# HIERARCHICAL MINIMAX PROBLEMS IN LOCALLY CONVEX HAUSDORFF TOPOLOGICAL VECTOR SPACES 

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#### Abstract

The minimax theorems have been studied by single-valued functions or vector-valued functions for a long time. Since the beginnings of the century, thanks to many pioneer scholars, the fields have developed the framework for studying the set-valued mappings. In this paper, new minimax theorems for non-continuous set-valued mappings under locally convex Hausdorff topological vector spaces settings are introduced. First, the validity of scalar minimax theorems for five scalar set-valued mappings are introduced. Second, three versions of minimax theorems for non-continuous set-valued mappings that possess hierarchical structures are also proposed. Finally, some examples are proposed and serve to illustrate our theorems.


## 1. Introduction and preliminaries

Minimax theorems are crucial in the methods and techniques of several mathematical applications, such as optimization theory, calculus of variations, mathematical programming, game theory, control theory, convex analysis, functional analysis, nonlinear analysis, nonsmooth analysis, set-valued analysis and variational analysis.

Recently, minimax theorems have been developed in the methodology of several non-continuous set-valued mappings which are so-called hierarchical structures. We refer the readers to $[3,4,5,6,7,8]$ which discuss the minimax theorems for hierarchical structures. The environment where several non-continuous set-valued mappings can work well to successfully express the hierarchical structures can be stated as follows: Let $X, Y$ be two nonempty sets in two Hausdorff topological vector spaces, $\mathbb{W}$ be a Hausdorff topological vector space, and $C \subset \mathbb{W}$ be a closed convex and pointed solid cone with apex at the origin; that is, $C$ is a closed set with nonempty interior, satisfying $\lambda C \subseteq C, \forall \lambda>0, C+C \subseteq C$, and $C \cap(-C)=\{0\}$. The relationship between hierarchical minimax theorems and scalar hierarchical minimax theorems has been discussed by several authors $[3,4,5,6,7,8]$. Notably, scalarization technology is a critical in explaining this relationship.

In order to discuss the validity of scalar minimax theorems, we state a variant form, Lemma A, of alternative principle due to Balaj[2].

Lemma A ([2]). Let $X, Y$ be two nonempty convex subsets, each in a locally convex Hausdorff topological vector spaces, one of them compact. Let $F_{i}: X \rightrightarrows Y$,
$1 \leq i \leq 5$, be set-valued mappings with $F_{i}(x) \subset F_{i+1}(x)$ for each $x \in X$ and for $1 \leq i \leq 3$ such that
(i) $F_{1}^{-1}: Y \rightrightarrows X$ and $F_{5}^{c}: X \rightrightarrows Y$ are upper semi-continuous mappings;
(ii) $F_{4}^{-1}(c o A) \subset F_{5}^{-1}(A)$ for each finite subset $A$ of $Y$, where co $A$ means the convex hull of the set $A$;
(iii) $F_{2}^{-1}(y)$ is convex for each $y \in Y$, and the both sets $F_{3}^{-1}(y)$ and $F_{4}^{c}(x)$ are compact for each $x \in X$ and for each $y \in Y$.
Then either there is a $y_{0} \in Y$ such that $F_{1}^{-1}\left(y_{0}\right)=\emptyset$, or

$$
\bigcap_{y \in Y} F_{5}^{-1}(y) \neq \emptyset .
$$

Here, the notations $S^{-1}$ and $S^{c}$ for a mapping $S: X \rightrightarrows Y$ are defined by

$$
x \in S^{-1}(y) \text { if and only if } y \in S(x)
$$

and

$$
S^{c}(x)=Y \backslash S(x)
$$

for some suitable $x \in X$ and $y \in Y$. The Lemma A uses some slight different descriptions from Lemma 2.3[8]. The following alternative principle is a variant form of Lemma A.

Lemma B. Let $X, Y$ be two nonempty convex subsets, each in a locally convex Hausdorff topological vector spaces, one of them compact. Let $F_{i}: X \rightrightarrows Y, 1 \leq i \leq$ 5, be set-valued mappings with $F_{i}(x) \subset F_{i+1}(x)$ for each $x \in X$ and for $1 \leq i \leq 3$ such that
(i) $F_{1}: X \rightrightarrows Y$ and $\left(F_{5}^{-1}\right)^{c}: Y \rightrightarrows X$ are upper semi-continuous mappings;
(ii) $F_{4}(c o A) \subset F_{5}(A)$ for each finite subset $A$ of $X$, where co $A$ means the convex hull of the set $A$;
(iii) $F_{2}(x)$ is convex for each $x \in X$, and the both sets $F_{3}(x)$ and $\left(F_{4}^{-1}\right)^{c}(y)$ are compact for each $x \in X$ and for each $y \in Y$.
Then either there is an $x_{0} \in X$ such that $F_{1}\left(x_{0}\right)=\emptyset$, or

$$
\bigcap_{x \in X} F_{5}(x) \neq \emptyset
$$

On the basis of Lemma A and Lemma B, some hierarchical minimax theorems for non-continuous set-valued mappings under locally convex Hausdorff topological vector space settings are introduced in this paper. In addition, the following notations and some established facts are applied throughout this paper.

Definition 1.1 ([5]). Let $X$ be a nonempty convex subset of a vector space, $Z$ a vector space, $C \subset Z$ a closed convex and pointed solid cone with apex at the origin. The mapping $S: X \rightrightarrows Z$ is
(i) above-C-quasi-convex on $X$ if the set

$$
\operatorname{Lev}_{S \leq}(z):=\{x \in X: S(x) \subset z-C\}
$$

is convex for all $z \in Z$.
(ii) above-properly $C$-quasi-convex (above-properly $C$-quasi-concave, respectively) on $X$ if for any $x_{1}, x_{2} \in X$ and any $\lambda \in[0,1]$, either

$$
\begin{gathered}
S\left(\lambda x_{1}+(1-\lambda) x_{2}\right) \subset S\left(x_{1}\right)-C \\
\left(S\left(x_{1}\right) \subset S\left(\lambda x_{1}+(1-\lambda) x_{2}\right)-C, \text { respectively }\right)
\end{gathered}
$$

or

$$
\begin{gathered}
S\left(\lambda x_{1}+(1-\lambda) x_{2}\right) \subset S\left(x_{2}\right)-C \\
\left(S\left(x_{2}\right) \subset S\left(\lambda x_{1}+(1-\lambda) x_{2}\right)-C, \text { respectively }\right)
\end{gathered}
$$

(iii) above-naturally $C$-quasi-convex (above-naturally $C$-quasi-concave, respectively) on $X$ if for any $x_{1}, x_{2} \in X$ and any $\lambda \in[0,1]$,

$$
\begin{gathered}
S\left(\lambda x_{1}+(1-\lambda) x_{2}\right) \subset \operatorname{co}\left\{S\left(x_{1}\right) \cup S\left(x_{2}\right)\right\}-C . \\
\left(\operatorname{co}\left\{S\left(x_{1}\right) \cup S\left(x_{2}\right)\right\} \subset S\left(\lambda x_{1}+(1-\lambda) x_{2}\right)-C, \text { respectively }\right)
\end{gathered}
$$

Definition $1.2([5])$. Let $A$ be a nonempty subset of $Z$, then a point $z \in A$ is called
(i) a minimal point of $A$ if $A \cap(z-C)=\{z\}$; $\operatorname{Min} A$ is the set of all minimal points of $A$;
(ii) a maximal point of $A$ if $A \cap(z+C)=\{z\}$; $\operatorname{Max} A$ is the set of all maximal points of $A$;
(iii) a weakly minimal point of $A$ if $A \cap(z-i n t C)=\emptyset ; \operatorname{Min}_{w} A$ is the set of all weakly minimal points of $A$; and
(iv) a weakly maximal point of $A$ if $A \cap(z+\operatorname{int} C)=\emptyset ; \operatorname{Max}_{w} A$ is the set of all weakly maximal points of $A$.

