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A DIFFERENTIAL EQUATION METHOD FOR SOLVING THE VARIATIONAL INEQUALITY PROBLEM WITH THE CYCLICALLY MONOTONE MAPPING

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Abstract. This paper presents a system of differential equations based on the projection operator for solving the variational inequality problem with the cyclically monotone mapping. Using an important inequality for the cyclically monotone mapping, we prove that any accumulation point of the trajectory of the differential equation system is a solution to the variational inequality problem. Numerical experiments are reported to verify the effectiveness of the differential equation approach for solving nonlinear convex programming problems whose KKT mappings are cyclically monotone.

1. Introduction

We consider the variational inequality problem, denoted by $\mathrm{VIP}(\Omega, F)$, which is to find a vector $x^* \in \Omega$ such that

$$\langle F(x^*), y - x^* \rangle \geq 0, \quad \forall\, y \in \Omega, \tag{1.1}$$

where $F : \Re^n \to \Re^n$ is a cyclically monotone mapping, Ω is a closed convex subset of $\Re^n$.

Various numerical methods have been explored for solving variational inequality problems, among them the Euclidean projector is undoubtedly one of the most fundamental and useful mathematical tools. Some classical results using projection operator for solving variational inequality problems can be found in [3]. It is worth noting that Bingsheng He and his collaborators have studied convergence properties of various numerical algorithms for VIs by using properties of the projection operator, for examples [6] and [7].

Applications of the differential equation methods to the nonlinear complementarity problems and the variational inequality problems can be found in [1], [2], [4], [5] and [9]. Based on the projection characterization of solutions to variational inequality problems, Gao, Liao and Qi [4] presented a differential equation system for solving variational inequality problems with linear and nonlinear constraints. Based on a simple projection and contraction method, He and Yang [5] gave a differential equation system for solving asymmetric linear variational inequalities in which the positive semidefiniteness of the asymmetric matrix is assumed. Based on an

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unconstrained reformulation, Liao, Qi and Qi [9] established a differential equation system for solving nonlinear complementarity problems.

In this paper, based on the projection operator, we establish a system of differential equations for solving the variational inequality problem (1.1). An important inequality for the cyclically monotone mapping is established. By using this inequality, we prove that the accumulation points of the trajectory of the differential equation system are the solutions to the variational inequality problem (1.1). Note that the KKT system of a convex optimization problem consists of a cyclically monotone mapping, we know that the method is applicable to convex programming.

The paper is organized as follows. The next section presents the system of differential equations based on the projection operator, and proves an important inequality and the convergence theorem for the differential equation approach. Section 3 reports the numerical experiments performed for nonlinear convex programming problems whose KKT mappings are cyclically monotone. The transient behaviors of the trajectories of the differential equation system for solving these problems are illustrated.

2. A differential equation system

The projection operator to a convex set is quite useful in reformulating the variational inequality (1.1) as an equation. Let C be a convex closed set, for every $x \in \Re^n$, there is a unique $\hat{x}$ in C such that

$$\|x - \hat{x}\| = \min\{\|x - y\| \mid y \in C\}.$$

The point $\hat{x}$ is the projection of x onto C, denoted by $\Pi_C(x)$. The projection operator $\Pi_C : \Re^n \to C$ is well defined over $\Re^n$ and it is a nonexpensive mapping.

Lemma 2.1 ([11]). *Let H be a real Hilbert space and $C \subset H$ be a closed convex set. For a given $z \in H$, $u \in C$ satisfies the inequality*

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in C,$$

if and only if $u - \Pi_C(z) = 0$.

Therefore, from Lemma 2.1, the variational inequality problem (1.1) is equivalent to finding a root of the following equation:

$$\Phi_\mu(x) := \Pi_\Omega(x - \mu F(x)) - x = 0, \tag{2.1}$$

where $\mu > 0$ and $\Pi_\Omega(\cdot)$ is the operator projection a vector onto the set Ω.

Next we recall the definition of cyclically monotone mapping.

