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SHIFTED BERNOULLI NUMBERS AND SHIFTED FUBINI NUMBERS

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Abstract. In this paper, we introduce the shifted Bernoulli numbers as a different natural extension of the classical Bernoulli numbers, in particular, in terms of continued fraction and determinant expressions. We give several arithmetical and combinatorial properties of shifted Bernoulli numbers. Fubini numbers form an integer sequence in which the nth term counts the number of weak orderings of a set with n elements. We also introduce the shifted Fubini numbers as one kind of their generalizations and show several similar properties. Though Bernoulli numbers and Fubini numbers do not seem to be so related to each other, the shifted Bernoulli numbers and shifted Fubini numbers have several similarities and relations.

1. Introduction

The classical Bernoulli numbers B_n are defined by the generating function:

$$\frac{x}{e^x-1}=\sum_{n=0}^\infty B_n\frac{x^n}{n!} \tag{1.1}$$

with $B_1=-1/2$.

It is known that Bernoulli numbers have a determinant expression ([7, p.53]):

$$B_n=(-1)^n n!\begin{vmatrix} \frac{1}{2!} & 1 & 0 & & \\ \frac{1}{3!} & \frac{1}{2!} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \cdots & \frac{1}{2!} & 1 \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \cdots & \frac{1}{3!} & \frac{1}{2!} \end{vmatrix}. \tag{1.2}$$

One of the natural questions is that if Bernoulli numbers B_n have the determinant expression in (1.2), then what kind of numbers $\breve{B}_n$ is to express, for example,

$$\breve{B}_n=(-1)^n n!\begin{vmatrix} \frac{1}{3!} & 1 & 0 & & \\ \frac{1}{4!} & \frac{1}{3!} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \cdots & \frac{1}{3!} & 1 \\ \frac{1}{(n+2)!} & \frac{1}{(n+1)!} & \cdots & \frac{1}{4!} & \frac{1}{3!} \end{vmatrix}.$$

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Such determinant expressions may be obvious or artificial for the readers with different backgrounds. However, there are motivations from Combinatorics, in particular, graph theory. In 1989, Cameron [4] considered the operator A defined on the set of sequences of non-negative integers as follows: for $x = \{x_n\}_{n\ge 1}$ and $z = \{z_n\}_{n\ge 1}$, set $Ax = z$, where

$$1+\sum_{n=1}^\infty z_n t^n = \left(1-\sum_{n=1}^\infty x_n t^n\right)^{-1}\,. \tag{1.3}$$

Many motivations and background together with many concrete examples (in particular, in the aspects of Graph theory) by this operator can be seen in [4]. Though this operator was used for non-negative integers in [4], it is possible to deal with rational numbers. In fact, recently, such an operator are used to introduced new numbers corresponding to harmonic numbers [13].

In this paper, we introduce the shifted Bernoulli and Fubini numbers as a different natural extension of the classical Bernoulli and Fubini numbers, in particular, in terms of determinant expressions. It does not seem to have any substantial connection between Bernoulli numbers and Fubini numbers. We give their several similar arithmetical and/or combinatorial properties which show the connections between two numbers, under the aspects of continued fractions and convolutions.

2. Definitions and basic properties

For nonnegative integers n and m, define the *shifted Bernoulli numbers* $B_n^{(m)}$ by

$$\frac{x^m}{e^x-E_m(x)+x^m}=\sum_{n=0}^\infty B_n^{(m)}\frac{x^n}{n!}\,, \tag{2.1}$$

where

$$E_m(x)=\sum_{n=0}^m\frac{x^n}{n!}$$

is the partial summation of e^x ([12]). When $m=1$, $B_n = B_n^{(1)}$ are the classical Bernoulli numbers defined by (1.1). When $m=0$, $B_n^{(0)}=(-1)^n$.

By the definition (2.1),

$$\begin{aligned}1&=\left(\sum_{n=0}^\infty B_n^{(m)}\frac{x^n}{n!}\right)\left(1+\sum_{l=1}^\infty\frac{x^l}{(l+m)!}\right)\\&=\sum_{n=0}^\infty B_n^{(m)}\frac{x^n}{n!}+\sum_{n=1}^\infty\sum_{k=0}^{n-1}\frac{B_k^{(m)}}{(n-k+m)!k!}x^n\,.\end{aligned}$$

Comparing the coefficients on both sides, we have $B_0^{(m)}=1$ and for $n\ge 1$

$$\frac{B_n^{(m)}}{n!}+\sum_{k=0}^{n-1}\frac{B_k^{(m)}}{(n-k+m)!k!}=0\,.$$

This is a recurrence relation among the shifted Bernoulli numbers.

Lemma 2.1. *For integers $n \ge 1$ and $m \ge 0$,*

$$B_n^{(m)} = -\sum_{k=0}^{n-1} \frac{n!}{(n-k+m)!k!} B_k^{(m)} .$$

with $B_0^{(m)} = 1$.

Remark. If $m = 1$ in Lemma 2.1, we have a famous recurrence formula for the classical Bernoulli numbers:

$$\frac{B_n}{n!} = -\sum_{k=0}^{n-1} \frac{B_k}{(n-k+1)!k!}$$

or

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0 .$$

If $m = 0$ in Lemma 2.1, we have the famous identity

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0 .$$

The shifted Bernoulli numbers have an explicit expression. This is proved by induction using Lemma 2.1.

Theorem 2.2. *For integers $n \ge 1$ and $m \ge 0$,*

$$B_n^{(m)} = n! \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\dots,i_k \ge 1}} \frac{1}{(i_1+m)! \cdots (i_k+m)!} .$$

Remark. If $m = 1$ in Theorem 2.2, we have

$$B_n = n! \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\dots,i_k \ge 1}} \frac{1}{(i_1+1)! \cdots (i_k+1)!} .$$

If $m = 0$ in Theorem 2.2, we have

$$(-1)^n = \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\dots,i_k \ge 1}} \frac{n!}{i_1! \cdots i_k!} .$$

Shifted Bernoulli numbers are naturally extended from the original Bernoulli numbers in terms of determinants. By expanding the determinant, we can prove this result by induction together with Lemma 2.1.

