonempty closed subset of M and a vector field $A \in \mathfrak{X}(K)$ be a maximal monotone. Let $\{\lambda_n\} \subset (0, \infty)$ be a real sequence with* $\lim_{n \longrightarrow \infty} \lambda_n = \lambda > 0$ *and a sequence $\{x_n\} \subset K$ with* $\lim_{n \longrightarrow \infty} x_n = x \in K$ *such that* $\lim_{n \longrightarrow \infty} J^A_{\lambda_n}(x_n) = y$. *Then, $y = J^A_\lambda(x)$.*

We then turn towards the theory of bifunctions, their resolvents and the related equilibrium problems.

Let K be a nonempty closed geodesic convex set in M and $F : K \times K \longrightarrow \mathbb{R}$ be a bifunction. We suppose the following assumptions:

(A1) for all $x \in K$, $F(x, x) \geq 0$;
(A2) F is monotone, that is, for all $x, y \in K$, $F(x, y) + F(y, x) \leq 0$;
(A3) For every $y \in K$, $x \mapsto F(x, y)$ is upper semicontinuous;
(A4) For every $x \in K$, $y \mapsto F(x, y)$ is geodesic convex and lower semicontinuous;
(A5) $x \mapsto F(x, x)$ is lower semicontinuous;
(A6) There exists a compact set $L \subseteq M$ such that

$$x \in K \setminus L \Longrightarrow [\exists y \in K \cap L \text{ such that } F(x, y) < 0].$$

Calao et al. [9] introduced the concept of resolvent of a bifunction on Hadamard manifold as follows: let $F : K \times K \longrightarrow \mathbb{R}$, the resolvent of a bifunction F is a multivalued operator $T^F_r : M \longrightarrow 2^K$ such that for all $x \in M$

$$T^F_r(x) = \Big\{ z \in K : F(z, y) - \frac{1}{r} \langle \exp_z^{-1} x, \exp_z^{-1} y \rangle \geq 0, \ \forall y \in K \Big\}.$$

Let us end the preliminary section with the following results which discuss the regularization of a given bifunction.

Theorem 2.15 ([9, 28]). *Let $F : K \times K \longrightarrow \mathbb{R}$ be a bifunction satisfying the following conditions:*

(1) *F is monotone;*
(2) *for all $r > 0$, T^F_r is properly defined, that is, the domain $D(T^F_r) \neq \emptyset$.*

Then for any $r > 0$,

(i) *the resolvent T^F_r is single-valued;*
(ii) *the resolvent T^F_r is firmly nonexpansive;*
(iii) *the fixed point set of T^F_r is the equilibrium point set of F,*

$$\mathrm{Fix}(T^F_r) = EP(F).$$

Moreover, if F satisfying conditions (A1)–(A4). Then, $D(T^F_r) = M$.

3. Main results

In this paper, K always denotes a nonempty closed geodesic convex subset of Hadamard manifold M, unless explicitly stated otherwise. Let $A \in \mathfrak{X}(K)$ be a vector field and $F : K \times K \longrightarrow \mathbb{R}$ be a bifunction. We consider the problem of finding $x \in K$ such that

$$x \in EP(F) \cap A^{-1}(\mathbf{0}), \tag{3.1}$$

that is, x is simultaneously an equilibrium point of F and a singularity of A. We suppose that $\Omega := EP(F) \cap A^{-1}(\mathbf{0}) \neq \emptyset$.

We first introduce the following iterative algorithm for computing the approximate solutions of problem (3.1).

Algorithm 3.1. Let $A \in \mathfrak{X}(K)$ be a vector field and $F : K \times K \longrightarrow \mathbb{R}$ be a bifunction. Choose an initial point $x_0 \in K$ and define $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ as follows:

$$y_n := \exp_{x_n} \alpha_n \exp^{-1}_{x_n} J^A_{\lambda_n}(x_n), \tag{3.2}$$

$$z_n \in K \text{ such that } F(z_n, t) - \frac{1}{r_n}\langle \exp^{-1}_{z_n} y_n, \exp^{-1}_{z_n} t\rangle \geq 0, \quad \forall t \in K, \tag{3.3}$$

$$x_{n+1} := \exp_{x_n} \beta_n \exp^{-1}_{x_n} z_n, \quad \forall n \in \mathbb{N}, \tag{3.4}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}$ and $\{r_n\}$ are given real positive sequences such that

(i) $0 < a \leq \alpha_n, \beta_n \leq b < 1, \quad \forall n \in \mathbb{N}$,
(ii) $0 < \hat{\lambda} \leq \lambda_n \leq \tilde{\lambda} < \infty, \quad \forall n \in \mathbb{N}$,
(iii) $\liminf_{n \longrightarrow \infty} r_n > 0$.

When $F \equiv 0$, the Algorithm (3.1) becomes the following algorithm for finding a solution of the problem (1.4).