Definition 2.2 ([8]). A mapping $T : \Re^n \to \Re^n$ is called cyclically monotone if one has

$$\langle x_1 - x_0, T(x_0)\rangle + \langle x_2 - x_1, T(x_1)\rangle + \cdots + \langle x_0 - x_m, T(x_m)\rangle \leq 0$$

for any set of points $\{x_0, x_1, \cdots, x_m\} \subset \Re^n$.

It is well known that $f(x)$ is continuously differentiable and convex if and only if $F(x) = \nabla f(x)$ is cyclically monotone.

The following inequality plays an important role in demonstrating the convergence theorem of the trajectory of the differential equation system.

Lemma 2.3. *Suppose that $F : \Re^n \to \Re^n$ is cyclically monotone and Lipschitz continuous with the constant L. Then the following inequality holds:*

$$\langle F(x_1) - F(x_3), x_2 - x_1\rangle \le 2L\|x_3 - x_2\|^2 \tag{2.2}$$

for any $x_1, x_2, x_3 \in \Re^n$.

Proof. Since F is a cyclically monotone mapping, we have for any $x_1, x_2, x_3 \in \Re^n$ that

$$\langle x_2 - x_1, F(x_1)\rangle + \langle x_3 - x_2, F(x_2)\rangle + \langle x_1 - x_3, F(x_3)\rangle \le 0 \tag{2.3}$$

and

$$\langle x_3 - x_2, F(x_2) - F(x_3)\rangle \le 0. \tag{2.4}$$

Summing (2.3) and (2.4), we get that

$$\langle x_2 - x_1, F(x_1)\rangle + \langle x_3 - x_2, F(x_2)\rangle + \langle x_1 - x_3, F(x_3)\rangle + \langle x_3 - x_2, F(x_2) - F(x_3)\rangle \le 0,$$

which means that

$$\begin{aligned}\langle x_2 - x_1, F(x_1)\rangle + \langle x_3 - x_2, F(x_2)\rangle + \langle x_1 - x_2, F(x_3)\rangle \\ +\langle x_2 - x_3, F(x_3)\rangle + \langle x_3 - x_2, F(x_2) - F(x_3)\rangle \le 0.\end{aligned}$$

From the above inequality, we obtain that

$$\langle x_2 - x_1, F(x_1) - F(x_3)\rangle + 2\langle x_3 - x_2, F(x_2) - F(x_3)\rangle \le 0. \tag{2.5}$$

Note that F is Lipschitz continuous with the constant L. It follows from (2.5) that

$$\begin{aligned}\langle x_2 - x_1, F(x_1) - F(x_3)\rangle &\le 2\langle x_2 - x_3, F(x_2) - F(x_3)\rangle \\ &\le 2\|x_2 - x_3\|\|F(x_2) - F(x_3)\| \\ &\le 2L\|x_2 - x_3\|^2.\end{aligned}$$

This completes the proof. □

Now we construct a differential equation system based on the equation (2.1) as follows.

$$\frac{dx}{dt} + x = \Pi_\Omega(x - \mu F(x)). \tag{2.6}$$

Note that Π_Ω is Lipschitz continuous when F is Lipschitz continuous, the existence and uniqueness of solutions for the differential equation system (2.6) can be easily obtained from [13].

For simplicity, we denote $\dot{x} = \frac{dx}{dt}$. According to Lemma 2.1, the differential equation system (2.6) can be equivalent to the variational inequality

$$\langle \dot{x} + \mu F(x), y - \dot{x} - x\rangle \ge 0, \quad \forall y \in \Omega. \tag{2.7}$$

Now we prove the convergence theorem of the trajectory $x(t)$ of the differential equation system (2.6).

