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SOME NOTES ON TWO FIXED POINT THEOREMS OF ĆIRIĆ TYPE

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Abstract. In this paper, we discuss two fixed point theorems of Ćirić type recently proved by Kumam *et al.* [A generalization of Ćirić fixed point theorems, Filomat, 29, 1549–1556 (2015)]. We show that their first result can be regarded a direct consequence of Walter's fixed point theorem [Remarks on a paper by F. Browder about contraction, Nonlinear Anal., 5, 21–25 (1981)]. Moreover, we establish an appropriate multivalued version of our theorem and deduced their second theorem with a weak assumption.

1. Introduction

Suppose that $X := (X, d)$ is a complete metric space and $\mathrm{BN}(X)$ is the set of all nonempty bounded subsets of X. For $A, B \in \mathrm{BN}(X)$, we define

$$D(A,B) := \inf\{d(a,b) : a \in A, b \in B\}$$
$$\rho(A,B) := \sup\{d(a,b) : a \in A, b \in B\}.$$

For $x \in X$, we also write $D(x, A) := D(\{x\}, A)$ and $\rho(x, A) := \rho(\{x\}, A)$.

Ćirić [1] proved the following two famous fixed point theorems.

Theorem 1.1. *Suppose that $q \in (0,1)$ and $T : X \to X$ satisfies the following condition:*

$$d(Tx,Ty) \le q \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$$

for all $x, y \in X$. Then the following statements are true:

(a) *T has a unique fixed point, that is, there exists a unique element $u \in X$ such that $u = Tu$.*
(b) *$d(T^n x, u) \le \frac{q^n}{1-q} d(x, Tx)$ for all $x \in X$ and $n \ge 1$. In particular, $\lim_{n\to\infty} d(T^n x, u) = 0$.*

Theorem 1.2. *Suppose that $q \in (0,1)$ and $F : X \to \mathrm{BN}(X)$ satisfies the following condition:*

$$\rho(Fx,Fy) \le q \max\{d(x,y), \rho(x,Fx), \rho(y,Fy), D(x,Fy), D(y,Fx)\}$$

for all $x, y \in X$. Then the following statements are true:

(a) *F has a unique fixed point, that is, there exists a unique element $u \in X$ such that $\{u\} = Fu$.*

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(b) *For each* $x \in X$*, there exists a sequence* $\{x_n\}$ *in* X *such that* $x_0 := x$, $x_{n+1} \in Fx_n$ *and* $d(x_n, u) \le \frac{q^{(1-a)n}}{1-q^{1-a}} d(x_0, x_1)$ *for all* $n \ge 0$*, where* $a \in (0,1)$*. In particular,* $\lim_{n\to\infty} d(x_n, u) = 0$.

Recently, Kumam *et al.* [2] proposed the following two results which generalize Theorems 1.1 and 1.2, respectively.

Theorem 1.3. *Suppose that* $q \in (0,1)$ *and* $T : X \to X$ *satisfies the following condition:*

$$d(Tx,Ty) \le q \max \left\{ \begin{array}{c} d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx), \\ d(x,T^2x), d(y,T^2x), d(Tx,T^2x), d(Ty,T^2x) \end{array} \right\}$$

for all $x, y \in X$*. Then the following statements are true:*

(a) T *has a unique fixed point* u.

(b) $d(T^n x, u) \le \frac{q^n}{1-q} d(x, Tx)$ *for all* $x \in X$ *and* $n \ge 1$*. In particular,* $\lim_{n\to\infty} d(T^n x, u) = 0$.

Theorem 1.4. *Suppose that* $q \in (0,1)$ *and* $F : X \to \mathrm{BN}(X)$ *satisfies the following condition:*

$$\rho(Fx,Fy) \le q \max \left\{ \begin{array}{c} d(x,y), \rho(x,Fx), \rho(y,Fy), D(x,Fy), D(y,Fx), \\ D(x,F^2x), D(y,F^2x), D(Fx,F^2x), D(Fy,F^2x) \end{array} \right\}$$

for all $x, y \in X$*. Here* $F^2x := \bigcup_{w\in Fx} Fw$*. Then the following statements are true:*

(a) F *has a unique fixed point* u.

(b) *For each* $x \in X$ *and* $a \in (0,1)$*, there exists a sequence* $\{x_n\}$ *such that* $x_0 := x$*,* $x_{n+1} \in Fx_n$ *and* $d(x_n, u) \le \frac{q^{(1-a)n}}{1-q^{1-a}} d(x_0, x_1)$ *for all* $n \ge 0$*. In particular,* $\lim_{n\to\infty} d(x_n, u) = 0$.

The purpose of the paper is to show that (1) Theorem 1.3 is not only a direct consequence of the result of Walter (Theorem 1.5 below) in 1981 [3] but also established under a weaker assumption; and (2) Theorem 1.4 can be established as a consequence of a multivalued version of our theorem.