Algorithm 3.2. Let $A \in \mathfrak{X}(K)$ be a vector field. Choose initial point $x_0 \in K$ and define $\{x_n\}$ as follows:

$$x_{n+1} := \exp_{x_n} \alpha_n \exp^{-1}_{x_n} J^A_{\lambda_n}(x_n), \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset (0, \infty)$ are the same as in Algorithm 3.1.

When $A \equiv \mathbf{0}$, the Algorithm (3.1) becomes the following algorithm for finding a solution of the problem (1.3).

Algorithm 3.3. Let $F : K \times K \longrightarrow \mathbb{R}$ be a bifunction. Choose initial point $x_0 \in K$ and define $\{x_n\}$ and $\{z_n\}$ as follows:

$$z_n \in K \text{ such that } F(z_n, t) - \frac{1}{r_n}\langle \exp^{-1}_{z_n} x_n, \exp^{-1}_{z_n} t\rangle \geq 0, \quad \forall t \in K,$$

$$x_{n+1} := \exp_{x_n} \beta_n \exp^{-1}_{x_n} z_n, \quad \forall n \in \mathbb{N},$$

where $\{\beta_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, \infty)$ are the same as in Algorithm 3.1.

Now we prove the convergence of any sequences generated by Algorithm **??** to a common solution of problem (3.1).

Theorem 3.4. *Suppose that a vector field $A \in \mathfrak{X}(K)$ be a maximal monotone and $F : K \times K \longrightarrow \mathbb{R}$ be a bifunction satisfying assumptions (A1)–(A6) with $\Omega \neq \emptyset$. Then, the sequence generated by Algorithm 3.1 converges to a solution of problem* (3.1).

Proof. It is sufficient to show by Lemma 2.6 that $\{x_n\}$ is Fejér convergent with respect to Ω and the cluster points of $\{x_n\}$ belongs to Ω. We divide the proof into the following four steps.

Step I. We show that $\{x_n\}$ is Fejér convergent with respect to Ω.

Let $\omega \in \Omega$. Then $\omega \in EP(F)$ and $\omega \in A^{-1}(\mathbf{0})$. By Theorem 2.15, we have $z_n = T^F_{r_n}(y_n)$ and

$$\begin{aligned} d(z_n,\omega) &= d(T^F_{r_n}(y_n), T^F_{r_n}(\omega)) \\ &\leq d(y_n,\omega), \quad \text{for } \omega \in \Omega. \end{aligned} \tag{3.5}$$

Since $\omega \in A^{-1}(\mathbf{0})$, Remark 2.10 gives $\omega = J^A_{\lambda_n}(\omega)$. Set $u_n := J^A_{\lambda_n}(x_n)$ and let $\triangle(\omega, x_n, u_n) \subseteq M$ be a geodesic triangle with vertices ω, x_n and u_n, and let $\triangle(\overline{\omega}, \overline{x_n}, \overline{u_n}) \subseteq \mathbb{R}^2$ be the corresponding comparison triangle. Then, we have

$$d(x_n,\omega) = \|\overline{x_n} - \overline{\omega}\|, \quad d(x_n,u_n) = \|\overline{x_n} - \overline{u_n}\| \text{ and } d(u_n,\omega) = \|\overline{u_n} - \overline{\omega}\|. \tag{3.6}$$

Recall from (3.2) that $y_n = \exp_{x_n} \alpha_n \exp^{-1}_{x_n} u_n$, then we have

$$\overline{y_n} = (1-\alpha_n)\overline{x_n} + \alpha_n \overline{u_n}.$$

From (2.4) and (2.5), we get

$$\angle_\omega(u_n, x_n) \leq \angle_{\overline{\omega}}(\overline{u_n}, \overline{x_n}) \tag{3.7}$$