Theorem 2.4. *Assume that the set of solutions to problem (1.1) is not empty, and F is cyclically monotone and Lipschitz continuous with the constant L. Then for any $x_0 \in \Re^n$ and $0 < \mu < \frac{1}{2L}$, the accumulation point of the trajectory $x(t)$ of the differential equation system (2.6) is a solution to the variational inequality problem (1.1).*

Proof. Let x^* be one of the solutions to problem (1.1). Setting $y = x^*$ in (2.7) and $y = x + \dot{x}$ in (1.1), we have

$$\langle \dot{x} + \mu F(x), x^* - \dot{x} - x \rangle \geq 0$$

and

$$\langle F(x^*), \dot{x} + x - x^* \rangle \geq 0.$$

We sum the above two inequalities to obtain the following

$$\langle \dot{x} + \mu(F(x) - F(x^*)), x^* - \dot{x} - x \rangle \geq 0. \tag{2.8}$$

Using the inequality (2.2), we have

$$\langle F(x) - F(x^*), x^* - x - \dot{x} \rangle \leq 2L\|\dot{x}\|^2. \tag{2.9}$$

Submitting (2.9) into (2.8), we get that

$$\langle \dot{x}, x - x^* + \dot{x} \rangle - 2\mu L\|\dot{x}\|^2 \leq 0,$$

which implies that

$$\frac{1}{2}\frac{d}{dt}\|x - x^*\|^2 + (1 - 2\mu L)\|\dot{x}\|^2 \leq 0. \tag{2.10}$$

It follows from $0 < \mu < \frac{1}{2L}$ that $1 - 2\mu L > 0$. By integrating inequality (2.10) over the range $[t_0, t]$, we obtain that

$$\frac{1}{2}\|x - x^*\|^2 + (1 - 2\mu L)\int_{t_0}^{t} \|\dot{x}\|^2 d\tau \leq \frac{1}{2}\|x_0 - x^*\|^2, \tag{2.11}$$

where $x_0 = x(t_0)$. It follows from (2.11) that

$$\|x(t) - x^*\|^2 \leq \|x_0 - x^*\|^2 \tag{2.12}$$

and

$$\int_{t_0}^{t} \|\dot{x}\|^2 d\tau < \infty, \quad t \to \infty. \tag{2.13}$$

The inequality (2.13) shows that there exists a subsequence $\{t_i\}$ such that $\|\dot{x}(t_i)\| \to 0, i \to \infty$. Otherwise, we assume that there exists an $\varepsilon > 0$ such that $\|\dot{x}\| \geq \varepsilon$ for all $t \geq t_0$, then we obtain a contradiction with (2.13).

The inequality (2.12) shows that $x(t)$ is bounded, hence the subsequence $\{x(t_i)\}$ is also bounded. We can choose a subsequence $\{x(t_{i_j})\}$ of the sequence $\{x(t_i)\}$ such that there exists $x' \in \Re^n$ and $x(t_{i_j}) \to x', j \to \infty$ and $\|\dot{x}(t_{i_j})\| \to 0, j \to \infty$.

Let us consider the inequality (2.7) for $x(t_{i_j})$ and take the limit as $j \to \infty$, then we have

$$\langle F(x'), y - x' \rangle \geq 0, \quad \forall y \in \Omega,$$

which implies that x' is the solution of the variational inequality problem (1.1). This completes the proof. □

Remark 2.5. Under the certain condition, we can prove that the convergent subsequence $\{x(t_{i_j})\}$ of the trajectory $x(t)$ of the differential equation system (2.6) is exponential stable [10].

In fact, if there exist $\alpha > 0, \omega > 0$ and $c > 0$ such that

$$\|\dot{x}(t_{i_j})\|^2 \geq \frac{\alpha\|x_0 - x^*\|^2}{t_{i_j} - t_0}, \tag{2.14}$$

and

$$0 < \mu \leq \frac{2\alpha - [1 - (ce^{-\omega t_{i_j}})^2]}{4L\alpha}. \tag{2.15}$$

From the inequality (2.11), we have

$$\frac{1}{2}\|x(t_{i_j}) - x^*\|^2 + (1 - 2\mu L)\int_{t_0}^{t_{i_j}} \|\dot{x}(t_{i_j})\|^2 d\tau \leq \frac{1}{2}\|x_0 - x^*\|^2, \tag{2.16}$$

where $x_0 = x(t_0)$.