To state Walter's Theorem, we recall the following notation: For $T : X \to X$ and $x, y \in X$, we write $\mathcal{O}(x) := \{x, Tx, T^2x, \dots\}$ and $\mathcal{O}(x,y) := \mathcal{O}(x) \cup \mathcal{O}(y)$.

Theorem 1.5. *Suppose that* $\varphi : [0,\infty) \to [0,\infty)$ *is a nondecreasing and continuous function and* $\varphi(t) < t$ *for all* $t > 0$*. Suppose that* $X := (X,d)$ *is a complete metric space and* $T : X \to X$ *is a mapping such that* $\operatorname{diam} \mathcal{O}(x) < \infty$ *for all* $x \in X$ *and*

$$d(Tx,Ty) \le \varphi(\operatorname{diam} \mathcal{O}(x,y))$$

for all $x, y \in X$*. Then* T *has a unique fixed point* u *and* $\lim_{n\to\infty} d(T^n x, u) = 0$ *for all* $x \in X$.

2. Results

2.1. **Theorem 1.3 is a consequence of Theorem 1.5.** Theorem 1.3 can be regarded as a direct consequence of Theorem 1.5. In fact, we prove even more.

Theorem 2.1. *Suppose that $\varphi : [0,\infty) \to [0,\infty)$ is a nondecreasing and continuous function such that $\varphi(t) < t$ for all $t > 0$ and $\lim_{t\to\infty}(t - \varphi(t)) = \infty$. Suppose that $T : X \to X$ is a mapping such that*

$$d(Tx,Ty) \le \varphi\left(\max\left\{\begin{array}{c} d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx), \\ d(x,T^2x), d(y,T^2x), d(Tx,T^2x), d(Ty,T^2x) \end{array}\right\}\right)$$

for all $x, y \in X$. Then the following statements are true:

(a) *For each $x \in X$, $\operatorname{diam}\mathcal{O}(x) < \infty$ and*

$$\operatorname{diam}\mathcal{O}(x) - \varphi(\operatorname{diam}\mathcal{O}(x)) \le d(x,Tx).$$

(b) *T has a unique fixed point u.*

(c) *$d(T^n x, u) \le \varphi^n(\operatorname{diam}\mathcal{O}(x))$ for all $x \in X$ and for all $n \ge 1$. In particular, $\lim_{n\to\infty} d(T^n x, u) = 0$.*

Proof. (a) Let $x \in X$. The statement holds trivially if $x = Tx$. We now assume that $x \ne Tx$. For convenience, we write $\mathcal{O}(x;n) := \{x, Tx, T^2x, \dots, T^n x\}$ where $n \ge 1$. Note that $\operatorname{diam}\mathcal{O}(x;n) \ge d(x,Tx) > 0$ for all $n \ge 1$. Moreover, for $1 \le i \le n-1$ and $1 \le j \le n$, we have

$$\begin{aligned} d(T^i x, T^j x) &= d(T(T^{i-1}x), T(T^{j-1}x)) \\ &\le \varphi\left(\max\left\{\begin{array}{c} d(T^{i-1}x, T^{j-1}x), d(T^{i-1}x, T^i x), d(T^{j-1}x, T^j x), \\ d(T^{i-1}x, T^j x), d(T^{j-1}x, T^i x), \\ d(T^{i+1}x, T^{i-1}x), d(T^{i+1}x, T^{j-1}x), \\ d(T^{i+1}x, T^i x), d(T^{i+1}x, T^j x) \end{array}\right\}\right) \\ &\le \varphi(\operatorname{diam}\mathcal{O}(x;n)) < \operatorname{diam}\mathcal{O}(x;n). \end{aligned}$$

Hence $\operatorname{diam}\mathcal{O}(x;n) = d(x, T^j x)$ for some $1 \le j \le n$. Now

$$\begin{aligned} \operatorname{diam}\mathcal{O}(x;n) = d(x,T^j x) &\le d(x,Tx) + d(Tx, T^j x) \\ &\le d(x,Tx) + \varphi(\operatorname{diam}\mathcal{O}(x;n)). \end{aligned}$$

In particular,

$$\operatorname{diam}\mathcal{O}(x;n) - \varphi(\operatorname{diam}\mathcal{O}(x;n)) \le d(x,Tx).$$

If $\lim_{n\to\infty}\operatorname{diam}\mathcal{O}(x;n) = \infty$, then it follows from $\lim_{t\to\infty}(t - \varphi(t)) = \infty$ that $d(x,Tx) = \infty$ which is impossible. Hence $\operatorname{diam}\mathcal{O}(x) = \lim_{n\to\infty}\operatorname{diam}\mathcal{O}(x;n) < \infty$ and the conclusion follows from the continuity of φ.