and

$$d(y_n,\omega) \leq \|\overline{y_n} - \overline{\omega}\|.$$

From the last inequality, (3.7) and $\alpha_n \in (0,1)$, then

$$\begin{aligned} d^2(y_n,\omega) &\leq \|\overline{y_n} - \overline{\omega}\|^2 \\ &= \|(1-\alpha_n)\overline{x_n} + \alpha_n\overline{u_n} - \overline{\omega}\|^2 \\ &= \|(\overline{x_n} - \overline{\omega}) - \alpha_n(\overline{x_n} - \overline{u_n})\|^2 \\ &= \|\overline{x_n} - \overline{\omega}\|^2 + \alpha_n^2\|\overline{x_n} - \overline{u_n}\|^2 - 2\alpha_n\|\overline{x_n} - \overline{\omega}\|\,\|\overline{x_n} - \overline{u_n}\|\cos\angle_{\overline{\omega}}(\overline{u_n}, \overline{x_n}) \\ &\leq \|\overline{x_n} - \overline{\omega}\|^2 + \alpha_n\|\overline{x_n} - \overline{u_n}\|^2 - 2\alpha_n\|\overline{x_n} - \overline{\omega}\|\,\|\overline{x_n} - \overline{u_n}\|\cos\angle_{\overline{\omega}}(\overline{u_n}, \overline{x_n}) \\ &= \|\overline{x_n} - \overline{\omega}\|^2 + \alpha_n\|\overline{x_n} - \overline{u_n}\|^2 - 2\alpha_n\langle \overline{x_n} - \overline{\omega}, \overline{x_n} - \overline{u_n}\rangle_{\mathbb{R}^2} \\ &= \|\overline{x_n} - \overline{\omega}\|^2 + (\alpha_n - 2\alpha_n)\|\overline{x_n} - \overline{u_n}\|^2 + 2\alpha_n\langle \overline{\omega} - \overline{u_n}, \overline{x_n} - \overline{u_n}\rangle_{\mathbb{R}^2} \\ &\leq d^2(x_n,\omega) - \alpha_n d^2(x_n,u_n) + 2\alpha_n\left\langle \exp^{-1}_{u_n}\omega, \exp^{-1}_{u_n}x_n\right\rangle. \end{aligned} \tag{3.8}$$

On the other hand, since $u_n := J^A_{\lambda_n}(x_n)$ and $J^A_{\lambda_n}$ is firmly nonexpansive, it follows from Proposition 2.13 that

$$\left\langle \exp^{-1}_{u_n}\omega, \exp^{-1}_{u_n}x_n\right\rangle \leq 0.$$

This together with (3.8) yields that

$$d^2(y_n,\omega) \leq d^2(x_n,\omega) - \alpha_n d^2(x_n,u_n) \tag{3.9}$$

$$\leq d^2(x_n,\omega). \tag{3.10}$$

Recall from (3.2) that $y_n = \exp_{x_n} \alpha_n \exp_{x_n}^{-1} u_n$, we get $d(x_n, y_n) = \alpha_n d(x_n, u_n)$. From (3.9), we obtain this

$$d^2(y_n, \omega) \leq d^2(x_n, \omega) - \frac{1}{\alpha_n} d^2(x_n, y_n). \tag{3.11}$$

For $n \in \mathbb{N}$, let $\gamma_n : [0,1] \longrightarrow M$ be a geodesic joining $\gamma_n(0) = x_n$ to $\gamma_n(1) = z_n$. Then, (3.4) can be written as $x_{n+1} = \gamma_n(\beta_n)$. By using geodesic convexity of Riemannian distance, (3.5) and (3.10), we get

$$\begin{aligned} d(x_{n+1}, \omega) &= d(\gamma_n(\beta_n), \omega) \\ &\leq (1-\beta_n) d(\gamma_n(0), \omega) + \beta_n d(\gamma_n(1), \omega) \\ &= (1-\beta_n) d(x_n, \omega) + \beta_n d(z_n, \omega) \\ &\leq (1-\beta_n) d(x_n, \omega) + \beta_n d(y_n, \omega) \\ &\leq (1-\beta_n) d(x_n, \omega) + \beta_n d(x_n, \omega) \\ &= d(x_n, \omega). \end{aligned} \tag{3.12}$$

Therefore, $\{x_n\}$ is Fejér convergent with respect to Ω.

Step II. We show that $\lim_{n \longrightarrow \infty} d(x_{n+1}, x_n) = 0$.

Fix $n \in \mathbb{N}$. Let $\triangle(x_n, z_n, \omega)$ be a geodesic triangle with vertices x_n, z_n and ω, and $\triangle(\overline{x_n}, \overline{z_n}, \overline{\omega})$ be the corresponding comparison triangle. Then, we have

$$d(x_n, \omega) = \|\overline{x_n} - \overline{\omega}\|, \quad d(z_n, \omega) = \|\overline{z_n} - \overline{\omega}\| \text{ and } d(z_n, x_n) = \|\overline{z_n} - \overline{x_n}\|.$$

Recall that $x_{n+1} := \exp_{x_n} \beta_n \exp_{x_n}^{-1} z_n$, so its comparison point is $\overline{x_{n+1}} = (1 - \beta_n)\overline{x_n} + \beta_n \overline{z_n}$. Using (2.5), (3.5), and (3.10), we get