The inequality (2.14) means that

$$(1 - 2\mu L)\int_{t_0}^{t_{i_j}} \|\dot{x}(t_{i_j})\|^2 d\tau \geq (1 - 2\mu L)\alpha\|x_0 - x^*\|^2.$$

Submitting the above inequality into (2.16), we get that

$$\|x(t_{i_j}) - x^*\|^2 \leq [1 - 2(1 - 2\mu L)\alpha]\|x_0 - x^*\|^2, \tag{2.17}$$

From the inequality (2.15) and (2.17), we deduce that

$$\|x(t_{i_j}) - x^*\| \leq ce^{-\omega t_{i_j}}\|x_0 - x^*\|,$$

which shows that the convergent subsequent $\{x(t_{i_j})\}$ of the trajectory $x(t)$ of the differential equation is exponential stable. Hence the conclusion of Theorem 2.4 has guaranteed the stability of the convergent subsequence.

3. Numerical results

In this section, we test four examples by our differential equation system (2.6). For each test problem, we also compare the numerical performance of the proposed differential equation system (2.6) from various initial points. The numerical implementation is coded by Matlab 7.8 running on a PC Intel Pentium IV of 2.93 GHz CPU and the ordinary differential equation solver adopted is ode45, which uses an Runge-Kutta (4,5) formula. The parameter is chosen as $\mu = 0.03$ in all examples.

Example 3.1 Consider the variational inequality problem

$$\langle Dx + b, y - x\rangle \geq 0, \quad \forall y \in [-5, 5]^4, \tag{3.1}$$

where $D = \begin{pmatrix} 4 & 2 & 2 & 1 \\ 2 & 4 & 0 & 1 \\ 2 & 0 & 2 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$ and $b = (-8, -6, -4, 3)^T$. Its solution is $x^* = (-5/6, 19/6, 5, -5)^T$.

Figure 1 describes the convergence behaviors of the trajectory $x(t)$ of the differential equation system (2.6) with nine random initial points converging to its solution $(-0.8333, 3.1666,\ 4.9998,\ -5.0000)^T$.

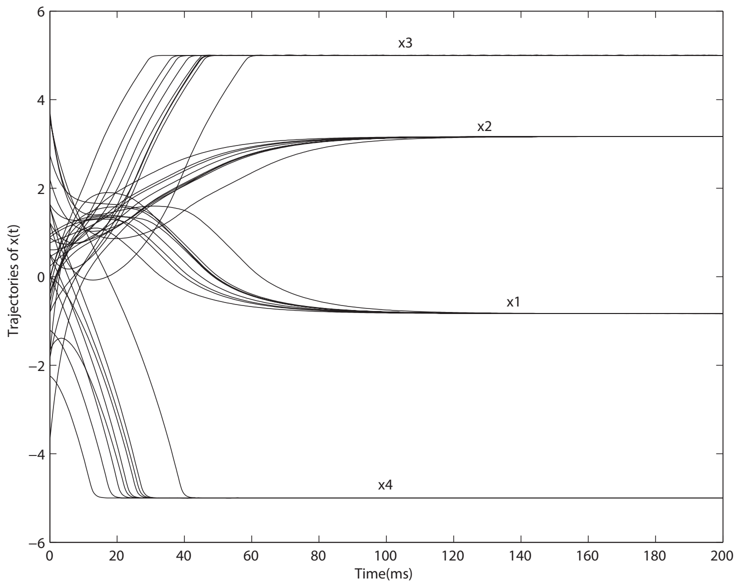

FIGURE 1. Transient behavior of $x(t)$ of the differential equation system (2.6) in Example 3.1 with nine random initial points

Example 3.2 Consider the following variational inequality problem

$$\langle F(x), y - x\rangle \geq 0, \quad \forall y \in \Omega, \tag{3.2}$$

where $F(x) = \begin{pmatrix} 3x_1^3 - 8 \\ x_2 - x_3 + x_2^3 + 3 \\ -x_2 + x_3 + 2x_3^3 - 3 \\ x_4 + 2x_4^3 \end{pmatrix}$, $\Omega = \{x \in \Re^4 \mid a \leq x \leq b\}$, $a = \{-1, 0, -2, -8\}^T$ and $b = \{6, 3, 5, 0\}^T$. Its solution is $x^* = (2, 0, 1, 0)^T$.