(b) It follows from Theorem 1.5 that T has a unique fixed point u. In fact, for each $x, y \in X$, it is clear that

$$\max\left\{\begin{array}{c} d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx), \\ d(x,T^2x), d(y,T^2x), d(Tx,T^2x), d(Ty,T^2x) \end{array}\right\} \le \operatorname{diam}\mathcal{O}(x,y).$$

(c) Let $x \in X$. Note from (a) that $\operatorname{diam}\mathcal{O}(Tx; n-1) \le \varphi(\operatorname{diam}\mathcal{O}(x;n))$ for all $n \ge 1$. In particular, letting $n \to \infty$ gives

$$\operatorname{diam}\mathcal{O}(Tx) \le \varphi(\operatorname{diam}\mathcal{O}(x)).$$

By induction, we obtain that $\operatorname{diam}\mathcal{O}(T^n x) \le \varphi^n(\operatorname{diam}\mathcal{O}(x))$ for all $n \ge 1$. Hence $d(T^n x, T^{n+k}x) \le \varphi^n(\operatorname{diam}\mathcal{O}(x))$ for all $n, k \ge 1$. It follows from Theorem 1.5 that $d(T^n x, u) \le \varphi^n(\operatorname{diam}\mathcal{O}(x))$. □

Remark 2.2. If we let $\varphi(t) := qt$ for all $t \ge 0$ where $q \in (0,1)$, then $(1-q)\operatorname{diam}\mathcal{O}(x) \le d(x, Tx)$ and hence we immediately obtain Theorem 1.3 via our Theorem 2.1.

Remark 2.3. Suppose that $\varphi(t) := t/(1+t)$ for all $t \ge 0$. It follows that φ is a nondecreasing and continuous function such that $\varphi(t) < t$ for all $t > 0$ and $\lim_{t\to\infty}(t - \varphi(t)) = \infty$. Moreover, there exists no $q \in (0,1)$ such that $\varphi(t) \le qt$ for all $t \ge 0$. In particular, our Theorem 2.1 is a genuine extension of Theorem 1.3.

2.2. **A further generalization of Theorem 1.4.** The following lemma is obvious.

Lemma 2.4. *Suppose that $\varphi : [0,\infty) \to [0,\infty)$ is a nondecreasing and continuous function such that $\varphi(t) < t$ for all $t > 0$ and $\lim_{t\to\infty}(t - \varphi(t)) = \infty$. Suppose that $\lambda \in (0,1)$. If $\psi(t) := (1-\lambda)t + \lambda\varphi(t)$ for all $t \ge 0$, then ψ is a nondecreasing and continuous function such that $\psi(t) < t$ for all $t > 0$ and $\lim_{t\to\infty}(t - \psi(t)) = \infty$.*

The following result is a consequence of our Theorem 2.1.

Theorem 2.5. *Suppose that $\varphi : [0,\infty) \to [0,\infty)$ is a nondecreasing and continuous function such that $\varphi(t) < t$ for all $t > 0$ and $\lim_{t\to\infty}(t - \varphi(t)) = \infty$. Suppose that $F : X \to \mathrm{BN}(X)$ is a mapping such that*

$$\rho(Fx, Fy) \le \varphi\left(\max\left\{\begin{array}{c} d(x,y), \rho(x,Fx), \rho(y,Fy), D(x,Fy), D(y,Fx), \\ D(x,F^2x), D(y,F^2x), D(Fx,F^2x), D(Fy,F^2x) \end{array}\right\}\right)$$

for all $x, y \in X$. Suppose that $\lambda \in (0,1)$ and $\psi(t) := (1-\lambda)t + \lambda\varphi(t)$ for all $t \ge 0$. Then the following statements are true.

(a) *For each $x \in X$ there exists a selection $Tx \in Fx$ such that the following condition holds:*

$$d(Tx, Ty) \le \psi\left(\max\left\{\begin{array}{c} d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx), \\ d(x,T^2x), d(y,T^2x), d(Tx,T^2x), d(Ty,T^2x) \end{array}\right\}\right)$$

for all $x, y \in X$.

(b) *F has a unique fixed point u.*

(c) *For each $x \in X$, there exists a sequence $\{x_n\}$ such that*
- *$x_0 := x$ and $x_{n+1} \in Fx_n$ for all $n \ge 0$;*
- *$\delta - \psi(\delta) \le d(x_0, x_1)$ where $\delta := \operatorname{diam}\{x_n : n \ge 0\} < \infty$;*
- *$d(x_n, u) \le \psi^n(\delta)$ for all $n \ge 0$ and hence $\lim_{n\to\infty} d(x_n, u) = 0$.*

Proof. (a) Note that $\varphi(t) < \psi(t) < t$ for all $t > 0$. Let $x \in X$. If $x \in Fx$, then let $Tx := x$. On the other hand, we assume that $x \notin Fx$, that is, $\rho(x, Fx) > 0$. We can choose $Tx \in Fx$ such that

$$\psi(d(x, Tx)) \ge \varphi(\rho(x, Fx)).$$

Otherwise, there exists $\{z_n\} \subset Fx$ such that $\lim_{n\to\infty} d(x, z_n) = \rho(x, Fx)$ and

$$\psi(d(x, z_n)) < \varphi(\rho(x, Fx)) \quad \text{for all } n \ge 1.$$

Since ψ is continuous, we have $\psi(\rho(x, Fx)) \leq \varphi(\rho(x, Fx))$. This implies that $\rho(x, Fx) = 0$ which is impossible.

