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APPROXIMATION OF ZEROS OF AN ACCRETIVE OPERATOR AND FIXED POINTS OF NONEXPANSIVE MAPPINGS IN A BANACH SPACE

KOJI AOYAMA AND MASASHI TOYODA

Abstract. Let A be an accretive operator on a strictly convex reflexive Banach space with a Gâteaux differential norm and J_t the resolvent of A for positive t. We show that if A has a zero and satisfies some appropriate conditions, then for any $x \in E$ the strong $\lim_{t\to\infty} J_t x$ exists and the limit is a zero of A. Then, using this result, we establish a strong convergence theorem for an explicit iterative method for approximating a common fixed point of a strongly nonexpansive sequence.

1. Introduction

Let E be a real Banach space, A an m-accretive operator on E with a zero, and J_t the resolvent of A for $t > 0$. Reich [23] established that if E is uniformly smooth, then for any $x \in E$ the strong $\lim_{t\to\infty} J_t x$ exists and the limit is a zero of A [23, Theorem 1]; see also Bruck [14] and Reich [19,21]. Reich [23] also pointed out that the assumptions on E and A can be weakened or replaced; see [23, Remark 1 and Note added in proof (p.291)] for more information. After that, Takahashi and Ueda [28] proved a generalization [28, Theorem 1] of Reich's result [23, Theorem 1] by using Banach limits.

These convergence results are also important for an explicit iterative method for nonexpansive mappings in a Banach space. For example, such convergence results lead to strong convergence of an iterative sequence $\{x_n\}$ generated by the following algorithm: $x_1 \in C$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) S_n x_n$$

for each positive integer n, where C is a closed convex subset of a Banach space E, u is a point in C, $\{\alpha_n\}$ is a sequence in $(0, 1]$, and $\{S_n\}$ is a sequence of nonexpansive self-mappings of C with a common fixed point; see [12] and see also [6, 23–25].

In this paper, motivated by Reich [23], and Takahashi and Ueda [28], we first prove a convergence theorem (Theorem 3.1) for an accretive operator on a strictly convex reflexive Banach space with a uniformly Gâteaux differential norm, which is a variant of strong convergence theorems in [23, 28]. Then, combining Theorem 3.1 with the results in [12], we also show some strong convergence theorems for an explicit iterative method for approximating a fixed point of nonexpansive mappings in a Banach space.

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2. Preliminaries

Throughout the present paper, E denotes a real Banach space with norm $\|\cdot\|$, E^* the dual of E, $\langle x, x^*\rangle$ the value of $x^* \in E^*$ at $x \in E$, $\mathbb{N}$ the set of positive integers, and $\mathbb{R}$ the set of real numbers. The norm of E^* is also denoted by $\|\cdot\|$. Strong convergence of a sequence $\{x_n\}$ in E to $x \in E$ is denoted by $x_n \to x$. The identity mapping on E is denoted by I; the (normalized) *duality mapping* of E is denoted by J, that is, it is a set-valued mapping of E into E^* defined by $Jx = \{x^* \in E^* \colon \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$ for $x \in E$.

Let S_E denote the unit sphere of E, that is, $S_E = \{x \in E \colon \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in S_E$. In this case, E is said to be *smooth*. It is known that the duality mapping J is single-valued and norm-to-weak* continuous if E is smooth; see [26]. The norm of E is said to be *uniformly Gâteaux differentiable* if for each $y \in S_E$ the limit (2.1) is attained uniformly for $x \in S_E$. A Banach space E is said to be *strictly convex* if $x, y \in S_E$ and $x \neq y$ imply $\|x + y\| < 2$; E is said to be *uniformly convex* if for any $\epsilon > 0$ there exists $\delta > 0$ such that $x, y \in S_E$ and $\|x - y\| \geq \epsilon$ imply $\|x + y\|/2 \leq 1 - \delta$. It is known that if E is uniformly convex, then E is strictly convex and reflexive; see [26].

Let E be a strictly convex reflexive Banach space and C a nonempty closed convex subset of E. It is known that for each $x \in E$ there exists a unique point $z \in C$ such that $\|x - z\| \leq \|x - y\|$ for all $y \in C$. Such a point z is denoted by Px. This defines a mapping P of E onto C and P is said to be the *metric projection* of E onto C.

Let C be a nonempty subset of E and $T \colon C \to E$ a mapping. The set of fixed points of T is denoted by $\mathrm{F}(T)$. A mapping T is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

Let C be a nonempty subset of E, K a nonempty subset of C, and Q a mapping of C onto K. Then Q is said to be a *retraction* if $Qx = x$ for all $x \in K$; Q is said to be *sunny* if $Q\big(Qx + \lambda(x - Qx)\big) = Qx$ holds whenever $x \in C$, $\lambda \geq 0$, and $Qx + \lambda(x - Qx) \in C$; K is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction [18] of C onto K; see also [15, 16].