Theorem 2.3. *For integers $n \ge 1$ and $m \ge 0$,*

$$B_n^{(m)} = (-1)^n n! \begin{vmatrix} \frac{1}{(m+1)!} & 1 & 0 & & \\ \frac{1}{(m+2)!} & \frac{1}{(m+1)!} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{1}{(m+n-1)!} & \frac{1}{(m+n-2)!} & \cdots & \frac{1}{(m+1)!} & 1 \\ \frac{1}{(m+n)!} & \frac{1}{(m+n-1)!} & \cdots & \frac{1}{(m+2)!} & \frac{1}{(m+1)!} \end{vmatrix} . \tag{2.2}$$

Remark. When $m = 1$ in Theorem 2.3, the result is reduced to (1.2). When $m = 0$ in Theorem 2.3, we have

$$\frac{1}{n!} = \begin{vmatrix} 1 & 1 & & & \\ \frac{1}{2!} & 1 & 0 & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \cdots & 1 & 1 \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \cdots & \frac{1}{2!} & 1 \end{vmatrix}$$

([7, §9]). This fact confirms that $B_n^{(0)} = (-1)^n$.

Proof of Theorem 2.3. For simplicity, put $\tilde{B}_n^{(m)} = (-1)^n B_n^{(m)}/n!$ and prove that

$$\tilde{B}_n^{(m)} = \begin{vmatrix} \frac{1}{(m+1)!} & 1 & 0 & & \\ \frac{1}{(m+2)!} & \frac{1}{(m+1)!} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{1}{(m+n-1)!} & \frac{1}{(m+n-2)!} & \cdots & \frac{1}{(m+1)!} & 1 \\ \frac{1}{(m+n)!} & \frac{1}{(m+n-1)!} & \cdots & \frac{1}{(m+2)!} & \frac{1}{(m+1)!} \end{vmatrix} . \tag{2.3}$$

By Theorem 2.2, we see that

$$\tilde{B}_1^{(m)} = \frac{1}{(m+1)!} .$$

Assume that (2.3) is valid up to $n-1$. By Lemma 2.1,

$$\tilde{B}_n^{(m)} = \sum_{k=0}^{n-1} \frac{(-1)^{n-k-1}}{(n-k+m)!} \tilde{B}_k^{(m)}$$

with $\tilde{B}_0^{(m)} = 1$. Expanding at the first row of the right-hand side of (2.3), we have

$$\frac{\tilde{B}_{n-1}^{(m)}}{(m+1)!} - \begin{vmatrix} \frac{1}{(m+2)!} & 1 & 0 & & \\ \frac{1}{(m+3)!} & \frac{1}{(m+1)!} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{1}{(m+n-1)!} & \frac{1}{(m+n-3)!} & \cdots & \frac{1}{(m+1)!} & 1 \\ \frac{1}{(m+n)!} & \frac{1}{(m+n-2)!} & \cdots & \frac{1}{(m+2)!} & \frac{1}{(m+1)!} \end{vmatrix}$$

$$= \frac{\tilde{B}_{n-1}^{(m)}}{(m+1)!} - \frac{\tilde{B}_{n-2}^{(m)}}{(m+2)!} + \cdots + (-1)^n \begin{vmatrix} \frac{1}{(m+n-1)!} & 1 \\ \frac{1}{(m+n)!} & \frac{1}{(m+1)!} \end{vmatrix}$$

$$= \sum_{k=0}^{n-1} \frac{(-1)^{n-k-1}}{(n-k+m)!} \tilde{B}_k^{(m)} = \tilde{B}_n^{(m)} .$$

□

2.1. Table of $B_n^{(m)}$.

n	0	1	2	3	4	5	6
$B_n^{(0)}$	1	-1	1	-1	1	-1	1
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$
$B_n^{(2)}$	1	$-\frac{1}{6}$	$-\frac{1}{36}$	$\frac{1}{180}$	$-\frac{11}{1080}$	$\frac{43}{9072}$	$-\frac{289}{90720}$
$B_n^{(3)}$	1	$-\frac{1}{24}$	$-\frac{19}{1440}$	$-\frac{53}{11520}$	$-\frac{3113}{2419200}$	$\frac{349}{2322432}$	$-\frac{174947}{232243200}$
$B_n^{(4)}$	1	$-\frac{1}{120}$	$-\frac{19}{7200}$	$-\frac{709}{672000}$	$-\frac{28813}{60480000}$	$-\frac{46721}{207360000}$	$-\frac{20744051}{203212800000}$

The following property is easily seen. In the later section, we shall see more relations, in particular, with shifted Fubini numbers.

Theorem 2.4. *For $m \ge 0$*

$$B_1^{(m)} = -\frac{1}{(m+1)!} .$$

2.2. Trudi's formula and inverse formula.

We shall use Trudi's formula [16, Vol.3, p.214],[18] to obtain different explicit expressions and inversion relations for the numbers $B_n^{(m)}$. If $a_0 = 1$, this formula is known as Brioschi's formula [3],[16, Vol.3, pp.208–209].

Lemma 2.5. *For a positive integer n, we have*

$$\begin{vmatrix} a_1 & a_0 & 0 & \cdots & \\ a_2 & a_1 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_{n-1} & & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & \cdots & a_2 & a_1 \end{vmatrix}$$

$$= \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n} (-a_0)^{n-t_1-\cdots-t_n} a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n},$$

where $\binom{t_1+\cdots+t_n}{t_1,\dots,t_n} = \frac{(t_1+\cdots+t_n)!}{t_1!\cdots t_n!}$ are the multinomial coefficients.