$$\begin{aligned} d^2(x_{n+1}, \omega) &\leq \|\overline{x_{n+1}} - \overline{\omega}\|^2 \\ &= \|(1-\beta_n)\overline{x_n} + \beta_n \overline{z_n} - \overline{\omega}\|^2 \\ &= \|(1-\beta_n)(\overline{x_n} - \overline{\omega}) + \beta_n(\overline{z_n} - \overline{\omega})\|^2 \\ &= (1-\beta_n)\|\overline{x_n} - \overline{\omega}\|^2 + \beta_n \|\overline{z_n} - \overline{\omega}\|^2 - \beta_n(1-\beta_n)\|\overline{x_n} - \overline{z_n}\|^2 \\ &= (1-\beta_n) d^2(x_n, \omega) + \beta_n d^2(z_n, \omega) - \beta_n(1-\beta_n) d^2(x_n, z_n) \\ &\leq (1-\beta_n) d^2(x_n, \omega) + \beta_n d^2(y_n, \omega) - \beta_n(1-\beta_n) d^2(x_n, z_n) \qquad (3.13) \\ &\leq (1-\beta_n) d^2(x_n, \omega) + \beta_n d^2(x_n, \omega) - \beta_n(1-\beta_n) d^2(x_n, z_n) \\ &= d^2(x_n, \omega) - \beta_n(1-\beta_n) d^2(x_n, z_n). \qquad (3.14) \end{aligned}$$

From (3.14), we also obtain

$$\beta_n(1-\beta_n) d^2(x_n, z_n) \leq d^2(x_n, \omega) - d^2(x_{n+1}, \omega),$$

and we further have

$$\begin{aligned} d^2(x_n, z_n) &= \frac{1}{\beta_n(1-\beta_n)}(d^2(x_n, \omega) - d^2(x_{n+1}, \omega)) \\ &\leq \frac{1}{a(1-b)}(d^2(x_n, \omega) - d^2(x_{n+1}, \omega)). \end{aligned}$$

Since $\{x_n\}$ is Fejér convergent with respect to Ω implies that $\lim_{n\longrightarrow\infty} d(x_n, \omega)$ exists. By letting $n \longrightarrow \infty$, we have

$$\lim_{n\longrightarrow\infty} d(x_n, z_n) = 0. \tag{3.15}$$

Recall that $x_{n+1} = \gamma_n(\beta_n)$ for all $n \in \mathbb{N}$, using the geodesic convexity of Riemannian distance, we obtain

$$\begin{aligned} d(x_{n+1}, x_n) &= d(\gamma_n(\beta_n), x_n) \\ &\leq (1-\beta_n)d(\gamma_n(0), x_n) + \beta_n d(\gamma_n(1), x_n) \\ &= (1-\beta_n)d(x_n, x_n) + \beta_n d(z_n, x_n) \\ &= \beta_n d(x_n, z_n) \\ &\leq b d(x_n, z_n). \end{aligned}$$

Letting $n \longrightarrow \infty$ and using (3.15), we get

$$\lim_{n\longrightarrow\infty} d(x_{n+1}, x_n) = 0. \tag{3.16}$$

Step III. We show that $\lim_{n\longrightarrow\infty} d(x_n, y_n) = 0$.

Using (3.11) and (3.13), we obtain

$$\begin{aligned} d^2(x_{n+1}, \omega) &\leq (1-\beta_n)d^2(x_n, \omega) + \beta_n(d^2(x_n, \omega) \\ &\quad - \frac{1}{\alpha_n} d^2(x_n, y_n)) - \beta_n(1-\beta_n)d^2(x_n, z_n) \\ &= (1-\beta_n)d^2(x_n, \omega) + \beta_n d^2(x_n, \omega) \\ &\quad - \frac{\beta_n}{\alpha_n} d^2(x_n, y_n) - \beta_n(1-\beta_n)d^2(x_n, z_n) \\ &= d^2(x_n, \omega) - \frac{\beta_n}{\alpha_n} d^2(x_n, y_n) - \beta_n(1-\beta_n)d^2(x_n, z_n). \end{aligned}$$

With some rearrangements we obtain

$$\frac{a}{b} d^2(x_n, y_n) \leq \frac{\beta_n}{\alpha_n} d^2(x_n, y_n) \leq d^2(x_n, \omega) - d^2(x_{n+1}, \omega) - \beta_n(1-\beta_n)d^2(x_n, z_n).$$

Since $\{x_n\}$ is Fejér convergence of with respect to Ω and (3.15) together imply that

$$\lim_{n\longrightarrow\infty} d(x_n, y_n) = 0. \tag{3.17}$$

Step IV. We show that the cluster points of $\{x_n\}$ belongs to Ω.

Since the sequence $\{x_n\}$ is Fejér convergent, by (ii) of Lemma 2.6, $\{x_n\}$ is bounded. Hence, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges to a cluster point x^* of $\{x_n\}$. From (3.17), we get $y_{n_i} \longrightarrow x^*$ as $i \longrightarrow \infty$. Also from (3.15), we have $z_{n_i} \longrightarrow x^*$ as $i \longrightarrow \infty$.