Whenever $C=\mathbb{R}_{+}$, and all mappings are single-valued, the Definition 1.1-1.2 can be reduced to the original ones. In this case, both $\operatorname{Max} A$ and $\operatorname{Max}_{w} A(\operatorname{Min} A$ and $\operatorname{Min}_{w} A$, respectively) can be denoted by max $A(\min A$, respectively).

Definition 1.3 ([1]). Let $U, V$ be Hausdorff topological spaces. A set-valued map $S: U \rightrightarrows V$ with nonempty values is said to be
(a) upper semi-continuous at $x_{0} \in U$ if for every $x_{0} \in U$ and for every open set $N$ containing $S\left(x_{0}\right)$, there exists a neighborhood $M$ of $x_{0}$ such that $S(M):=\bigcup_{x \in M} S(x) \subset N ;$
(b) lower semi-continuous at $x_{0} \in U$ if for every $x_{0} \in U$ and for every open set $N$ with $S\left(x_{0}\right) \cap N \neq \emptyset$, there exists a neighborhood $M$ of $x_{0}$ such that $S(x) \cap N \neq \emptyset$ for all $x \in M$; and
(c) continuous at $x_{0} \in U$ if $S$ is upper semi-continuous as well as lower semicontinuous at $x_{0}$.
(d) closed if its graph is closed.

The following example includes two such types of semi-continuous set-valued mappings, which will be used again in the sequel.

Example 1. Let $X=Y=[0,1]$ and $C=\mathbb{R}_{+}$, and set-valued mappings $U, V$ : $X \times Y \rightrightarrows \mathbb{R}$ be defined by

$$
\begin{gathered}
U(x, y):= \begin{cases}\{t: t \in[0,2]\}, & x=0, y=0 \\
\{t: t \in[1,2]\}, & x=0,0<y \leq 1 \\
\{t: t \in[0,1]\}, & 0<x \leq 1, y=0 \\
\{1\}, & 0<x, y \leq 1\end{cases} \\
V(x, y):=\left[x^{3}-1+y^{2}, x^{2}\right] .
\end{gathered}
$$

Thus $U$ is upper semi-continuous mapping on $X \times Y$ and $V$ is a lower semicontinuous mapping on $X \times Y$.

Proof. We first claim that $U$ is upper semi-continuous on $X \times Y$. If $(\bar{x}, \bar{y})=(0,0)$. For any open set $N \supset U(\bar{x}, \bar{y})=[0,2]$, we choose an open set $M=[0, \delta) \times[0, \varepsilon)$, for some small numbers $0<\delta, \varepsilon<1$. Then, for any $(x, y) \in M$,

$$
U(x, y)= \begin{cases}\{t: t \in[0,2]\}, & x=0, y=0 \\ \{t: t \in[1,2]\}, & x=0,0<y<\varepsilon \\ \{t: t \in[0,1]\}, & 0<x<\delta, y=0 \\ \{1\}, & 0<x<\delta, 0<y<\varepsilon\end{cases}
$$

We can see that, for any cases, $U(x, y) \subset U(\bar{x}, \bar{y})$. Hence $U(x, y) \subset N$. If $\bar{x}=$ $0,0<\bar{y}<1$, then for any open set $N \supset U(\bar{x}, \bar{y})=[1,2]$, we choose an open set $M=[0, \delta) \times(\bar{y}-\varepsilon / 2, \bar{y}+\varepsilon / 2)$, for some small numbers $0<\delta<1$ and $0<\varepsilon<$ $\min \{1,2 \bar{y}, 2(1-\bar{y})\}$. Then, for any $(x, y) \in M$,

$$
U(x, y)= \begin{cases}\{t: t \in[1,2]\}, & x=0, y \in(\bar{y}-\varepsilon / 2, \bar{y}+\varepsilon / 2) \\ \{1\}, & 0<x<\delta, y \in(\bar{y}-\varepsilon / 2, \bar{y}+\varepsilon / 2)\end{cases}
$$

Hence, $U(x, y) \subset N$. If $\bar{x}=0, \bar{y}=1$, then for any open set $N \supset U(\bar{x}, \bar{y})=[1,2]$, we choose an open set $M=[0, \delta) \times(\varepsilon, 1]$, for some small numbers $0<\delta, \varepsilon<1$, such that for all $(x, y) \in M, U(x, y) \subset N$. Hence, $U$ is upper semi-continuous on $\{0\} \times(0,1]$. By similar arguments for the both cases, $0<\bar{x} \leq 1, \bar{y}=0$ and $0<\bar{x}, \bar{y} \leq 1$, we can get that $U$ is upper semi-continuous at $(\bar{x}, \bar{y})$.

Next, we claim that $V$ is lower semi-continuous on $X \times Y$. Indeed, for any $(\bar{x}, \bar{y}) \in X \times Y$ and for any open set $N$ with $N \cap V(\bar{x}, \bar{y}) \neq \emptyset$. First, if $\bar{x}^{2} \in N$, there exists a positive number $\varepsilon$ such that $\left(\bar{x}^{2}-\varepsilon, \bar{x}^{2}+\varepsilon\right) \subset N$. Since the mapping $x \mapsto x^{2}$ is continuous, there exists a positive number $\delta$ such that, for all $x$ with $|x-\bar{x}|<\delta$, $\left|x^{2}-\bar{x}^{2}\right|<\varepsilon$. We choose an open neighborhood $M=(\bar{x}-\delta, \bar{x}+\delta) \times(\bar{y}-\delta, \bar{y}+\delta)$ of $(\bar{x}, \bar{y})$ such that for all $(x, y) \in M$, the intersection $N \cap V(x, y)$ is nonempty because $x^{2} \in\left(\bar{x}^{2}-\varepsilon, \bar{x}^{2}+\varepsilon\right)$. Secondly, if $\bar{x}^{3}-1+\bar{y}^{2} \in N$, there exists a positive number $\varepsilon$ such that $\left(\bar{x}^{3}-1+\bar{y}^{2}-\varepsilon, \bar{x}^{3}-1+\bar{y}^{2}+\varepsilon\right) \subset N$. Since the mapping $(x, y) \mapsto x^{3}-1+y^{2}$ is continuous on $X \times Y$, there exist positive number $\delta$ and open set $M=(\bar{x}-\delta, \bar{x}+\delta) \times(\bar{y}-\delta, \bar{y}+\delta)$ such that $\left|x^{3}-1+y^{2}-\left(\bar{x}^{3}-1+\bar{y}^{2}\right)\right|<\epsilon$. Hence $N \cap V(x, y) \neq \emptyset$. Thirdly, if $\bar{x}^{3}-1+\bar{y}^{2} \notin N$ and $\bar{x}^{3}-1+\bar{y}^{2} \notin \partial N$, where
$\partial N$ means the boundary of $N$. For any $\varepsilon>0$ with $\bar{x}^{3}-1+\bar{y}^{2}+\varepsilon \notin N$. Then we choose an open set as the same as in the second step, then for all $(x, y) \in M$, $N \cap V(x, y) \neq \emptyset$. Fourthly, if $\bar{x}^{3}-1+\bar{y}^{2} \notin N$ but $\bar{x}^{3}-1+\bar{y}^{2} \in \partial N$. We can choose an open set $N_{1} \subset N$ with $N_{1} \cap V(\bar{x}, \bar{y}) \neq \emptyset$ and $\bar{x}^{3}-1+\bar{y}^{2} \notin N_{1} \cup \partial N_{1}$. Then the same process of third step is followed by $N \cap V(x, y) \supset N_{1} \cap V(x, y) \neq \emptyset$. Finally, for the both cases, $\bar{x}^{2} \notin N \cup \partial N$ and $\bar{x}^{2} \in \partial N \backslash N$, we can see $N \cap V(x, y) \neq \emptyset$ by using a similar method of third or fourth step. From above argument, we prove that $V$ is lower semi-continuous on $X \times Y$.