In addition, there exists the following inversion formula (see, e.g. [14]), which is based upon the relation:

$$\sum_{k=0}^{n} (-1)^{n-k} \alpha_k D(n-k) = 0 \quad (n \ge 1) .$$

Lemma 2.6. *If $\{\alpha_n\}_{n\ge 0}$ is a sequence defined by $\alpha_0 = 1$ and*

$$\alpha_n = \begin{vmatrix} D(1) & 1 & & \\ D(2) & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ D(n) & \cdots & D(2) & D(1) \end{vmatrix}, \quad \textit{then } D(n) = \begin{vmatrix} \alpha_1 & 1 & & \\ \alpha_2 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \alpha_n & \cdots & \alpha_2 & \alpha_1 \end{vmatrix}.$$

By applying these lemmata to Theorem 2.3, we obtain an explicit expression for shifted Bernoulli numbers.

Theorem 2.7. *For $n \ge m \ge 1$, we have*

$$B_n^{(m)} = n! \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n} \times (-1)^{t_1+\cdots+t_n} \left(\frac{1}{(m+1)!}\right)^{t_1} \cdots \left(\frac{1}{(m+n)!}\right)^{t_n}.$$

By applying the inversion relation in Lemma 2.6 to Theorem 2.3, we have the following.

Theorem 2.8. *For $n \ge 1$, we have*

$$\frac{(-1)^n}{(n+m)!} = \begin{vmatrix} B_1^{(m)} & 1 & 0 & & \\ \frac{B_2^{(m)}}{2!} & B_1^{(m)} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{B_{n-1}^{(m)}}{(n-1)!} & \frac{B_{n-2}^{(m)}}{(n-2)!} & \cdots & B_1^{(m)} & 1 \\ \frac{B_n^{(m)}}{n!} & \frac{B_{n-1}^{(m)}}{(n-1)!} & \cdots & \frac{B_2^{(m)}}{2!} & B_1^{(m)} \end{vmatrix}.$$

3. Continued fraction expansions

The generating function of shifted Bernoulli numbers can be expressed in continued fractions. It is known that any real number α can be expressed uniquely as the simple continued fraction expansion:

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}, \tag{3.1}$$

where a_0 is an integer and $a_1, a_2, \dots$ are positive integers. Though the expression is not unique, there exist general continued fraction expansions for real or complex numbers, and in general, analytic functions $f(x)$:

$$f(x)=a_0(x)+\cfrac{b_1(x)}{a_1(x)+\cfrac{b_2(x)}{a_2(x)+\cfrac{b_3(x)}{a_3(x)+\ddots}}}\,. \tag{3.2}$$

In [2] several continued fraction expansions for Bernoulli numbers are given. For example,

$$\sum_{n=1}^{\infty}B_{2n}(4x)^n=\cfrac{x}{1+\frac{1}{2}+\cfrac{x}{\frac{1}{2}+\frac{1}{3}+\cfrac{x}{\frac{1}{3}+\frac{1}{4}+\cfrac{x}{\ddots}}}}\,. \tag{3.3}$$

More general continued fractions expansions for analytic functions are recorded, for example, in [19].

Let the n-th convergent of the continued fraction expansion of (3.2) be

$$\frac{P_n(x)}{Q_n(x)}=a_0(x)+\cfrac{b_1(x)}{a_1(x)+\cfrac{b_2(x)}{a_2(x)+\ddots+\cfrac{b_n(x)}{a_n(x)}}}\,. \tag{3.4}$$

There exist the fundamental recurrence formulas:

$$\begin{aligned} P_n(x)&=a_n(x)P_{n-1}(x)+b_n(x)P_{n-2}(x)\ (n\ge 1),\\ Q_n(x)&=a_n(x)Q_{n-1}(x)+b_n(x)Q_{n-2}(x)\ (n\ge 1), \end{aligned} \tag{3.5}$$

with $P_{-1}(x)=1$, $Q_{-1}(x)=0$, $P_0(x)=a_0(x)$ and $Q_0(x)=1$.

From the definition in (2.1), shifted Bernoulli numbers satisfy the relation

$$\left(1+\sum_{n=0}^{\infty}\frac{x^{n+1}}{(m+n+1)!}\right)\left(\sum_{n=0}^{\infty}B_n^{(m)}\frac{x^n}{n!}\right)=1\,.$$

Thus,

$$P'_M(x)=1,\quad Q'_M(x)=1+\sum_{i=0}^{M-1}\frac{x^{i+1}}{(m+i+1)!}$$

or

$$P_M(x)=(m+M)!,\quad Q_M(x)=(m+M)!\left(1+\sum_{i=0}^{M-1}\frac{x^{i+1}}{(m+i+1)!}\right)$$

yield that

$$Q'_M(x)\sum_{n=0}^{\infty}B_n^{(m)}\frac{x^n}{n!}\sim P'_M(x)\quad(M\to\infty)$$

or

$$Q_M(x)\sum_{n=0}^\infty B_n^{(m)}\frac{x^n}{n!}\sim P_M(x)\quad (M\to\infty)\,.$$

Notice that the n-th convergent p_n/q_n of the simple continued fraction (3.1) of a real number α yields the approximation property

$$|q_n\alpha-p_n|<\frac{1}{q_{n+1}}\,.$$

Now,

$$\frac{P_0(x)}{Q_0(x)}=\frac{m!}{m!}=1,\quad \frac{P_1(x)}{Q_1(x)}=\frac{(m+1)!}{(m+1)!+x}=1-\frac{x}{(m+1)!+x}\,,$$

$$\frac{P_2(x)}{Q_2(x)}=\frac{(m+2)!}{(m+2)!+(m+2)x+x^2}=1-\cfrac{x}{(m+1)!+x-\cfrac{(m+1)!x}{m+2+x}}$$

and $P_n(x)$ and $Q_n(x)$ $(n\ge 3)$ satisfy the recurrence relations

$$P_n(x)=(m+n+x)P_{n-1}(x)-(m+n-1)xP_{n-2}(x)$$
$$Q_n(x)=(m+n+x)Q_{n-1}(x)-(m+n-1)xQ_{n-2}(x)$$

(They are proved by induction). Since by (3.5) for $n\ge 3$

$$a_n(x)=m+n+x\quad\text{and}\quad b_n(x)=-(m+n-1)x\,,$$

we have the following continued fraction expansion.