The solution trajectory $x(t)$ of the differential equation system (2.6) converging to $(2.0000, 0.0000,\ 1.0000,\ 0.0000)^T$ for the variational inequality problem (3.2) with six random initial points are illustrated in Figure 2.

Example 3.3 Consider the nonlinear convex programming problem

$$\min f(x) \quad \text{s.t.} \quad x \in \Re^5_+,$$

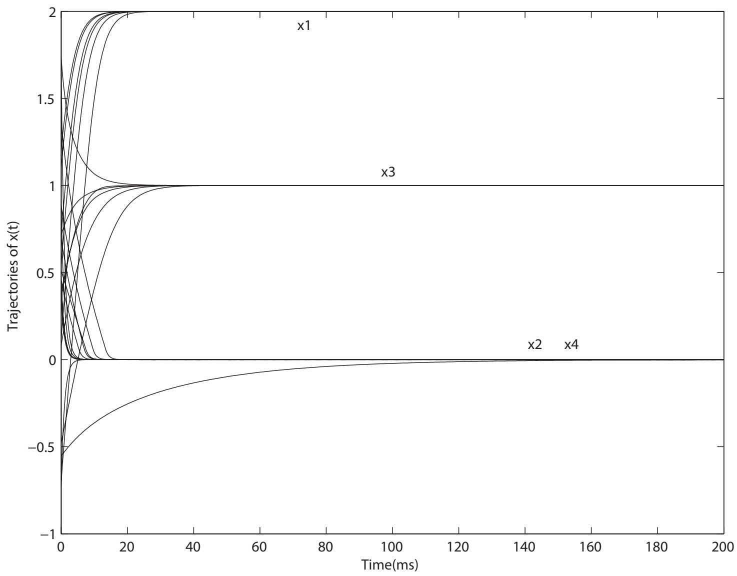

FIGURE 2. Transient behavior of $x(t)$ of the differential equation system (2.6) in Example 3.2 with six random initial points

where $f(x) = x_1^2/2 + x_2^2/2 + x_3^2/2 + x_4^2/2 + x_5^2/2 + x_1x_2x_3x_4x_5/50 - 3x_2 - x_3 + x_4/2$. Its optimal solution is $x^* = (0, 3, 1, 0, 0)^T$.

This problem can be transformed into the following nonlinear complementarity problem

$$F(x) = \begin{pmatrix} x_1 + x_2x_3x_4x_5/50 \\ x_2 + x_1x_3x_4x_5/50 - 3 \\ x_3 + x_1x_2x_4x_5/50 - 1 \\ x_4 + x_1x_2x_3x_5/50 + 1/2 \\ x_5 + x_1x_2x_3x_4/50 \end{pmatrix}, \tag{3.3}$$

which have been considered in [12].

For the complementarity problem (3.3), Figure 3 describes the convergence behaviors of the trajectory $x(t)$ of the differential equation system (2.6) with eight random initial points converging to its solution $(0.0000, 3.0000, 1.0000, 0.0000, 0.0000)^T$.

Example 3.4 Consider the nonlinear convex programming problem

$$\min \exp\Big(\sum_{i=1}^{5}(x_i - i + 2)^2\Big) \quad \text{s.t.} \quad x \in \mathcal{K}^5,$$

where $\mathcal{K}^5 := \{x \in \Re^5 \mid x_n \geq \|x^t\|\}$, $x^t = (x_1, x_2, \cdots, x_{n-1})$ and $\|\cdot\|$ stands for the Euclidean norm. Its optimal solution is $x^* = (-1, 0, 1, 2, 3)^T$.