We firstly prove that $x^* \in EP(F)$. By $z_n = T^F_{r_n}(y_n)$, we get

$$F(z_n, y) - \frac{1}{r_n}\langle \exp^{-1}_{z_n} y_n, \exp^{-1}_{z_n} y\rangle \geq 0, \quad \forall y \in K.$$

Since the bifunction F is monotone, we obtain

$$-\frac{1}{r_n}\langle \exp_{z_n}^{-1} y_n, \exp_{z_n}^{-1} y\rangle \geq F(y, z_n).$$

Replacing n by n_i, we get

$$-\frac{1}{r_{n_i}}\langle \exp_{z_{n_i}}^{-1} y_{n_i}, \exp_{z_{n_i}}^{-1} y\rangle \geq F(y, z_{n_i}). \tag{3.18}$$

Recall that

$$\lim_{i\longrightarrow\infty} \| \exp_{z_{n_i}}^{-1} y_{n_i}\| = \lim_{i\longrightarrow\infty} d(y_{n_i}, z_{n_i}) = 0,$$

so we get $\exp_{z_{n_i}}^{-1} y_{n_i} \longrightarrow \mathbf{0}$ as $i \longrightarrow \infty$. Using $\liminf_{i\longrightarrow\infty} r_{n_i} > 0$ and $y \mapsto F(x, y)$ is lower semicontinuous, and letting $i \longrightarrow \infty$ into (3.18), we get

$$0 \geq \liminf_{i\longrightarrow\infty} F(y, z_{n_i}) \geq F(y, x^*), \quad \forall y \in K.$$

Let $\gamma : [0, 1] \longrightarrow M$ be the geodesic joining $\gamma(0) = x^*$ to $\gamma(1) = y \in K$. Since K is geodesic convex, then $\gamma(t) \in K$ and $F(\gamma(t), x^*) \leq 0$ for all $t \in [0, 1]$. From $y \mapsto F(x, y)$ is geodesic convex, we have, for $t > 0$, the following

$$\begin{aligned} 0 = F(\gamma(t), \gamma(t)) &\leq tF(\gamma(t), y) + (1 - t)F(\gamma(t), x^*) \\ &\leq tF(\gamma(t), y). \end{aligned}$$

Dividing by t and since $x \mapsto F(x, y)$ is upper semicontinuous, we see that

$$\begin{aligned} 0 &\leq \limsup_{t\longrightarrow 0^+} F(\gamma(t), y) \\ &\leq F(x^*, y). \end{aligned}$$

Since $y \in K$ is chosen arability, $x^* \in EP(F)$.

Next, we prove that $x^* \in A^{-1}(\mathbf{0})$. Since $\{\alpha_n\} \subset (0, 1)$ satisfying $0 < a \leq \alpha_n \leq b < 1$, $\frac{1}{\alpha_n} d(x_n, y_n) = d(x_n, u_n)$, and $\lim_{n\longrightarrow\infty} d(x_n, y_n) = 0$, we may see that

$$\lim_{n\longrightarrow\infty} d(x_n, u_n) = 0. \tag{3.19}$$

Since $\hat{\lambda} \leq \lambda_n \leq \tilde{\lambda}$, we may assume without the loss of generality that $\lim_{i\longrightarrow\infty} \lambda_{n_i} = \lambda$ for some subsequence $\{\lambda_{n_i}\}$ of $\{\lambda_n\}$ and some $\lambda \in [\hat{\lambda}, \tilde{\lambda}]$. Recall that $u_n = J^A_{\lambda_n}(x_n)$. Then by (3.19) and Lemma 2.14, we obtain $\lim_{i\longrightarrow\infty} u_{n_i} = x^*$ and that $x^* = J^A_{\lambda}(x^*)$. From Remark 2.10, we obtain $x^* \in A^{-1}(\mathbf{0})$. Therefore, we get $x^* \in \Omega$. By a (iii) of Lemma 2.6, the sequence $\{x_n\}$ generated by Algorithm 3.1 converges to a solution of the problem (3.1). The proof is therefore completed. □

Next, we have the following results of Theorem 3.4 as follows:

Corollary 3.5. *Suppose that a vector field $A \in \mathfrak{X}(K)$ be a maximal monotone with $A^{-1}(\mathbf{0}) \neq \emptyset$. Then, the sequence generated by Algorithm 3.2 converges to a solution of problem* (1.4).

Corollary 3.6. *Suppose that $F : K \times K \longrightarrow \mathbb{R}$ be a bifunction satisfying assumptions (A1)–(A6) with $EP(F) \neq \emptyset$. Then, the sequence generated by Algorithm 3.3 converges to a solution of problem* (1.3).

4. Applications

In this section, we derive our algorithm for finding minimizers of minimization problems, and also give the algorithm for finding saddle points of minimax problems in Hadamard manifolds.