Theorem 3.1.

$$\sum_{n=0}^\infty B_n^{(m)}\frac{x^n}{n!}=1-\cfrac{x}{(m+1)!+x-\cfrac{(m+1)!x}{m+2+x-\cfrac{(m+2)x}{m+3+x-\cfrac{(m+3)x}{m+4+x-\ddots}}}}\,.$$

When $m=1$ in Theorem 3.1, we have a continued fraction expansion concerning the original Bernoulli numbers. Other expressions can be found, for instance, in [2].

Corollary 3.2.

$$\sum_{n=0}^\infty B_n\frac{x^n}{n!}=1-\cfrac{x}{2+x-\cfrac{2x}{3+x-\cfrac{3x}{4+x-\cfrac{4x}{5+x-\ddots}}}}\,.$$

When $m = 0$ in Theorem 3.1, we have a continued fraction expansion of the exponential function. Another expression can be found, for instance, in [10, p.207] and [19, (91.3)].

Corollary 3.3.

$$e^{-x} = 1 - \cfrac{x}{1 + x - \cfrac{x}{2 + x - \cfrac{2x}{3 + x - \cfrac{3x}{4 + x - \ddots}}}}\,.$$

4. Convolution identities

The following identity on sums of products of two Bernoulli numbers $B_n = B_n^{(1)}$ is known as Euler's formula:

$$\sum_{k=0}^{n} \binom{n}{k} B_k B_{n-k} = -nB_{n-1} - (n-1)B_n \quad (n \ge 1)\,. \tag{4.1}$$

We can give a more general result for shifted Bernoulli numbers $B_n^{(m)}$.

Theorem 4.1. *For integers $n \ge 0$ and $m \ge 1$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} B_k^{(m)} B_{n-k}^{(m)}$$
$$= -\frac{n!}{m^2 \cdot m!} \sum_{l=0}^{n-1} \left(\frac{m! - 1}{m \cdot m!} \right)^{n-l-1} \frac{l(m! - 1) + m}{l!} B_l^{(m)} - \frac{n-m}{m} B_n^{(m)}\,.$$

Remark. If $m = 1$ in Theorem 4.1, we have the identity (4.1).

Proof of Theorem 4.1. For simplicity, in (2.1), put

$$b(x) := \left(1 + \sum_{l=1}^{\infty} \frac{x^l}{(l+m)!} \right)^{-1} = \sum_{n=0}^{\infty} B_n^{(m)} \frac{x^n}{n!}\,.$$

Then, we have

$$\begin{aligned} b'(x) &= -b(x)^2 \sum_{l=1}^{\infty} \frac{l x^{l-1}}{(l+m)!} \\ &= -b(x)^2 \left(\sum_{l=1}^{\infty} \frac{x^{l-1}}{(l+m-1)!} - m \sum_{l=1}^{\infty} \frac{x^{l-1}}{(l+m)!} \right) \\ &= -b(x)^2 \left(\frac{1}{m!} + \frac{x-m}{x} \big(b(x)^{-1} - 1 \big) \right) \end{aligned}$$

$$= \frac{(m!-1)x - m\cdot m!}{m!x} b(x)^2 - \frac{x-m}{x} b(x)\,.$$

Thus,

$$\begin{aligned}
b(x)^2 &= \frac{m!x}{(m!-1)x - m\cdot m!} b'(x) + \frac{m!(x-m)}{(m!-1)x - m\cdot m!} b(x) \\
&= -\frac{x}{m}\frac{1}{1-\frac{m!-1}{m\cdot m!}x} b'(x) - \frac{x-m}{m}\frac{1}{1-\frac{m!-1}{m\cdot m!}x} b(x) \\
&= -\frac{x}{m}\sum_{l=0}^\infty \left(\frac{m!-1}{m\cdot m!}\right)^l x^l \sum_{n=1}^\infty B_n^{(m)}\frac{x^{n-1}}{(n-1)!} \\
&\quad - \frac{x-m}{m}\sum_{l=0}^\infty \left(\frac{m!-1}{m\cdot m!}\right)^l x^l \sum_{n=0}^\infty B_n^{(m)}\frac{x^n}{n!} \\
&= -\frac{1}{m}\sum_{l=0}^\infty \left(\frac{m!-1}{m\cdot m!}\right)^l \sum_{n=0}^\infty \frac{n!}{(n-l-1)!} B_{n-l}^{(m)}\frac{x^n}{n!} \\
&\quad - \frac{1}{m}\sum_{l=0}^\infty \left(\frac{m!-1}{m\cdot m!}\right)^l \sum_{n=0}^\infty \frac{n!}{(n-l-1)!} B_{n-l-1}^{(m)}\frac{x^n}{n!} \\
&\quad + \sum_{l=0}^\infty \left(\frac{m!-1}{m\cdot m!}\right)^l \sum_{n=0}^\infty \frac{n!}{(n-l)!} B_{n-l}^{(m)}\frac{x^n}{n!} \\
&= -\frac{1}{m}\sum_{n=0}^\infty\sum_{l=0}^n \left(\frac{m!-1}{m\cdot m!}\right)^l \frac{n!(n-l-m)}{(n-l)!} B_{n-l}^{(m)}\frac{x^n}{n!} \\
&\quad - \frac{1}{m}\sum_{n=0}^\infty\sum_{l=0}^{n-1} \left(\frac{m!-1}{m\cdot m!}\right)^l \frac{n!}{(n-l-1)!} B_{n-l-1}^{(m)}\frac{x^n}{n!}\,.
\end{aligned}$$

On the other hand,

$$b(x)^2 = \sum_{n=0}^\infty\sum_{k=0}^n \binom{n}{k} B_k^{(m)} B_{n-k}^{(m)}\frac{x^n}{n!}\,.$$