Note that $S$ is upper semicontinuous at $x_{0}$ and $S\left(x_{0}\right)$ is compact, then for any net $\left\{x_{\nu}\right\} \subset U, x_{\nu} \rightarrow x_{0}$, and for any net $y_{\nu} \in S\left(x_{\nu}\right)$ for each $\nu$, there exists $y_{0} \in S\left(x_{0}\right)$ and a subnet $\left\{y_{\nu_{\alpha}}\right\}$ such that $y_{\nu_{\alpha}} \rightarrow y_{0}$. Furthermore, $S$ is lower semicontinuous at $x_{0}$ if for any net $\left\{x_{\nu}\right\} \subset U, x_{\nu} \rightarrow x_{0}, y_{0} \in S\left(x_{0}\right)$ implies that there exists subnet $y_{\nu_{k}} \in S\left(x_{\nu_{k}}\right)$ such that $y_{\nu_{k}} \rightarrow y_{0}$. For more details, we refer the reader to [1].

The following lemma clarifies the relationship of quasi-convexities between the $G$ and $\max G(x)$ mappings.

Lemma 1.4 ([6], Lemma 1). Suppose that $X$ is a nonempty convex subset of a topological vector space, with a set-valued mapping of $G: X \mapsto \mathbb{R}$ where $\max G(x)$ exists for each $x \in X$, then the mapping $G: X \mapsto \mathbb{R}$ is above- $\mathbb{R}_{+}$-quasi-convex if and only if the mapping $x \mapsto \max G(x)$ is a quasi-convex function.

The preceding notable properties indicate that above-properly $C$-quasi-concave mapping (above-naturally $C$-quasi-convex mapping, respectively) is more general than above-naturally $C$-quasi-concave mapping (above-properly $C$-quasi-convex mapping, respectively).

Lemma 1.5. Let $X$ be a nonempty convex subset of a vector space, $Z$ a vector space, and $C \subset Z$ be a closed convex and pointed solid cone with an apex at the origin. If the set-valued mapping $S: X \rightrightarrows Z$ is above-naturally $C$-quasi-concave on $X$, then $S$ is above-properly $C$-quasi-concave. Furthermore, every above-properly $C$-quasi-convex mapping is an above-naturally $C$-quasi-convex.

The proof of Lemma 1.5 can be directly derived from definitions.

## 2. Hierarchical structures for scalar set-valued mappings

The following scalar minimax theorem is based on Lemma A.
Theorem 2.1. Let $X, Y$ be two nonempty compact convex subsets of locally convex Hausdorff topological vector spaces, respectively. Furthermore, let $G_{i}: X \times$ $Y \rightrightarrows \mathbb{R}$, for $i=1,2,3,4,5$, be a set-valued mapping, where $\max G_{i}(x, y)$ and $\min \bigcup_{x \in X} G_{5}(x, y)$ exist for each $(x, y) \in X \times Y$, such that $\max G_{i}(x, y) \leq$ $\max G_{i+1}(x, y)$ for all $(x, y) \in X \times Y$ and for $i=1,2,3,4$. Finally, suppose that the following conditions are satisfied:
(i) either $y \mapsto G_{1}(x, y)$ or $y \mapsto G_{2}(x, y)$ is above-properly $\mathbb{R}_{+}$-quasi-concave on $Y$ for each $x \in X$, whereas $x \mapsto G_{4}(x, y)$ is above- $\mathbb{R}_{+}$-quasi-convex on $X$ for each $y \in Y$;
(ii) $(x, y) \mapsto G_{1}(x, y)$ is an upper semi-continuous mapping on $X \times Y$ and $y \mapsto G_{2}(x, y)$ is an upper semi-continuous mapping on $Y$ for each $x \in X$; meanwhile, $x \mapsto G_{1}(x, y)$ and $x \mapsto G_{3}(x, y)$ are lower semi-continuous mappings on $X$ for each $y \in Y$, and $(x, y) \mapsto G_{5}(x, y)$ is a lower semi-continuous mapping on $X \times Y$; and
(iii) for each $w \in Y$, there is an $x_{w} \in X$ such that

$$
\max G_{5}\left(x_{w}, w\right) \leq \max \bigcup_{y \in Y} \min \bigcup_{x \in X} G_{5}(x, y)
$$

Thus, the following relationship holds:

$$
\begin{equation*}
\min \bigcup_{x \in X} \max \bigcup_{y \in Y} G_{1}(x, y) \leq \max \bigcup_{y \in Y} \min \bigcup_{x \in X} G_{5}(x, y) \tag{S-H}
\end{equation*}
$$

Proof. For any $t_{1}, t_{2}, t_{3}, t_{4} \in \mathbb{R}$ with $t_{1}<t_{2}<t_{3}<t_{4}$. Define $F_{1}: X \rightrightarrows Y$ by

$$
\begin{equation*}
F_{1}(x):=\left\{y \in Y: \max G_{5}(x, y) \leq t_{1}\right\} \tag{2.1}
\end{equation*}
$$

for all $x \in X$, and $F_{2}, F_{3}, F_{4}, F_{5}$ are similarly, with the triples $\left(G_{5}, \leq, t_{1}\right)$ inside the braces in (2.1) replaced by $\left(G_{4},<, t_{2}\right),\left(G_{3}, \leq, t_{3}\right),\left(G_{2},<, t_{4}\right)$ and $\left(G_{1},<, t_{4}\right)$, respectively. Because $\max G_{i}(x, y) \leq \max G_{i+1}(x, y)$ for all $(x, y) \in X \times Y$ and for $i=1,2,3,4, F_{i}(x) \subset F_{i+1}(x)$ for $i=1,2,3,4$.

Note that the graphs of both mappings $F_{1}^{-1}$ and $F_{5}^{c}$ are closed. Indeed, for any sequence $\left(y_{n}, x_{n}\right) \in \operatorname{Graph}\left(F_{1}^{-1}\right):=\left\{(y, x): \max G_{5}(x, y) \leq t_{1}\right\}$ with $\left(y_{n}, x_{n}\right) \rightarrow$ $\left(y_{0}, x_{0}\right)$. Then, for all $n, \max G_{5}\left(x_{n}, y_{n}\right) \leq t_{1}$. From the lower semi-continuity of $G_{5}$ and Lemma 2.5[5], $\max G_{5}$ is lower semi-continuous on $X \times Y$ and $\max G_{5}\left(x_{0}, y_{0}\right) \leq$ $t_{1}$. Hence, $\left(y_{0}, x_{0}\right) \in \operatorname{Graph}\left(F_{1}^{-1}\right)$ and the mapping $F_{1}^{-1}$ is closed. By a similar argument, the mapping $F_{5}^{c}$ is also closed. Therefore, these two mappings are upper semi-continuous.