This problem can be transformed into the following variational inequality

$$\langle F(x), y - x\rangle \geq 0, \quad \forall y \in \mathcal{K}^5, \tag{3.4}$$

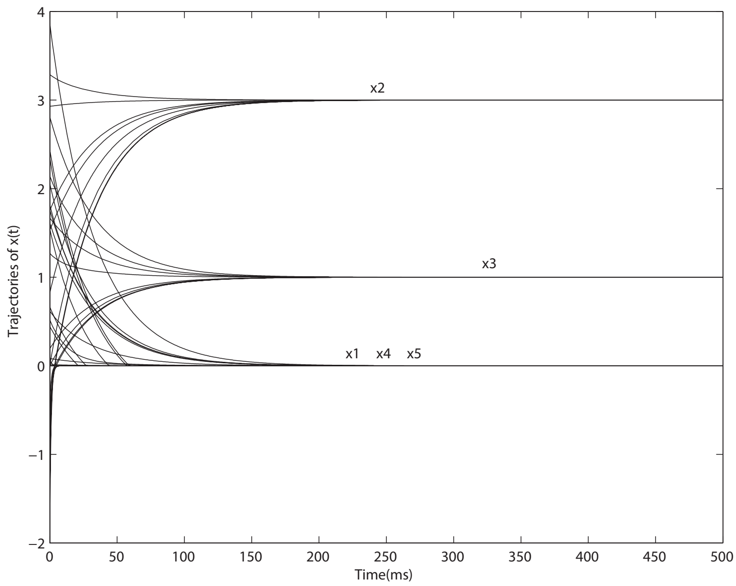

FIGURE 3. Transient behavior of $x(t)$ of the differential equation system (2.6) in Example 3.3 with eight random initial points

where $F(x) = 2\exp(\sum_{i=1}^{5}(x_i - i + 2)^2)\begin{pmatrix} x_1+1 \\ x_2 \\ x_3-1 \\ x_4-2 \\ x_5-3 \end{pmatrix}$.

The solution trajectory $x(t)$ of the differential equation system (2.6) converging to $(-1.0000,\ -0.0000,\ 1.0000,\ 1.9999, 2.9999)^T$ for the variational inequality problem (3.4) with five random initial points are showed in Figure 4.

4. Conclusions

In this paper, we establish a system of differential equations based on the projection operator for solving the variational inequality problems with cyclically monotone mappings. An important inequality is proved for the cyclically monotone mapping based on which it proved that the accumulate points of the trajectory of the differential equation system are the solutions to the variational inequality problem.

However, we have a problem worth studying whether the method converges when the mapping F is only monotone rather than cyclically monotone.

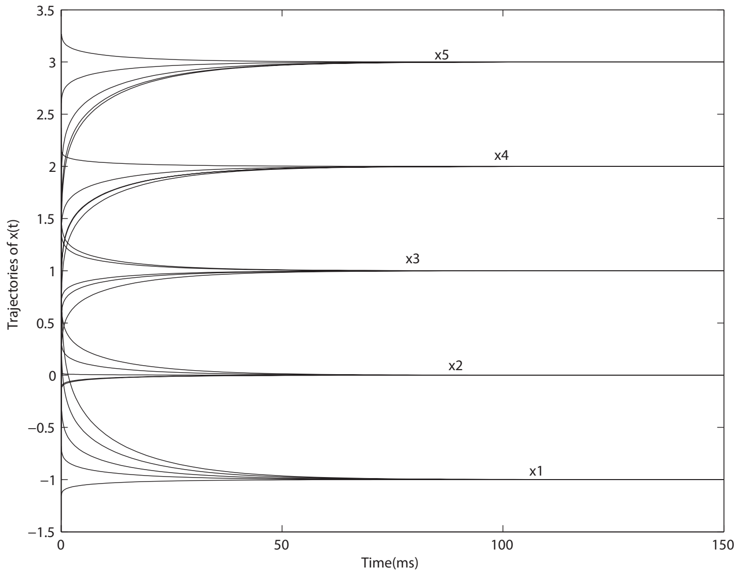

Figure 4. Transient behavior of $x(t)$ of the differential equation system (2.6) in Example 3.4 with five random initial points

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