4.1. **Minimization problems.** Let $g : M \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and geodesic convex function. Consider the minimization problem:

$$\min_{x \in M} g(x). \tag{4.1}$$

We denote S_g the solution set of (4.1), that is,

$$S_g = \{x \in M : g(x) \leq g(y), \quad \forall y \in M\}.$$

Definition 4.1. Let $g : M \longrightarrow \mathbb{R}$ be a geodesic convex and $x \in M$. A vector $s \in T_x M$ is called a *subgradient* of g at x if and only if

$$g(y) \geq g(x) + \langle s, \exp_x^{-1} y\rangle, \quad \forall y \in M. \tag{4.2}$$

The set of all subgradients of g, denoted by $\partial g(x)$ is called the *subdifferential* of g at x, which is closed geodesic convex (possibly empty) set.

Lemma 4.2. [15] *Let $g : M \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and geodesic convex function. Then, the subdifferential ∂g of g is a maxiaml monotone vector field.*

It is easy to see that

$$x \in S_g \Longleftrightarrow \mathbf{0} \in \partial g(x).$$

Recall that ∂g is maximal monotone if $g : M \longrightarrow \mathbb{R} \cup \{+\infty\}$ is proper lower semicontinuous and geodesic convex. Applying Algorithm 3.1 to the multivalued vector field ∂g, we obtain the following results for the convex minimization problem (4.1).

Theorem 4.3. *Suppose that $g : M \longrightarrow \mathbb{R}$ be a proper lower semicontinuous and geodesic convex function and $F : K \times K \longrightarrow \mathbb{R}$ be a bifunction satisfying assumptions (A1)–(A6) with $EP(F) \cap S_g \neq \emptyset$. Let $\{x_n\}$ be a sequence in $D(g)$ generated as*

$$\begin{aligned}
&y_n := \exp_{x_n} \alpha_n \exp_{x_n}^{-1} J_{\lambda_n}^{\partial g}(x_n),\\
&z_n \in K \text{ such that } F(z_n, t) - \frac{1}{r_n}\langle \exp_{z_n}^{-1} y_n, \exp_{z_n}^{-1} t\rangle \geq 0, \quad \forall t \in K,\\
&x_{n+1} := \exp_{x_n} \beta_n \exp_{x_n}^{-1} z_n, \quad \forall n \in \mathbb{N},
\end{aligned}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}$ and $\{r_n\}$ are real positive sequences such that

(i) $0 < a \leq \alpha_n, \beta_n \leq b < 1, \quad \forall n \in \mathbb{N}$,
(ii) $0 < \hat{\lambda} \leq \lambda_n \leq \tilde{\lambda} < \infty, \quad \forall n \in \mathbb{N}$,
(iii) $\liminf_{n \longrightarrow \infty} r_n > 0$.

Then, the sequence $\{x_n\}$ converges to a solution of the problem $EP(F) \cap S_g$.

Corollary 4.4. *Let $g : M \longrightarrow \mathbb{R}$ be a proper lower semicontinuous and geodesic convex function with $S_g \neq \emptyset$. Let $\{x_n\}$ be a sequence in $D(g)$ generated as*

$$x_{n+1} := \exp_{x_n} \alpha_n \exp_{x_n}^{-1} J_{\lambda_n}^{\partial g}(x_n), \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ and $\{\lambda_n\}$ are real positive sequences such that

(i) $0 < a \leq \alpha_n \leq b < 1, \quad \forall n \in \mathbb{N},$

(ii) $0 < \hat{\lambda} \leq \lambda_n \leq \tilde{\lambda} < \infty, \quad \forall n \in \mathbb{N}.$

Then, the sequence $\{x_n\}$ converges to a solution of the problem (4.1).

4.2. **Saddle points in minimax problems.**

In this subsection, we first recall the formulation of saddle point problems in the frame work of Hadamard manifolds. Then, we derive the proposed algorithm for finding the saddle point.

Let M_1 and M_2 be the Hadamard manifolds, and K_1 and K_2 the geodesic convex subset of M_1 and M_2, respectively. A function $H : K_1 \times K_2 \longrightarrow \mathbb{R}$ is called a *saddle function* if

(a) $H(x, \cdot)$ is geodesic convex on K_2 for all $x \in K_1$ and

(b) $H(\cdot, y)$ is geodesic concave, i.e., $-H(\cdot, y)$ is geodesic convex on K_1 for all $y \in K_2$.