By comparing the coefficients on both sides, we get

$$\begin{aligned}
&\sum_{k=0}^n \binom{n}{k} B_k^{(m)} B_{n-k}^{(m)} \\
&= -\frac{n!}{m}\sum_{l=0}^n \left(\frac{m!-1}{m\cdot m!}\right)^l \frac{n-l-m}{(n-l)!} B_{n-l}^{(m)} \\
&\quad - \frac{n!}{m}\sum_{l=0}^{n-1} \left(\frac{m!-1}{m\cdot m!}\right)^l \frac{1}{(n-l-1)!} B_{n-l-1}^{(m)} \\
&= -\frac{n!}{m}\sum_{l=0}^n \left(\frac{m!-1}{m\cdot m!}\right)^{n-l} \frac{l-m}{l!} B_l^{(m)}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{n!}{m}\sum_{l=0}^{n-1}\left(\frac{m!-1}{m\cdot m!}\right)^{n-l-1}\frac{1}{l!}B_l^{(m)}\\
&= -\frac{n!}{m}\sum_{l=0}^{n-1}\left(\frac{m!-1}{m\cdot m!}\right)^{n-l-1}\left(\frac{m!-1}{m\cdot m!}(l-m)+1\right)\frac{B_l^{(m)}}{l!}\\
&- \frac{n!}{m}\frac{n-m}{n!}B_n^{(m)}\\
&= -\frac{n!}{m^2\cdot m!}\sum_{l=0}^{n-1}\left(\frac{m!-1}{m\cdot m!}\right)^{n-l-1}\frac{l(m!-1)+m}{l!}B_l^{(m)} - \frac{n-m}{m}B_n^{(m)}\,.
\end{aligned}
$$

□

5. Fubini numbers

The Fubini numbers [5, p.32, p.228] (or the ordered Bell numbers) form an integer sequence in which the nth term counts the number of weak orderings of a set with n elements. By using the Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$, Fubini numbers are defined by

$$F_n = \sum_{k=0}^{n} k! \left\{ {n \atop k} \right\}\,.$$

They can be expanded involving binomial coefficients or given by an infinite series:

$$F_n = \sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n = \frac{1}{2}\sum_{m=0}^{\infty}\frac{m^n}{2^m}\,.$$

The first Fubini numbers are presented as

$$\{F_n\}_{n=0}^{\infty} = 1, 1, 3, 13, 75, 541, 4683, 47293, 545835, 7087261, 102247563, \dots$$

[17, A000670]. The (exponential) generating function of Fubini numbers is given by

$$\frac{1}{2-e^x} = \sum_{n=0}^{\infty} F_n \frac{x^n}{n!}\,. \tag{5.1}$$

The Fubini numbers satisfy the recurrence relation [8]:

$$F_n = \sum_{j=1}^{n}\binom{n}{j}F_{n-j}\,. \tag{5.2}$$

Several generalizations of the Fubini numbers have been proposed and studied. A typical one is the higher-order Fubini numbers $\mathfrak{F}_n^{(r)}$ (e.g., [6, 15]) defined by

$$\left(\frac{1}{2-e^x}\right)^r = \sum_{n=0}^{\infty}\mathfrak{F}_n^{(r)}\frac{x^n}{n!}\,.$$

In this section, we propose a different type of generalized Fubini numbers, similarly to shifted Bernoulli numbers, in terms of determinants.

In [14], a determinant expression of Fubini numbers is given:

Proposition 5.1. *For $n \ge 1$, we have*

$$F_n = n! \begin{vmatrix} 1 & 1 & 0 & & \\ -\frac{1}{2!} & 1 & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{(-1)^{n-2}}{(n-1)!} & \frac{(-1)^{n-3}}{(n-2)!} & \cdots & 1 & 1 \\ \frac{(-1)^{n-1}}{n!} & \frac{(-1)^{n-2}}{(n-1)!} & \cdots & -\frac{1}{2!} & 1 \end{vmatrix}.$$

Now, for $m \ge 0$, we define the *shifted Fubini numbers* $F_n^{(m)}$ by

$$F_n^{(m)} = n! \begin{vmatrix} \frac{1}{(m+1)!} & 1 & 0 & & \\ -\frac{1}{(m+2)!} & \frac{1}{(m+1)!} & 1 & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{(-1)^{n-2}}{(m+n-1)!} & \frac{(-1)^{n-3}}{(m+n-2)!} & \cdots & \frac{1}{(m+1)!} & 1 \\ \frac{(-1)^{n-1}}{(m+n)!} & \frac{(-1)^{n-2}}{(m+n-1)!} & \cdots & -\frac{1}{(m+2)!} & \frac{1}{(m+1)!} \end{vmatrix}. \tag{5.3}$$

When $m = 0$, $F_n = F_n^{(0)}$ are the original Fubini numbers.

If we expand the determinant on the right-hand side of (5.3), we have

$$\begin{aligned} \frac{F_n^{(m)}}{n!} &= \frac{F_{n-1}^{(m)}}{(m+1)!(n-1)!} \\ &\quad - \begin{vmatrix} -\frac{1}{(m+2)!} & 1 & 0 & & \\ \frac{1}{(m+3)!} & \frac{1}{(m+1)!} & 1 & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{(-1)^{n-2}}{(m+n-1)!} & \frac{(-1)^{n-4}}{(m+n-3)!} & \cdots & \frac{1}{(m+1)!} & 1 \\ \frac{(-1)^{n-1}}{(m+n)!} & \frac{(-1)^{n-3}}{(m+n-2)!} & \cdots & -\frac{1}{(m+2)!} & \frac{1}{(m+1)!} \end{vmatrix} \\ &= \frac{F_{n-1}^{(m)}}{(m+1)!(n-1)!} + \frac{F_{n-2}^{(m)}}{(m+2)!(n-2)!} + \cdots \\ &\quad + (-1)^{n-2} \begin{vmatrix} \frac{(-1)^{n-2}}{(m+n-1)!} & 1 \\ \frac{(-1)^{n-1}}{(m+n)!} & \frac{1}{(m+1)!} \end{vmatrix} \\ &= \sum_{k=0}^{n-1} \frac{F_k^{(m)}}{(n-k+m)!k!}, \end{aligned}$$

where we put $F_0^{(m)} = 1$. Therefore, we have the recurrence relation for shifted Fubini numbers.

Lemma 5.2. *For integers $n \ge 1$ and $m \ge 0$,*

$$F_n^{(m)} = \sum_{k=0}^{n-1} \frac{n!}{(n-k+m)!k!} F_k^{(m)}.$$

with $F_0^{(m)} = 1$.