Next, for any finite subset $A$ of $Y$, we claim that $F_{4}^{-1}(c o A) \subset F_{5}^{-1}(A)$. Assume that for each $x \in X$, the mapping $y \mapsto G_{1}(x, y)$ above-properly $\mathbb{R}_{+}$-quasiconcave on $Y$. Let $w \in F_{4}^{-1}(c o A)$, then $w \in F_{4}^{-1}\left(\sum_{i=1}^{n} \alpha_{i} y_{i}\right)$ for some $y_{i} \in A$, $\alpha_{i} \in[0,1]$, for $i=1,2 \ldots, n$ and $\sum_{i=1}^{n} \alpha_{i}=1$. Then, $\max G_{1}\left(w, \sum_{i=1}^{n} \alpha_{i} y_{i}\right) \leq$ $\max G_{2}\left(w, \sum_{i=1}^{n} \alpha_{i} y_{i}\right)<t_{4}$. By the fact of Proposition 3.5[5], the mapping $y \mapsto$ $\max G_{1}(x, y)$ is properly quasi-concave for each $x \in X$. Hence, at least one $i \in$ $\{1,2, \ldots n\}$ such that $\max G_{1}\left(w, y_{i}\right) \leq \max G_{1}\left(w, \sum_{i=1}^{n} \alpha_{i} y_{i}\right)$ holds. That is, at least one $i \in\{1,2, \ldots n\}$ such that $\max G_{1}\left(w, y_{i}\right)<t_{4}$ holds. Hence, $w \in F_{5}^{-1}(A)$. Therefore, $F_{4}^{-1}(c o A) \subset F_{5}^{-1}(A)$.

Finally, let us to claim that condition (iii) of Lemma A is valid. Indeed, because the mapping $x \mapsto G_{4}(x, y)$ is above- $\mathbb{R}_{+}$-quasi-convex on $X$ for each $y \in Y$, the mapping $x \mapsto \max G_{4}(x, y)$ is quasi-convex on $X$ for each $y \in Y$ by Lemma 1.4. This implies that the set $F_{2}^{-1}(y)$ is convex for each $y \in Y$. Finally, we can deduce the following situation without difficulty that, for all $x \in X$ and $y \in Y$, the both sets $F_{3}^{-1}(y)$ and $F_{4}^{c}(x)$ are compact from the facts that the mapping $y \mapsto G_{2}(x, y)$ is upper semi-continuous on $Y$ for each $x \in X$, the mapping $x \mapsto G_{3}(x, y)$ is lower semi-continuous on $X$ for each $y \in Y$ and Lemma 2.5[5].

Then, all conditions of Lemma A are satisfied, and hence we know that either there is a $y_{0} \in Y$ such that $F_{1}^{-1}\left(y_{0}\right)=\emptyset$, or

$$
\bigcap_{y \in Y} F_{5}^{-1}(y) \neq \emptyset
$$

That is, either there is a $y_{0} \in Y$ such that $\max G_{5}\left(x, y_{0}\right)>t_{1}$ for all $x \in X$, or there exists an $x_{0} \in X$ such that $\max G_{1}\left(x_{0}, y\right)<t_{4}$ for all $y \in Y$. Nevertheless, the first assertion does not hold because of condition (iii) and $t_{1}<t_{4}$. This implies that $\max \bigcup_{y \in Y} \max G_{1}\left(x_{0}, y\right) \leq t_{4}$. By Lemma 1.1[7], we have $\max \bigcup_{y \in Y} G_{1}\left(x_{0}, y\right) \leq t_{4}$. Therefore, $\min \bigcup_{x \in X} \max \bigcup_{y \in Y} G_{1}(x, y) \leq t_{4}$, which means $(S-H)$ is valid.

Accordingly, above- $\mathbb{R}_{+}$-quasi-convexity, is more general than above-naturally $\mathbb{R}_{+}{ }^{-}$ quasiconvexity which is used in Theorem 3.1[8]. Furthermore, the truth is determined with Theorem 2.1, which is a new version of scalar minimax results, when the readers compare it with other theorems in literatures $[5,3,7,4,6,8]$.

Corollary 2.2. Let $X, Y$ be two nonempty compact convex subsets of locally convex Hausdorff topological vector spaces, respectively. Let $G_{i}: X \times Y \rightrightarrows \mathbb{R}$, for $i=$ $1,2,3,4,5$, be a set-valued mapping where $\max G_{i}(x, y)$ exists for each $(x, y) \in X \times$ $Y$, such that $\max G_{i}(x, y) \leq \max G_{i+1}(x, y)$ for all $(x, y) \in X \times Y$ and for $i=$ $1,2,3,4$. Assume that the set $\bigcup_{x \in X} G_{5}(x, y)$ is compact for all $(x, y) \in X \times Y$. Finally, suppose that the following conditions are satisfied:
(i) either $y \mapsto G_{1}(x, y)$ or $y \mapsto G_{2}(x, y)$ is above-properly $\mathbb{R}_{+}$-quasi-concave on $Y$ for each $x \in X$, whereas $x \mapsto G_{4}(x, y)$ is above- $\mathbb{R}_{+}$-quasi-convex on $X$ for each $y \in Y$;
(ii) $(x, y) \mapsto G_{1}(x, y)$ is an upper semi-continuous mapping on $X \times Y$ and $y \mapsto G_{2}(x, y)$ is an upper semi-continuous mapping on $Y$ for each $x \in X$; meanwhile, $(x, y) \mapsto G_{1}(x, y)$ and $x \mapsto G_{3}(x, y)$ are lower semi-continuous mappings on $X$ for each $y \in Y$, and $(x, y) \mapsto G_{5}(x, y)$ is a lower semicontinuous mapping on $X \times Y$; and
(iii) for each $w \in Y$, there is an $x_{w} \in X$, such that

$$
\max G_{5}\left(x_{w}, w\right) \leq \max \bigcup_{y \in Y} \min \bigcup_{x \in X} G_{5}(x, y)
$$

Thus, the relationship (S-H) holds.
Proof. We observe that the compactness of both sets $\bigcup_{y \in Y} G_{1}(x, y)$ and $\bigcup_{x \in X} G_{5}(x, y)$ for all $(x, y) \in X \times Y$ will guarantee that the existences of $\max \bigcup_{y \in Y} G_{1}(x, y)$ and $\min \bigcup_{x \in X} G_{5}(x, y)$ exist for each $(x, y) \in X \times Y$. The conclusion completes directly from Theorem 2.1.

The following corollary is a special case of Theorem 2.1, which assumes that $G_{1}=F_{1}, G_{2}=F_{2}, G_{3}=G_{4}=F_{2}, G_{5}=F_{4}$ in Theorem 2.1.

Corollary 2.3. Let $X, Y$ be two nonempty compact convex subsets of locally convex Hausdorff topological vector spaces, respectively. Let $F_{i}: X \times Y \rightrightarrows \mathbb{R}$, for $i=$ $1,2,3,4$, be a set-valued mapping, where $\max F_{i}(x, y)$ and $\min \bigcup_{x \in X} F_{4}(x, y)$ exist
for each $(x, y) \in X \times Y$, such that $\max F_{i}(x, y) \leq \max F_{i+1}(x, y)$ for all $(x, y) \in$ $X \times Y$ and for $i=1,2,3$. Finally, suppose that the following conditions are satisfied:
(i) either $y \mapsto F_{1}(x, y)$ or $y \mapsto F_{2}(x, y)$ is above-properly $\mathbb{R}_{+}$-quasi-concave on $Y$ for each $x \in X$, whereas $x \mapsto F_{3}(x, y)$ is above- $\mathbb{R}_{+}$-quasi-convex on $X$ for each $y \in Y$;
(ii) $(x, y) \mapsto F_{1}(x, y)$ is an upper semi-continuous mapping on $X \times Y$, and $y \mapsto F_{2}(x, y)$ is an upper semi-continuous mapping on $Y$ for each $x \in$ $X$; meanwhile, $x \mapsto F_{1}(x, y)$ and $x \mapsto F_{3}(x, y)$ are lower semi-continuous mappings on $X$ for each $y \in Y$, and $(x, y) \mapsto F_{4}(x, y)$ is a lower semicontinuous mapping on $X \times Y$; and
(iii) for each $w \in Y$, there is an $x_{w} \in X$, such that

$$
\max F_{4}\left(x_{w}, w\right) \leq \max \bigcup_{y \in Y} \min \bigcup_{x \in X} F_{4}(x, y)
$$