A point $\tilde{z} = (\tilde{x}, \tilde{y})$ is said to be a *saddle point* of H if

$$H(x, \tilde{y}) \leq H(\tilde{x}, \tilde{y}) \leq H(\tilde{x}, y), \quad \forall z = (x, y) \in K_1 \times K_2.$$

We denote SPP to the set of saddle points of H. Let $V_H : K_1 \times K_2 \longrightarrow 2^{TM_1} \times 2^{TM_2}$ be a multivalued vector field associated with saddle function H, defined by

$$V_H(x, y) = \partial(-H(\cdot, y))(x) \times \partial(H(x, \cdot))(y), \quad \forall (x, y) \in K_1 \times K_2. \tag{4.3}$$

The product space $M = M_1 \times M_2$ is a Hadamard manifold and the tangent space of M at $z = (x, y)$ is $T_zM = T_xM_1 \times T_yM_2$. For further details, see [23, Page 239]. The corresponding metric given by

$$\langle w, w' \rangle = \langle u, u' \rangle + \langle v, v' \rangle, \quad \forall w = (u, v), w' = (u', v') \in T_zM.$$

A geodesic in the product manifold M is the product of two geodesic in M_1 and M_2. Then, for any two point $z = (x, y)$ and $z' = (x', y')$ in M, we have

$$\exp_z^{-1} z' = \exp_{(x,y)}^{-1}(x', y') = (\exp_x^{-1} x', \exp_y^{-1} y').$$

A vector field $V : M_1 \times M_2 \longrightarrow 2^{TM_1} \times 2^{TM_2}$ is said to be monotone if and only if for any $z = (x, y), z' = (x', y'), w = (u, v) \in V(z)$ and $w' = (u', v') \in V(z')$, we have

$$\langle u, \exp_x^{-1} x' \rangle + \langle v, \exp_y^{-1} y' \rangle \leq \langle u', -\exp_{x'}^{-1} x \rangle + \langle v', -\exp_{y'}^{-1} y \rangle.$$

Theorem 4.5. [15] *Let H be a saddle function on $K = K_1 \times K_2$ and V_H the multivalued vector field defined by* (4.3). *Then, V_H is maximal monotone.*

One can check that a point $\tilde{z} = (\tilde{x}, \tilde{y}) \in K$ is a saddle point of H if and only if it is a singularity of V_H. Applying Algorithm (3.1) to multivalued vector field V_H associated with the saddle function H, we get the following result.

Theorem 4.6. *Suppose that $H : K = K_1 \times K_2 \longrightarrow \mathbb{R}$ be a saddle function and $V_H : K_1 \times K_2 \longrightarrow 2^{TM_1} \times 2^{TM_2}$ be the associated maximal monotone vector field. Assume that $F : K \times K \longrightarrow \mathbb{R}$ be a bifunction satisfying assumptions (A1)–(A6) with $EP(F) \cap SSP \neq \emptyset$. Choose initial point $x_0 \in K \times K$ and define $\{x_n\}, \{y_n\}$ and $\{z_n\}$ as follows:*

$$\begin{aligned} &y_n := \exp_{x_n} \alpha_n \exp_{x_n}^{-1} J_{\lambda_n}^{V_H}(x_n), \\ &z_n \in K \text{ such that } F(z_n, t) - \frac{1}{r_n}\langle \exp_{z_n}^{-1} y_n, \exp_{z_n}^{-1} t\rangle \geq 0, \quad \forall t \in K, \\ &x_{n+1} := \exp_{x_n} \beta_n \exp_{x_n}^{-1} z_n, \quad \forall n \in \mathbb{N}, \end{aligned}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}$ and $\{r_n\}$ are real positive sequences such that

(i) $0 < a \leq \alpha_n, \beta_n \leq b < 1, \quad \forall n \in \mathbb{N}$,
(ii) $0 < \hat{\lambda} \leq \lambda_n \leq \tilde{\lambda} < \infty, \quad \forall n \in \mathbb{N}$,
(iii) $\liminf_{n\longrightarrow\infty} r_n > 0$.

Then, the sequence $\{x_n\}$ converges to a solution of the problem $EP(F) \cap SPP$.

Corollary 4.7. *Suppose that $H : K = K_1 \times K_2 \longrightarrow \mathbb{R}$ be a saddle function and $V_H : K_1 \times K_2 \longrightarrow 2^{TM_1} \times 2^{TM_2}$ be the associated maximal monotone vector field with $SSP \neq \emptyset$. Choose initial point $x_0 \in K$ and define $\{x_n\}$ as follows:*

$$x_{n+1} := \exp_{x_n} \alpha_n \exp_{x_n}^{-1} J_{\lambda_n}^{V_H}(x_n), \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$and $\{\lambda_n\}$ are real positive sequences such that

(i) $0 < a \leq \alpha_n \leq b < 1, \quad \forall n \in \mathbb{N}$,
(ii) $0 < \hat{\lambda} \leq \lambda_n \leq \tilde{\lambda} < \infty, \quad \forall n \in \mathbb{N}$.

Then, the sequence $\{x_n\}$ converges to a saddle point of H.