Remark. When $m = 0$ in Lemma 5.2, we have the relation (5.2).

By Lemma 5.2, we have the generating function of shifted Fubini numbers.

Theorem 5.3. *For* $m \ge 0$*, we have*

$$\frac{x^m}{x^m - e^x + E_m(x)} = \sum_{n=0}^\infty F_n^{(m)} \frac{x^n}{n!}\,,$$

where

$$E_m(x) = \sum_{n=0}^m \frac{x^n}{n!}$$

is the partial summation of e^x.

Proof. By Lemma 5.2, we have

$$\begin{aligned}
&\frac{x^m - e^x + E_m(x)}{x^m} \sum_{n=0}^\infty F_n^{(m)} \frac{x^n}{n!} \\
&= \left(1 - \sum_{l=1}^\infty \frac{x^l}{(l+m)!}\right)\left(\sum_{n=0}^\infty F_n^{(m)} \frac{x^n}{n!}\right) \\
&= \sum_{n=0}^\infty F_n^{(m)} \frac{x^n}{n!} - \sum_{n=1}^\infty \sum_{j=0}^{n-1} \frac{F_j^{(m)}}{(m+n-j)!j!} x^n \\
&= \sum_{n=0}^\infty F_n^{(m)} \frac{x^n}{n!} - \sum_{n=1}^\infty F_n^{(m)} \frac{x^n}{n!} \\
&= F_0^{(m)} = 1\,.
\end{aligned}$$

□

Similarly to Theorem 2.2, the shifted Fubini numbers have an explicit expression.

Theorem 5.4. *For integers* $n \ge 1$ *and* $m \ge 0$*,*

$$F_n^{(m)} = n! \sum_{k=1}^n \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge 1}} \frac{1}{(i_1+m)!\cdots(i_k+m)!}\,.$$

5.1. **Table of $F_n^{(m)}$, and mutual relations between $B_n^{(m)}$ and $F_n^{(m)}$.** Table of $F_n^{(m)}$ is given below.

n	0	1	2	3	4	5	6
F_n	1	1	3	13	75	541	4683
$F_n^{(1)}$	1	$\frac{1}{2}$	$\frac{5}{6}$	2	$\frac{191}{30}$	$\frac{76}{3}$	$\frac{5081}{42}$
$F_n^{(2)}$	1	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{29}{180}$	$\frac{263}{1080}$	$\frac{4157}{9072}$	$\frac{93881}{90720}$
$F_n^{(3)}$	1	$\frac{1}{24}$	$\frac{29}{1440}$	$\frac{149}{11520}$	$\frac{24967}{2419200}$	$\frac{115567}{11612160}$	$\frac{377909}{33177600}$
$F_n^{(4)}$	1	$\frac{1}{120}$	$\frac{7}{2400}$	$\frac{2687}{2016000}$	$\frac{44027}{60480000}$	$\frac{31627}{69120000}$	$\frac{66233749}{203212800000}$

The following properties are easily seen.

Theorem 5.5. *For $m \ge 0$*

$$F_1^{(m)} = \frac{1}{(m+1)!}\,.$$

There are some relations between shifted Bernoulli numbers and shifted Fubini numbers.

Theorem 5.6. *For $m \ge 0$*

$$\begin{aligned}
B_0^{(m)} + F_0^{(m)} &= 2\,,\\
-B_0^{(m)} + F_0^{(m)} &= 0\,,\\
B_1^{(m)} + F_1^{(m)} &= 0\,,\\
-B_1^{(m)} + F_1^{(m)} &= \frac{2}{(m+1)!}\,,\\
B_2^{(m)} + F_2^{(m)} &= \left(\frac{2}{(m+1)!}\right)^2\,,\\
-B_2^{(m)} + F_2^{(m)} &= \frac{4}{(m+2)!}\,,\\
B_3^{(m)} + F_3^{(m)} &= \frac{4!}{(m+1)!(m+2)!}\,,\\
-B_3^{(m)} + F_3^{(m)} &= \frac{2\cdot 3!\big(((m+1)!)^3 + (m+3)!\big)}{((m+1)!)^3(m+3)!}\,.
\end{aligned}$$

Proof. From the definition, we know that $B_0^{(m)} = F_0^{(m)} = 1$. From Theorem 2.4 and Theorem 5.5, we know that

$$-B_1^{(m)} = F_1^{(m)} = \frac{1}{(m+1)!}\,.$$

Hence, by Lemma 2.1 and Lemma 5.2, we have

$$B_2^{(m)} + F_2^{(m)} = \sum_{k=0}^{1} \frac{2}{(2-k+m)!k!}(-B_k^{(m)} + F_k^{(m)})$$

$$= \frac{2}{(m+1)!}\left(\frac{1}{(m+1)!} + \frac{1}{(m+1)!}\right)$$
$$= \left(\frac{2}{(m+1)!}\right)^2$$

and

$$-B_2^{(m)} + F_2^{(m)} = \sum_{k=0}^{1} \frac{2}{(2-k+m)!k!}(B_k^{(m)} + F_k^{(m)})$$
$$= \frac{2}{(m+2)!}(1+1)$$
$$= \frac{4}{(m+2)!}\,.$$

Further, by using these results too, we have

$$B_3^{(m)} + F_3^{(m)} = \sum_{k=0}^{2} \frac{3!}{(3-k+m)!k!}(-B_k^{(m)} + F_k^{(m)})$$
$$= \frac{3!}{(m+2)!}\left(\frac{1}{(m+1)!} + \frac{1}{(m+1)!}\right) + \frac{3!}{(m+1)!2!}\frac{4}{(m+2)!}$$
$$= \frac{4!}{(m+1)!(m+2)!}$$

and

$$-B_3^{(m)} + F_3^{(m)} = \sum_{k=0}^{2} \frac{3!}{(3-k+m)!k!}(B_k^{(m)} + F_k^{(m)})$$
$$= \frac{3!}{(m+3)!}(1+1) + \frac{3!}{(m+1)!2!}\left(\frac{2}{(m+1)!}\right)^2$$
$$= \frac{2\cdot 3!\big(((m+1)!)^3 + (m+3)!\big)}{((m+1)!)^3(m+3)!}\,.$$

□

Remark. We can continue to get the expressions of $B_n^{(m)} + F_n^{(m)}$ and $-B_n^{(m)} + F_n^{(m)}$ for $n \ge 4$ too. However, the results become more complicated.