Thus, the relationship holds:

$$
\min \bigcup_{x \in X} \max \bigcup_{y \in Y} F_{1}(x, y) \leq \max \bigcup_{y \in Y} \min \bigcup_{x \in X} F_{4}(x, y)
$$

It is noteworthy to compare Corollary 2.3 with Theorem 3.1[8]. We noted that the convexity of $F_{3}$ is weaker than that used in Theorem 3.1[8]. Furthermore, we did not require the compactness of the both sets, namely $\bigcup_{x \in X} F_{4}(x, y)$ and $\bigcup_{y \in Y} F_{1}(x, y)$. However, the conditions on $F_{2}$ are highly different. As far as we know, no relationship exists between the closed and upper semi-continuous mappings unless they possess compact values or compact codomain or closed values. Moreover, $F_{1}$ is weaker than that used in Theorem 3.1[8] because closed values are not required. Notably, above-properly $\mathbb{R}_{+}$-quasi-concavity is weaker than abovenaturally $\mathbb{R}_{+}$-quasi-concavity; indeed, such a property is valid according to Lemma 1.5. Next, we propose another version of scalar hierarchical minimax result.

Theorem 2.4. Let $X, Y$ be two nonempty compact convex subsets of locally convex Hausdorff topological vector spaces, respectively. Let $G_{i}: X \times Y \rightrightarrows \mathbb{R}$, for $i=$ $1,2,3,4,5$, be a set-valued mapping, such that both $\bigcup_{y \in Y} G_{1}(x, y)$ and $\bigcup_{x \in X} G_{5}(x, y)$ are nonempty and compact, and the values $\max G_{5}(x, y)$ and $G_{i}(x, y) \subset G_{i+1}(x, y)-$ $\mathbb{R}_{+}$exist for all $(x, y) \in X \times Y$ and for $i=1,2,3,4$. Finally, suppose that the following conditions are satisfied:
(i) $y \mapsto G_{2}(x, y)$ is above-properly $\mathbb{R}_{+}$-quasi-concave on $Y$ for each $x \in X$, whereas $x \mapsto G_{4}(x, y)$ is above-naturally $\mathbb{R}_{+}$-quasi-convex on $X$ for each $y \in Y$;
(ii) $(x, y) \mapsto G_{1}(x, y)$ is a lower semi-continuous mapping on $X \times Y, y \mapsto$ $G_{3}(x, y)$ is a lower semi-continuous mapping on $Y$ for each $x \in X ; x \mapsto$ $G_{4}(x, y)$ is lower semi-continuous on $X$ for each $y \in Y$, and $(x, y) \mapsto$ $G_{5}(x, y)$ is a lower semi-continuous mapping on $X \times Y$; and
(iii) for each $w \in Y$, there is an $x_{w} \in X$ such that

$$
\max G_{5}\left(x_{w}, w\right) \leq \max \bigcup_{y \in Y} \min \bigcup_{x \in X} G_{5}(x, y)
$$

Thus, the relationship (S-H) holds.
Proof. For any $\alpha, \beta \in \mathbb{R}$, where $\alpha>\beta . F_{1}: X \rightrightarrows Y$ is defined by

$$
\begin{equation*}
F_{1}(x)=\left\{y \in Y: \exists g \in G_{1}(x, y), g \geq \alpha\right\} \tag{2.2}
\end{equation*}
$$

for all $x \in X . F_{2}, F_{3}, F_{4}, F_{5}$ are similar, with the triples $\left(G_{1}, \geq, \alpha\right)$ inside the braces in (2.2) replaced by $\left(G_{2}, \geq, \alpha\right),\left(G_{3}, \geq, \alpha\right),\left(G_{4},>, \beta\right)$, and $\left(G_{5},>, \beta\right)$, respectively. We performed the following four steps to confirm that all conditions of Lemma B hold:

Step 1. $F_{i}(x) \subset F_{i+1}(x)$ is calculated for $i=1,2,3,4$ and for each $x \in X$.
Step 2. $F_{4}(\operatorname{co} A) \subset F_{5}(A)$ is determined for each finite subset $A$ of $X$ to be true, where $\operatorname{co} A$ indicates the convex hull of the set $A$;
Step 3. $F_{1}: X \rightrightarrows Y$ and $\left(F_{5}^{-1}\right)^{c}: Y \rightrightarrows X$ are demonstrated to be upper semicontinuous mappings; and
Step 4. $F_{2}(x)$ is confirmed to be the convex for each $x \in X$; both the $F_{3}(x)$ and $\left(F_{4}^{-1}\right)^{c}(y)$ sets are compact for each $x \in X$ and $y \in Y$.
Because we can use a similar method in Theorem 3.1[8] to determine that Steps 1 and 2 are valid, we omitted their proof.

Next, we claim that $F_{1}: X \rightrightarrows Y$ has a closed graph. Let the net $\left(x_{\nu}, y_{\nu}\right) \in$ $\operatorname{Graph}\left(F_{1}\right)$ with $\left(x_{\nu}, y_{\nu}\right) \rightarrow\left(x_{0}, y_{0}\right)$. Subsequently, $y_{\nu} \in F_{1}\left(x_{\nu}\right)$, and $g_{\nu} \in G_{1}\left(x_{\nu}, y_{\nu}\right)$ exists, such that $g_{\nu} \geq \alpha$. Because $(x, y) \mapsto G_{1}(x, y)$ is a lower semi-continuous mapping on $X \times Y$, for any $g_{0} \in G_{1}\left(x_{0}, y_{0}\right)$ there exists a subnet $g_{\nu_{k}} \in G_{1}\left(x_{\nu_{k}}, y_{\nu_{k}}\right)$ such that $g_{\nu_{k}} \rightarrow g_{0}$. Hence, $g_{0} \geq \alpha$ and $y_{0} \in F_{1}\left(x_{0}\right)$; therefore, the graph of $F_{1}$ is closed. Moreover, because $Y$ is compact, $F_{1}$ is an upper semi-continuous mapping with compact values. Similarly, $F_{5}^{c}$ is also closed mapping; therefore, both $F_{1}$ and $F_{5}^{c}$ are upper semi-continuous mappings, proving that Step 3 is valid.

For the final step, we determine that for any $x \in X$ and $y_{1}, y_{2} \in F_{2}(x)$, there exist $g_{1} \in G_{2}\left(x, y_{1}\right), g_{2} \in G_{2}\left(x, y_{2}\right)$ such that $g_{1} \geq \alpha$ and $g_{2} \geq \alpha$. On the basis of the above-properly $\mathbb{R}_{+-}$quasi-concavity of $G_{2}$ in $y$, for any $\lambda \in[0,1]$, either $G_{2}\left(x, y_{1}\right) \subset G_{2}\left(x, \lambda y_{1}+(1-\lambda) y_{2}\right)-\mathbb{R}_{+}$or $G_{2}\left(x, y_{2}\right) \subset G_{2}\left(x, \lambda y_{1}+(1-\lambda) y_{2}\right)-\mathbb{R}_{+}$. Therefore, there exist $g_{\lambda} \in G_{2}\left(x, \lambda y_{1}+(1-\lambda) y_{2}\right)$ and $r_{\lambda} \geq 0$, such that, either $g_{1}=g_{\lambda}-r_{\lambda}$ or $g_{2}=g_{\lambda}-r_{\lambda}$. This implies that $g_{\lambda} \geq \alpha$. Thus, $\lambda y_{1}+(1-\lambda) y_{2} \in F_{2}(x)$, and $F_{2}(x)$ is convex for all $x \in X$.