Moreover, for each $x, y \in X$, we have the following inequalities:

$$D(x, Fy) \leq d(x, Ty); \qquad D(y, Fx) \leq d(y, Tx); \qquad D(x, F^2x) \leq d(x, T^2x);$$
$$D(y, F^2x) \leq d(y, T^2x); \quad D(Fx, F^2x) \leq d(Tx, T^2x); \quad D(Fy, F^2x) \leq d(Ty, T^2x).$$

This implies that

$$\begin{aligned} d(Tx, Ty) &\leq \rho(Fx, Fy) \\ &\leq \varphi\left(\max\left\{\begin{array}{c} d(x, y), \rho(x, Fx), \rho(y, Fy), D(x, Fy), D(y, Fx), \\ D(x, F^2x), D(y, F^2x), D(Fx, F^2x), D(Fy, F^2x) \end{array}\right\}\right) \\ &\leq \psi\left(\max\left\{\begin{array}{c} d(x, y), \rho(x, Tx), \rho(y, Ty), d(x, Ty), d(y, Tx), \\ d(x, T^2x), d(y, T^2x), d(Tx, T^2x), d(Ty, T^2x) \end{array}\right\}\right). \end{aligned}$$

(b) It follows from our Theorem 2.1 that T has a unique fixed point u. Note that, since $u = Tu \in Fu$, we have $u = Tu = T^2u = T(Tu) \in F(Tu) \subset \bigcup_{z \in Fu} Fz = F^2u$ and

$$D(u, Fu) = D(u, F^2u) = D(Fu, F^2u) = 0.$$

In particular, it follows from the assumption that

$$\begin{aligned} \rho(u, Fu) &\leq \rho(Fu, Fu) \\ &\leq \varphi\left(\max\left\{\begin{array}{c} d(u, u), \rho(u, Fu), D(u, Fu), \\ D(u, F^2u), D(Fu, F^2u) \end{array}\right\}\right) \\ &= \varphi(\rho(u, Fu)). \end{aligned}$$

Hence $\rho(u, Fu) = 0$, that is, $Fu = \{u\}$.

(c) Let $x \in X$. Define $x_0 := x$ and $x_{n+1} := Tx_n$ for all $n \geq 0$. It follows from our Theorem 2.1 that $\delta := \operatorname{diam}\{x_n : n \geq 0\} < \infty$ and $\delta - \psi(\delta) \leq d(x_0, Tx_0) = d(x_0, x_1)$. Moreover, $d(x_n, u) = d(T^n x_0, u) \leq \psi^n(\delta)$ for all $n \geq 0$. □

Remark 2.6. Suppose that all the assumptions of Theorem 2.5 hold where $\varphi(t) := qt$ for all $t \geq 0$ where $q \in (0, 1)$. Then there exists $u \in X$ such that $Fu = \{u\}$. Let $x \in X$ and $\lambda \in (0, 1)$. Then, by Theorem 2.5, there exists a sequence $\{x_n\} \subset X$ such that

- $x_0 := x$ and $x_{n+1} \in Fx_n$ for all $n \geq 0$;
- $d(x_n, u) \leq \psi^n(\delta)$ where $\delta := \operatorname{diam}\{x_n : n \geq 0\}$ and $\psi(t) := (1 - \lambda)t + \lambda qt$.

Note that $\delta \leq \frac{d(x_0, x_1)}{\lambda(1-q)}$. It follows that

$$d(x_n, u) \leq \frac{(1 - \lambda(1 - q))^n}{\lambda(1 - q)} d(x_0, x_1)$$

for all $n \geq 0$. We now compare our estimate with the one in Theorem 1.4. It is easy to see that for each $a, \lambda \in (0, 1)$ there are $a', \lambda' \in (0, 1)$ such that

$$\frac{(1 - \lambda'(1 - q))^n}{\lambda'(1 - q)} < \frac{q^{(1-a)n}}{1 - q^{1-a}} \quad \text{and} \quad \frac{q^{(1-a')n}}{1 - q^{1-a'}} < \frac{(1 - \lambda(1 - q))^n}{\lambda(1 - q)}$$

for all $n \geq 1$.

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