By applying Trudi's formula, similarly to Theorem 2.7 and Theorem 2.8, we have the following combinatorial expression and the inversion expression, respectively.

Theorem 5.7. *For $n \ge m \ge 1$, we have*

$$F_n^{(m)} = n! \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n}$$
$$\times (-1)^{n-t_1-\cdots-t_n}\left(\frac{1}{(m+1)!}\right)^{t_1}\left(\frac{-1}{(m+2)!}\right)^{t_2}\cdots\left(\frac{(-1)^{n-1}}{(m+n)!}\right)^{t_n}\,.$$

Theorem 5.8. *For $n \ge 1$, we have*

$$\frac{(-1)^{n-1}}{(n+m)!}=\left|\begin{array}{ccccc} F_1^{(m)} & 1 & 0 & & \\ \frac{F_2^{(m)}}{2!} & F_1^{(m)} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{F_{n-1}^{(m)}}{(n-1)!} & \frac{F_{n-2}^{(m)}}{(n-2)!} & \cdots & F_1^{(m)} & 1 \\ \frac{F_n^{(m)}}{n!} & \frac{F_{n-1}^{(m)}}{(n-1)!} & \cdots & \frac{F_2^{(m)}}{2!} & F_1^{(m)} \end{array}\right|.$$

5.2. **Continued fraction expansions.** The generating function of shifted Fubini numbers can be expressed in continued fractions as similarity to that of shifted Bernoulli numbers.

From the expression in Theorem 5.3 below, shifted Fubini numbers satisfy the relation

$$\left(1-\sum_{n=0}^\infty\frac{x^{n+1}}{(m+n+1)!}\right)\left(\sum_{n=0}^\infty F_n^{(m)}\frac{x^n}{n!}\right)=1\,.$$

Thus,

$$P_M(x)=(m+M)!,\quad Q_M(x)=(m+M)!\left(1-\sum_{i=0}^{M-1}\frac{x^{i+1}}{(m+i+1)!}\right)$$

yield that

$$Q_M(x)\sum_{n=0}^\infty F_n^{(m)}\frac{x^n}{n!}\sim P_M(x)\quad(M\to\infty)\,.$$

Now,

$$\frac{P_0(x)}{Q_0(x)}=\frac{m!}{m!}=1,\quad \frac{P_1(x)}{Q_1(x)}=\frac{(m+1)!}{(m+1)!-x}=1+\frac{x}{(m+1)!-x}\,,$$
$$\frac{P_2(x)}{Q_2(x)}=\frac{(m+2)!}{(m+2)!-(m+2)x-x^2}=1+\cfrac{x}{(m+1)!-x-\cfrac{(m+1)!x}{m+2+x}}$$

and $P_n(x)$ and $Q_n(x)$ ($n\ge 3$) satisfy the recurrence relations

$$P_n(x)=(m+n+x)P_{n-1}(x)-(m+n-1)xP_{n-2}(x)$$
$$Q_n(x)=(m+n+x)Q_{n-1}(x)-(m+n-1)xQ_{n-2}(x)\,.$$

Since by (3.5) for $n\ge 3$

$$a_n(x)=m+n+x\quad\text{and}\quad b_n(x)=-(m+n-1)x\,,$$

we have the following continued fraction expansion.

Theorem 5.9.

$$\sum_{n=0}^{\infty} F_n^{(m)} \frac{x^n}{n!} = 1 + \cfrac{x}{(m+1)! - x - \cfrac{(m+1)!x}{m+2+x - \cfrac{(m+2)x}{m+3+x - \cfrac{(m+3)x}{m+4+x - \ddots}}}}.$$

When $m = 0$ in Theorem 5.9, we have a continued fraction expansion concerning the original Fubini numbers. Other expressions can be found in the generating functions in [17, A000670].

Corollary 5.10.

$$\sum_{n=0}^{\infty} F_n \frac{x^n}{n!} = 1 + \cfrac{x}{1 - x - \cfrac{x}{2+x - \cfrac{2x}{3+x - \cfrac{3x}{4+x - \ddots}}}}.$$

5.3. **Convolution identities.** Finally, similarly to Theorem 4.1, we show the convolution identities for two shifted Fubini numbers. The identity for $m = 0$ can be also seen in [11, (18)].

Theorem 5.11. *For integers $n \geq 0$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} F_k F_{n-k} = \frac{1}{2}(F_{n+1} + F_n).$$

For integers $n \geq 0$ and $m \geq 1$, we have

$$\sum_{k=0}^{n} \binom{n}{k} F_k^{(m)} F_{n-k}^{(m)}$$
$$= -\frac{n!}{m^2 \cdot m!} \sum_{l=0}^{n-1} \left(\frac{m!+1}{m \cdot m!} \right)^{n-l-1} \frac{l(m!+1)+m}{l!} F_l^{(m)} - \frac{n-m}{m} F_n^{(m)}.$$

Proof. For simplicity, put

$$f(x) := \left(1 - \sum_{l=1}^{\infty} \frac{x^l}{(l+m)!} \right)^{-1} = \sum_{n=0}^{\infty} F_n^{(m)} \frac{x^n}{n!}.$$

Then, we have

$$f'(x) = f(x)^2 \sum_{l=1}^{\infty} \frac{l x^{l-1}}{(l+m)!}$$

$$= f(x)^2\left(\sum_{l=1}^\infty \frac{x^{l-1}}{(l+m-1)!} - m\sum_{l=1}^\infty \frac{x^{l-1}}{(l+m)!}\right)$$
$$= f(x)^2\left(\frac{1}{m!} - \frac{x-m}{x}\big(f(x)^{-1}-1\big)\right)$$
$$= \frac{(m!+1)x - m\cdot m!}{m!x}f(x)^2 - \frac{x-m}{x}f(x)\,.$$