Let us claim that, for each $x \in X$, the set $F_{3}(x)$ is compact. Indeed, let the net $z_{\nu} \in F_{3}(x)$ where $z_{\nu} \rightarrow z_{0}$. Thus, there exists $g_{\nu} \in G_{3}\left(x, z_{\nu}\right)$ where $z_{\nu} \geq \alpha$. For any $g_{0} \in G_{3}\left(x, z_{0}\right)$, According to the lower semi-continuity of $G_{3}$ in $y$, there also exists a subnet $g_{\nu_{k}} \in G_{3}\left(x, z_{\nu_{k}}\right)$ such that $g_{\nu_{k}} \rightarrow g_{0}$. Hence, $g_{0} \geq \alpha$, and $z_{0} \in F_{3}(x)$. Therefore, the set $F_{3}(x)$ is closed, and hence is compact because $Y$ is compact. According to the lower semi-continuity of $G_{4}$ in $x$ and a similar argument, the set $\left(F_{4}^{-1}\right)^{c}(y)$ is compact for all $y \in Y$; this confirms the validity of the final step.

With Steps 1 to 4, we have proved that all of the Lemma B conditions hold. Thus, the conclusion of Lemma B is true. Consequently, there either exists an $x_{0} \in X$, such that $F_{1}\left(x_{0}\right)=\emptyset$, or a $y_{0} \in Y$, such that $\bigcap_{x \in X} F_{5}(x) \neq \emptyset$, for all $x \in X$. That
is, either there exists an $x_{0} \in X$, such that $G_{1}\left(x_{0}, y\right) \subset(-\infty, \alpha)$, for all $y \in Y$, or there exists a $y_{0} \in Y$, such that $G_{5}\left(x, y_{0}\right) \cap(\beta, \infty) \neq \emptyset$, for all $x \in X$.

For any $\alpha>\beta>\max \bigcup_{y \in Y} \min \bigcup_{x \in X} G_{5}(x, y)$ and $y_{0} \in Y$, by (iii), there exists an $x_{y_{0}} \in X$, such that $\max G_{5}\left(x_{y_{0}}, y_{0}\right)<\beta$. Then, $G_{5}\left(x_{y_{0}}, y_{0}\right) \cap(\beta, \infty)=\emptyset$. Therefore, there exists an $x_{0} \in X$ such that $G_{1}\left(x_{0}, y\right) \subset(-\infty, \alpha)$, for all $y \in Y$. This implies that

$$
\max \bigcup_{y \in Y} G_{1}\left(x_{0}, y\right) \leq \alpha,
$$

and hence,

$$
\min \bigcup_{x \in X} \max \bigcup_{y \in Y} G_{1}\left(x_{0}, y\right) \leq \alpha
$$

Thus, the relationship ( $S-H$ ) holds.
Here, we compare Theorem 2.4 with Theorem 3.1[8] in several aspects. First, all mapping we used are neither upper semi-continuous nor closed mapping. Second, rather than using above-naturally $\mathbb{R}_{+-}$-quasi-concavity, we adopt the more general above-properly $\mathbb{R}_{+}$-quasi-concavity condition. Third, the numbers of set-valued mappings are different. Furthermore, the methods for the proof in these theorems are quite different. For instance, Steps 3 and 4 provide the proof for Theorem 2.4. We illustrate our theorems through the following two examples.

Example 2. Let $X=Y=[0,1]$ and $C=\mathbb{R}_{+}$. The set-valued mappings $G_{i}$ : $X \times Y \rightrightarrows \mathbb{R}$, where $i=1,2,3,4,5$, are defined by

$$
G_{1}(x, y):=U(x, y),
$$

where $U$ is the same as in Example 1.

$$
\begin{aligned}
& G_{2}(x, y):: \begin{cases}\{t: t \in[0,2]\}, & y=0, \\
\{t: t \in[1,3.1]\}, & y=1, \\
\left\{t: t \in\left[1,3-(y-1)^{2}\right]\right\}, & 0<y<1 ;\end{cases} \\
& G_{3}(x, y):= \begin{cases}\{3.1\}, & x=0, \\
\{t: t \in[3.1-y, 3.1+y]\}, & 0<x \leq 1 ;\end{cases} \\
& G_{4}(x, y):= \begin{cases}\{t: t \in[4.1-y, 4.1+y]\}, & x=0, \\
\{4.1+y\}, & 0<x \leq 1 ;\end{cases} \\
& G_{5}(x, y):=\left[6,6+4 y^{2}\left(x-x^{2}\right]\right.
\end{aligned}
$$

for all $(x, y) \in X \times Y$. We review the concavity of $G_{2}$ herein but leave the review of $G_{4}$ to the readers. For any $y_{1}, y_{2} \in Y, y_{1}<y_{2}$ and $\lambda \in[0,1]$,
$G_{2}\left(x, \lambda y_{1}+(1-\lambda) y_{2}\right):= \begin{cases}G_{2}\left(x, y_{2}\right), & \lambda=0, \\ G_{2}\left(x, y_{1}\right), & \lambda=1, \\ \left\{t: t \in\left[1,3-\left(\lambda y_{1}+(1-\lambda) y_{2}-1\right)^{2}\right]\right\}, & 0<\lambda<1 .\end{cases}$
Then,

$$
G_{2}\left(x, \lambda y_{1}+(1-\lambda) y_{2}\right):
$$

$$
= \begin{cases}\left\{t: t \in\left[1,3-\left(y_{2}-1\right)^{2}\right]\right\}, & 0<y_{2}<1, \lambda=0 \\ \{t: t \in[1,3.1]\}, & y_{2}=1, \lambda=0 \\ \{t: t \in[0,2]\}, & y_{1}=0, \lambda=1 \\ \left\{t: t \in\left[1,3-\left(y_{1}-1\right)^{2}\right]\right\}, & 0<y_{1}<1, \lambda=1 \\ \left\{t: t \in\left[1,3-\left(\lambda y_{1}+(1-\lambda) y_{2}-1\right)^{2}\right]\right\}, & 0<\lambda<1\end{cases}
$$

For both cases of $0<y_{2}<1, \lambda=0$ and $y_{2}=1, \lambda=0, G_{2}\left(x, y_{2}\right)=G_{2}\left(x, \lambda y_{1}+\right.$ $\left.(1-\lambda) y_{2}\right)$. Hence, $G_{2}\left(x, y_{2}\right) \subset G_{2}\left(x, \lambda y_{1}+(1-\lambda) y_{2}\right)-\mathbb{R}_{+}$. For both cases of $y_{1}=0, \lambda=1$ and $0<y_{1}<1, \lambda=1, G_{2}\left(x, y_{1}\right)=G_{2}\left(x, \lambda y_{1}+(1-\lambda) y_{2}\right)$. Hence, $G_{2}\left(x, y_{1}\right) \subset G_{2}\left(x, \lambda y_{1}+(1-\lambda) y_{2}\right)-\mathbb{R}_{+}$. The final case is $0<\lambda<1$, where

$$
G_{2}\left(x, y_{1}\right):= \begin{cases}\{t: t \in[0,2]\}, & y_{1}=0 \\ \left\{t: t \in\left[1,3-\left(y_{1}-1\right)^{2}\right]\right\}, & 0<y_{1}<1\end{cases}
$$

Hence, $G_{2}\left(x, y_{1}\right) \subset G_{2}\left(x, \lambda y_{1}+(1-\lambda) y_{2}\right)-\mathbb{R}_{+}$. Therefore, the mapping of $y \mapsto$ $G_{2}(x, y)$ is above-properly $\mathbb{R}_{+}$-quasi-concave on $Y$ for each $x \in X$. This proves that condition (i) of Theorem 2.1 is valid.