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References

[1] R. L. Adler, J.-P. Dedieu, J. Y. Margulies, M. Martens and M. Shub, *Newton's method on Riemannian manifolds and a geometric model for the human spine*, IMA J. Numer. Anal. **22** (2002), 359–390.
[2] S. Al-Homidan, Q. H. Ansari and F. Babu, *Halpern- and Mann-type algorithms for fixed points and inclusion problems on Hadamard manifolds*, **40** (2019), 621–653.
[3] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994), 123–145.
[4] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999.

[5] M. P. do Carmo, *Riemannian Geometry*, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 1992.
[6] S. S. Chang, *Set-valued variational inclusions in Banach spaces*, J. Math. Anal. Appl. **248** (2000), 438–454.
[7] S. S. Chang, *Existence and approximation of solutions for set-valued variational inclusions in Banach space*, Nonlinear Anal. **47** (2001), 583–594.
[8] S. S. Chang, Y. J. Cho, B. S. Lee and I. H. Jung, *Generalized set-valued variational inclusions in Banach spaces*, J. Math. Anal. Appl. **246** (2000), 409–422.
[9] V. Colao, G. López, G. Marino and V. Martín-Márquez, *Equilibrium problems in Hadamard manifolds*, J. Math. Anal. Appl. **388** (2012), 61–77.
[10] J. X. da Cruz Neto, O. P. Ferreira and L. R. Lucambio Pérez, *Monotone point-to-set vector fields*, Balkan J. Geom. Appl. **5** (2000), 69–79.
[11] K. Fan, *A minimax inequality and applications*, in: Inequalities, III (Proc. Third Sympos., Univ. California, Los Angeles, Calif., 1969; dedicated to the memory of Theodore S. Motzkin), Academic Press, New York, 1972, pp. 103–113.
[12] O. P. Ferreira and P. R. Oliveira, *Proximal point algorithm on Riemannian manifolds*, Optimization **51** (2002), 257–270.
[13] S. Kamimura and W. Takahashi, *Approximating solutions of maximal monotone operators in Hilbert spaces*, J. Approx. Theory **106** (2000), 226–240.
[14] I. V. Konnov, *Equilibrium models and Variational Inequalities*, Mathematics in Science and Engineering, vol. 210, Elsevier B. V., Amsterdam, 2007.
[15] C. Li, G. López and V. Martín-Márquez, *Monotone vector fields and the proximal point algorithm on Hadamard manifolds*, J. Lond. Math. Soc. (2) **79** (2009), 663–683.
[16] C. Li, G. López, V. Martín-Márquez and J.-H. Wang, *Resolvents of set-valued monotone vector fields in Hadamard manifolds*, Set-Valued Var. Anal. **19** (2011), 361–383.
[17] W. Li, Y.-B. Xiao, N.-J. Huang and Y. J. Cho, *A class of differential inverse quasi-variational inequalities in finite dimensional spaces*, J. Nonlinear Sci. Appl. **10** (2017), 4532–4543.
[18] H. Manaka and W. Takahashi, *Weak convergence theorems for maximal monotone operators with nonspreading mappings in a Hilbert space*, Cubo. A Math. J. **13** (2010), 11–24.
[19] B. Martinet, *Régularisation d'inéquations variationnelles par approximations successives*, Rev. Française Informat. Recherche Opérationnelle **4** (1970), 154–158.
[20] L. D. Muu and W. Oettli, *Convergence of an adaptive penalty scheme for finding constrained equilibria*, Nonlinear Anal. **18** (1992), 1159–1166.
[21] S. Z. Németh, *Monotone vector fields*, Publ. Math. Debrecen **54** (1999), 437–449.
[22] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optimization **14** (1976), 877–898.
[23] T. Sakai, *Riemannian Geometry*, Translations of Mathematical Monographs, vol.149, American Mathematical Society, Providence, RI, 1996.
[24] S. T. Smith, *Optimization techniques on Riemannian manifolds*, in: Hamiltonian and Gradient flows, Algorithms and Control, Fields Inst. Commun., vol. 3, Amer. Math. Soc., Providence, RI, 1994, pp. 113–136.
[25] G.-J. Tang and N.-J. Huang, *An inexact proximal point algorithm for maximal monotone vector fields on Hadamard manifolds*, Oper. Res. Lett. **41** (2013), 586–591.
[26] C. Udrişte, *Convex Functions and Optimization Methods on Riemannians Manifolds*, Mathematics and its Applications, vol. 297, Kluwer Academic Publishers Group, Dordrecht, 1994.
[27] Y.-M. Wang, Y.-B. Xiao, X. Wang and Y. J. Cho, *Equivalence of well-posedness between systems of hemivariational inequalities and inclusion problems*, J. Nonlinear Sci. Appl. **9** (2016), 1178–1192.
[28] X. Wang, G. López, C. Li and J.-C. Yao, *Equilibrium problems on Riemannian manifolds with applications*, J. Math. Anal. Appl. **473** (2019), 866–891.

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