Thus, when $m=0$, by $f'(x) = 2f(x)^2 - f(x)$, we get

$$f(x)^2 = \frac{1}{2}f'(x) + \frac{1}{2}f(x)$$
$$= \frac{1}{2}\sum_{n=0}^\infty F_{n+1}\frac{x^n}{n!} + \frac{1}{2}\sum_{n=0}^\infty F_n\frac{x^n}{n!}\,.$$

When $m\ge 1$, we get

$$f(x)^2 = \frac{m!x}{(m!+1)x - m\cdot m!}f'(x) + \frac{m!(x-m)}{(m!+1)x - m\cdot m!}f(x)$$
$$= -\frac{1}{m}\sum_{n=0}^\infty\sum_{l=0}^n\left(\frac{m!+1}{m\cdot m!}\right)^l \frac{n!(n-l-m)}{(n-l)!}F_{n-l}^{(m)}\frac{x^n}{n!}$$
$$- \frac{1}{m}\sum_{n=0}^\infty\sum_{l=0}^{n-1}\left(\frac{m!+1}{m\cdot m!}\right)^l \frac{n!}{(n-l-1)!}F_{n-l-1}^{(m)}\frac{x^n}{n!}\,.$$

Therefore, we obtain

$$\sum_{k=0}^n\binom{n}{k}F_k^{(m)}F_{n-k}^{(m)} = -\frac{n!}{m}\sum_{l=0}^n\left(\frac{m!+1}{m\cdot m!}\right)^l\frac{n-l-m}{(n-l)!}F_{n-l}^{(m)}$$
$$-\frac{n!}{m}\sum_{l=0}^{n-1}\left(\frac{m!+1}{m\cdot m!}\right)^l\frac{1}{(n-l-1)!}F_{n-l-1}^{(m)}$$
$$= -\frac{n!}{m}\sum_{l=0}^n\left(\frac{m!+1}{m\cdot m!}\right)^{n-l}\frac{l-m}{l!}F_l^{(m)}$$
$$-\frac{n!}{m}\sum_{l=0}^{n-1}\left(\frac{m!+1}{m\cdot m!}\right)^{n-l-1}\frac{1}{l!}F_l^{(m)}$$
$$= -\frac{n!}{m}\sum_{l=0}^{n-1}\left(\frac{m!+1}{m\cdot m!}\right)^{n-l-1}\left(\frac{m!+1}{m\cdot m!}(l-m)+1\right)\frac{F_l^{(m)}}{l!}$$
$$-\frac{n!}{m}\frac{n-m}{n!}F_n^{(m)}$$
$$= -\frac{n!}{m^2\cdot m!}\sum_{l=0}^{n-1}\left(\frac{m!+1}{m\cdot m!}\right)^{n-l-1}\frac{l(m!+1)-m}{l!}F_l^{(m)}$$
$$-\frac{n-m}{m}F_n^{(m)}\,.$$

□

References

[1] M. Aoki, T. Komatsu and G. K. Panda, *Several properties of hypergeometric Bernoulli numbers*, J. Inequal. Appl. **2019** (2019), Paper No.113. 24p.

[2] T. Arakawa, T. Ibukiyama and M. Kaneko, *Bernoulli Numbers and Zeta Functions. With an appendix by Don Zagier*, Springer Monographs in Mathematics, Springer, Tokyo, 2014.

[3] F. Brioschi, *Sulle funzioni Bernoulliane ed Euleriane*, Annali di Mat., i. (1858), 260–263; Opere Mat., i. pp. 343–347.

[4] P. J. Cameron, *Some sequences of integers*, Discrete Math. **75** (1989), 89–102.

[5] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, 1974.

[6] T. Diagana and H. Maïga, *Some new identities and congruences for Fubini numbers*, J. Number Theory **173** (2017), 547–569.

[7] J. W. L. Glaisher, *Expressions for Laplace's coefficients, Bernoullian and Eulerian numbers etc. as determinants*, Messenger (2) **6** (1875), 49–63.

[8] O. A. Gross, *Preferential arrangements*, Amer. Math. Monthly **69** (1962), 4–8.

[9] S. Hu and T. Komatsu, *Explicit expressions for the related numbers of higher order Appell polynomials*, Quaest. Math. (to appear). https://doi.org/10.2989/16073606.2019.1596174

[10] W. B. Jones and W. J. Thron, *Continued fractions. Analytic theory and applications*, Encyclopedia of Mathematics and its Applications, Vol. 11, Addison-Wesley, Reading, 1980.

[11] L. Kargin, *Some formulae for products of geometric polynomials with applications*, J. Integer Seq. **20** (2017), Article 17.4.4 (15 pages).

[12] T. Komatsu, K. Liptai and I. Mező, *Incomplete poly-Bernoulli numbers associated with incomplete Stirling numbers*, Publ. Math. Debrecen **88** (2016), 357–368.

[13] T. Komatsu and A. Pizarro-Madariaga, *Harmonic numbers associated with inversion numbers in terms of determinants*, Turkish J. Math. **43** (2019), 340–354.

[14] T. Komatsu and J. L. Ramirez, *Some determinants involving incomplete Fubini numbers*, An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. **26** (2018), 143–170.

[15] Z.-R. Li, *Computational formulae for generalized mth order Bell numbers and generalized mth order ordered Bell numbers* (Chinese), J. Shandong Univ. Nat. Sci. **42** (2007), 59–63.

[16] T. Muir, *The Theory of Determinants in the Historical Order of Development*, Four volumes, Dover Publications, New York, 1960.

[17] N. J. A. Sloane, *The on-line encyclopedia of integer sequences*, available at oeis.org. (2020).

[18] N. Trudi, *Intorno ad alcune formole di sviluppo*, Rendic. dell' Accad. Napoli, (1862), 135–143.

[19] H. S. Wall, *Analytic Theory of Continued Fractions*, Chelsea Publ., New York, 1948.

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