We claim that the continuities of $G_{1}, G_{2}, G_{3}$, and $G_{5}$ satisfied condition (ii) of Theorem 2.1. Example 1 claimed that $(x, y) \mapsto G_{1}(x, y)$ is an upper semi-continuous mapping on $X \times Y$, and the other cases will leave the proofs to the readers. Therefore, the condition (ii) of Theorem 2.1 is true.

In addition, note that

$$
\begin{aligned}
\max G_{1}(x, y) & = \begin{cases}2, & x=0 \\
1, & 0<x \leq 1,\end{cases} \\
\max G_{2}(x, y) & = \begin{cases}3-(y-1)^{2}, & 0<y<1, \\
2, & y=0 \\
3.1, & y=1,\end{cases} \\
\max G_{3}(x, y) & = \begin{cases}3.1, & x=0 \\
3.1+y, & 0<x \leq 1,\end{cases} \\
\max G_{4}(x, y) & =4.1+y, \\
\max G_{5}(x, y) & =6+4 y^{2}\left(x-x^{2}\right), \\
\max \bigcup_{y \in Y} G_{1}(x, y) & = \begin{cases}2, & x=0 \\
1, & 0<x \leq 1,\end{cases} \\
\min \bigcup_{x \in X} G_{5}(x, y) & =6
\end{aligned}
$$

for all $(x, y) \in X \times Y$.
Condition (iii) of Theorem 2.1 is true if we take $x_{w}=1 / 3$ when $w=0$ and $x_{w}=1$ when $w \neq 0$. Then, all conditions of Theorem 2.1 are fulfilled. Thus, the relationship $(S-H)$ is valid, and $\min \bigcup_{x \in X} \max \bigcup_{y \in Y} G_{1}(x, y)=1 \leq 6=$ $\max \bigcup_{y \in Y} \min \bigcup_{x \in X} G_{5}(x, y)$.

Example 3. Let $X=Y=[0,1]$ and $C=\mathbb{R}_{+}$. The set-valued mappings $G_{i}$ : $X \times Y \rightrightarrows \mathbb{R}$, where $i=1,2,3,4,5$, are defined by

$$
G_{1}(x, y):=V(x, y),
$$

where $V$ is the same as in Example 1;

$$
\begin{aligned}
& G_{2}(x, y):= \begin{cases}\{t: t \in[0,1]\}, & y=0, \\
\{t: t \in[1,2.1]\}, & y=1, \\
\left\{t: t \in\left[1,2-(y-1)^{2}\right]\right\}, & 0<y<1 ;\end{cases} \\
& G_{3}(x, y):= \begin{cases}\{2.1\}, & y=1 / 2, \\
\{t: t \in[2.1,3.1]\}, & y \neq 1 / 2 ;\end{cases} \\
& G_{4}(x, y):= \begin{cases}\{3.1\}, & x=0, \\
\{t: t \in[2.1,3.1]\}, & 0<x \leq 1 ;\end{cases} \\
& G_{5}(x, y):= \begin{cases}\{3.1\}, & x=0, y=0, \\
\{t: t \in[3.1,3.1+y(1-x]\}, & \text { others }\end{cases}
\end{aligned}
$$

for all $(x, y) \in X \times Y$. Thus, we know that $G_{i}(x, y) \subset G_{i+1}(x, y)-\mathbb{R}_{+}$is valid for all $(x, y) \in X \times Y$ and for $i=1,2,3,4$; moreover, the lower semi-continuities of $G_{1}, G_{3}, G_{4}$, and $G_{5}$ are fulfilled. Indeed, Example 1 indicates that $G_{1}$ is a lower semi-continuous mapping on $X \times Y$. We can derive the other properties are valid by using similar methods; herein, we have omitted them and leave the proofs to the reader. Furthermore, we leave the readers to review whether the convexities of $G_{2}$ and $G_{4}$ are valid. Notably, these properties can be derived from definitions.

Note that

$$
\max G_{5}(x, y)=\left\{\begin{array}{lr}
3.1, & x=0, y=0 \\
3.1+y(1-x), & \text { otherwise }
\end{array}\right.
$$

exists for all $x \in X, y \in Y$. Both sets $\cup_{y \in Y} G_{1}(x, y)=[-1,1]$ and $\cup_{x \in X} G_{5}(x, y)=$ $[3.1,3.1+y]$ are compact for all $x \in X, y \in Y$. Finally, to determine whether condition (iii) of Theorem 2.4 is true. For any $w \in Y$, let us select

$$
x_{w}= \begin{cases}1, & w \neq 0 \\ 1 / 2, & w=0\end{cases}
$$

Then, $\max G_{5}\left(x_{w}, w\right)=3.1$; hence, condition (iii) is valid. In short, all conditions of Theorem 2.4 are fulfilled and the relation ( $S-H$ ) holds. Indeed, $\min \bigcup_{x \in X} \max \bigcup_{y \in Y} G_{1}(x, y)=0 \leq 3.1=\max \bigcup_{y \in Y} \min \bigcup_{x \in X} G_{5}(x, y)$.

## 3. Hierarchical structures for set-valued mappings

In this section, we propose some hierarchical structures of minimax theorems in locally convex topological vector spaces. Why and how such structures exist has been discussed and developed previously $[5,3,7,4,6,8]$, and we leave their proofs to the reader.

First, let $\mathbb{W}$ be a Hausdorff topological vector space, $C \subset \mathbb{W}$ be a closed convex and pointed cone with its apex at the origin and $\operatorname{int} C \neq \emptyset$. Let $C^{*}:=\left\{g \in \mathbb{W}^{*}\right.$ : $g(c) \geq 0$ for all $c \in C\}$, where $\mathbb{W}^{*}$ is the set of all continuous linear functionals on $\mathbb{W}$. We also apply the notations $\operatorname{Max} \Lambda, \operatorname{Min} \Lambda, \operatorname{Max}_{w} \Lambda$, and $\operatorname{Min}_{w} \Lambda$ of a nonempty set
$\Lambda$ in $\mathbb{W}$, which represent the set of maximal points, minimal points, weakly maximal points, and weakly minimal points of $\Lambda[5]$, respectively. The notation " $\mathcal{A} \preceq \mathcal{B}$ " $[6]$ indicates

$$
\operatorname{Max}_{w} \mathcal{A} \subset \operatorname{Max}_{w} \mathcal{B}-C
$$

for two nonempty sets $\mathcal{A}$ and $\mathcal{B}$ in $\mathbb{W}$. In the rest of this section, we describe three versions of hierarchical minimax theorems. The first one is concerned about the relationship $\left(H_{1}\right)$ is as follows:

Theorem 3.1. Let $X, Y$ be two nonempty compact convex subsets of locally convex Hausdorff topological vector spaces, respectively. Moreover, let $\mathbb{W}$ be a complete locally convex Hausdorff topological vector space. Let $F_{i}: X \times Y \rightrightarrows \mathbb{W}$, for $i=1,2,3,4,5$, be set-valued mappings where $\operatorname{Max}_{w} F_{i}(x, y)$ is nonempty for each $(x, y) \in X \times Y$, such that $F_{i}(x, y) \preceq F_{i+1}(x, y)$, both values, $\max \xi F_{i}(x, y)$, and $\max \bigcup_{x \in X} \xi F_{5}(x, y)$ exist for all $(x, y) \in X \times Y$, for any $\xi \in C^{\star}$, and for $i=1,2,3,4$. Finally, suppose that the following conditions are satisfied:
(i) either $y \mapsto F_{1}(x, y)$ or $y \mapsto F_{2}(x, y)$ is above-properly $C$-quasi-concave on $Y$ for each $x \in X$, whereas $x \mapsto F_{4}(x, y)$ is above-naturally $C$-quasi-convex on $X$ for each $y \in Y$;
(ii) $(x, y) \mapsto F_{1}(x, y)$ is an upper semi-continuous mapping on $X \times Y$ and $y \mapsto F_{2}(x, y)$ is an upper semi-continuous mapping on $Y$ for each $x \in X$, meanwhile, $x \mapsto F_{1}(x, y)$ and $x \mapsto F_{3}(x, y)$ are lower semi-continuous mappings on $X$ for each $y \in Y$, and $(x, y) \mapsto F_{5}(x, y)$ is continuous on $X \times Y$;
(iii) for any $\xi \in C^{\star}$ and for each $w \in Y$, there is an $x_{w} \in X$ such that

$$
\max \xi F_{5}\left(x_{w}, w\right) \leq \max \bigcup_{y \in Y} \min \bigcup_{x \in X} \xi F_{5}(x, y) ; \text { and }
$$

(iv) for each $y \in Y$,

$$
\operatorname{Max} \bigcup_{y \in Y} \operatorname{Min}_{w} \bigcup_{x \in X} F_{5}(x, y) \subset \operatorname{Min}_{w} \bigcup_{x \in X} F_{5}(x, y)+C
$$

Thus, the following relationship holds:
$\left(H_{1}\right) \quad \operatorname{Minco}\left(\bigcup_{x \in X} \operatorname{Max}_{w} \bigcup_{y \in Y} F_{1}(x, y)\right) \subset \operatorname{Max} \bigcup_{y \in Y} \operatorname{Min}_{w} \bigcup_{x \in X} F_{5}(x, y)+C$.
Notably, the relationship between Theorem 3.1 and Propositions 3.12-3.13[5] is crucial, because these propositions facilitate understanding the relationship that $\left(H_{1}\right)$ holds. The second relationship $\left(H_{2}\right)$ is as follows:

Theorem 3.2. Let $X, Y$ be nonempty compact convex subsets of locally convex Hausdorff topological vector spaces, respectively. Let $\mathbb{W}$ be a Hausdorff topological vector space. Under the framework of Theorem 3.1, excluding condition (iii), and any Gerstewitz function $\varphi_{k w}[5]$ and for each $y \in Y$, there is an $x_{y} \in X$, such that

$$
\begin{equation*}
\max \varphi_{k w} F_{5}\left(x_{y}, y\right) \leq \max \bigcup_{y \in Y} \min \bigcup_{x \in X} \varphi_{k w} F_{5}(x, y) \tag{iii'}
\end{equation*}
$$

Thus, the following relationship is valid:

$$
\begin{equation*}
\operatorname{Min} \bigcup_{x \in X} \operatorname{Max}_{w} \bigcup_{y \in Y} F_{1}(x, y) \subset \operatorname{Max} \bigcup_{y \in Y} \operatorname{Min}_{w} \bigcup_{x \in X} F_{5}(x, y)+C \tag{2}
\end{equation*}
$$

We encourage readers to review Proposition 3.9[5] and Remark 3.15[5] to further understand the reason that relationship $\left(H_{2}\right)$ is valid in Theorem 3.2. The third relationship $\left(\mathrm{H}_{3}\right)$ is as follows:

Theorem 3.3. Under the framework of Theorem 3.2, excluding condition (iv), the following equation is valid:
$\left(H_{3}\right) \operatorname{Max} \bigcup_{y \in Y} \operatorname{Min}_{w} \bigcup_{x \in X} F_{5}(x, y) \subset \operatorname{Min} \bigcup_{x \in X} \operatorname{Max}_{w} \bigcup_{y \in Y} F_{1}(x, y)+\mathbb{W} \backslash(C \backslash\{0\})$.
The results of Theorems 3.1-3.3 are closely related to Theorem 2.1. By contrast, the results of Theorems 3.4-3.6 are highly dependent on Theorem 2.4.

Theorem 3.4. Let $X, Y$ be two nonempty compact convex subsets of locally convex Hausdorff topological vector spaces, respectively. Let $\mathbb{W}$ be a complete locally convex Hausdorff topological vector space. Moreover, let $F_{i}: X \times Y \rightrightarrows \mathbb{W}$, for $i=1,2,3,4,5$, be set-valued mappings where the value $\max \xi F_{5}(x, y)$ exists for each $(x, y) \in X \times Y$, such that both $\bigcup_{y \in Y} F_{1}(x, y)$ and $\bigcup_{x \in X} F_{5}(x, y)$ are nonempty compact sets, and $F_{i}(x, y) \preceq F_{i+1}(x, y)$ for all $(x, y) \in X \times Y$ and for $i=1,2,3,4$. Finally, suppose that the following conditions are satisfied:
(i) $y \mapsto F_{2}(x, y)$ is above-properly $C$-quasi-concave on $Y$ for each $x \in X$, whereas $x \mapsto F_{4}(x, y)$ is above-naturally $C$-quasi-convex on $X$ for each $y \in Y$;
(ii) $(x, y) \mapsto F_{1}(x, y)$ and $(x, y) \mapsto F_{5}(x, y)$ are lower semi-continuous mappings on $X \times Y, y \mapsto F_{3}(x, y)$ is a lower semi-continuous mapping on $Y$ for each $x \in X$, an $x \mapsto F_{4}(x, y)$ is a lower semi-continuous mapping on $X$ for each $y \in Y$;
(iii) for any $\xi \in C^{\star}$ and for each $w \in Y$, there is an $x_{w} \in X$ such that

$$
\max \xi F_{5}\left(x_{w}, w\right) \leq \max \bigcup_{y \in Y} \min \bigcup_{x \in X} \xi F_{5}(x, y) ; \text { and }
$$

(iv) for each $y \in Y$,

$$
\operatorname{Max} \bigcup_{y \in Y} \operatorname{Min}_{w} \bigcup_{x \in X} F_{5}(x, y) \subset \operatorname{Min}_{w} \bigcup_{x \in X} F_{5}(x, y)+C
$$

Thus, the relationship $\left(H_{1}\right)$ holds.
Theorem 3.5. Let $X, Y$ be nonempty compact convex subsets of locally convex Hausdorff topological spaces, respectively. Let $\mathbb{W}$ be a Hausdorff topological vector space. Under the framework of Theorem 3.4, excluding condition (iii), and any Gerstewitz function $\varphi_{k w}$ and for each $y \in Y$, there is an $x_{y} \in X$ such that
(iii")

$$
\max \varphi_{k w} F_{5}\left(x_{y}, y\right) \leq \max \bigcup_{y \in Y} \min \bigcup_{x \in X} \varphi_{k w} F_{5}(x, y)
$$

Thus, the relationship $\left(\mathrm{H}_{2}\right)$ is valid.
Theorem 3.6. Under the framework of Theorem 3.5, excluding condition (iv), equation $\left(H_{3}\right)$ is valid.

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