Using [15, Lemma 13.1] or [26, Lemma 5.1.6], we obtain the following; we give the proof for the sake of completeness.

Lemma 2.1. *Let E be a smooth Banach space, C a subset of E, K a nonempty subset of C, and $Q \colon C \to K$ a mapping. Suppose that*

$$\langle x - Qx, J(z - Qx)\rangle \leq 0$$

for all $x \in C$ and $z \in K$. Then Q is a sunny nonexpansive retraction of C onto K.

Proof. We first show that Q is a retraction. Let $z \in K$. Then $z \in C$. Thus, by assumption, we have $\|z - Qz\|^2 = \langle z - Qz, J(z - Qz)\rangle \leq 0$, and hence $z = Qz$. As a result, we conclude that Q is a retraction of C onto K.

We next show that Q is nonexpansive. Let $x, y \in C$. Then $Qx, Qy \in K$. Since

$$\langle x - Qx, J(Qy - Qx)\rangle \le 0 \text{ and } \langle y - Qy, J(Qx - Qy)\rangle \le 0,$$

we have $\langle Qx - Qy - (x - y), J(Qx - Qy)\rangle \le 0$, and hence,

$$\|Qx - Qy\|^2 \le \langle x - y, J(Qx - Qy)\rangle \le \|x - y\| \, \|Qx - Qy\| .$$

This shows that Q is nonexpansive.

We lastly show that Q is sunny. Let $x \in C$ and $t \ge 0$ such that $z_t = Qx + t(x - Qx) \in C$. It is enough to show that $Qz_t = Qx$. Taking into account $Qx, Qz_t \in K$, we obtain

$$\langle x - Qx, J(Qz_t - Qx)\rangle \le 0 \text{ and } \langle z_t - Qz_t, J(Qx - Qz_t)\rangle \le 0. \tag{2.2}$$

Since $t(x - Qx) = z_t - Qx$, it follows from (2.2) that

$$\begin{aligned} 0 &\ge t\,\langle x - Qx, J(Qz_t - Qx)\rangle + \langle Qz_t - z_t, J(Qz_t - Qx)\rangle \\ &= \langle t(x - Qx) + Qz_t - z_t, J(Qz_t - Qx)\rangle = \|Qz_t - Qx\|^2 . \end{aligned}$$

Therefore, $Qz_t = Qx$. This completes the proof. □

Let A be a set-valued mapping of E into E. The domain of A is denoted by $\mathrm{D}(A)$, the range of A by $\mathrm{R}(A)$, and the set of zeros of A by $A^{-1}0$, that is, $\mathrm{D}(A) = \{x \in E \colon Ax \ne \emptyset\}$, $\mathrm{R}(A) = \bigcup\{Ax \colon x \in \mathrm{D}(A)\}$, and $A^{-1}0 = \{x \in \mathrm{D}(A) \colon 0 \in Ax\}$. We say that A is an *accretive* operator on E if for any $x, y \in \mathrm{D}(A)$, $u \in Ax$, and $v \in Ay$ there exists $j \in J(x - y)$ such that $\langle u - v, j\rangle \ge 0$; an accretive operator A is *m-accretive* if $\mathrm{R}(I + \lambda A) = E$ for all $\lambda > 0$.

Example 2.2 ([26, Theorem 4.6.4])**.** Let C be a nonempty closed convex subset of a Banach space E and $T \colon C \to C$ a nonexpansive mapping. Then $I - T$ is an accretive operator on E and $C = \mathrm{D}(I - T) \subset \mathrm{R}\big(I + \lambda(I - T)\big)$ for all $\lambda > 0$.

Let A be an accretive operator on E and λ a positive real number. It is known that $(I + \lambda A)^{-1}$ is a single-valued mapping of $\mathrm{R}(I + \lambda A)$ onto $\mathrm{D}(A)$. The mapping $(I + \lambda A)^{-1}$ is said to be the *resolvent* of A and is denoted by J_λ. It is also known that J_λ is nonexpansive, $\mathrm{F}(J_\lambda) = A^{-1}0$, and

$$\frac{x - J_\lambda x}{\lambda} \in AJ_\lambda x \tag{2.3}$$

for all $\lambda > 0$ and $x \in \mathrm{R}(I + \lambda A)$; see [26].

Using (2.3), we obtain the following lemma:

Lemma 2.3. *Let E be a smooth Banach space, A an accretive operator on E with a zero, $\{s_n\}$ a positive sequence such that $s_n \to \infty$, $x \in \bigcap_{\lambda > 0} \mathrm{R}(I + \lambda A)$, and $z \in A^{-1}0$. Suppose that $J_{s_n} x \to y \in E$. Then $\langle y - x, J(y - z)\rangle \le 0$.*

Proof. Since

$$\frac{(I - J_{s_n})x}{s_n} \in AJ_{s_n} x$$

by (2.3), $0 \in Az$, and A is accretive, it follows that

$$\left\langle \frac{x - J_{s_n} x}{s_n} - 0, J\left(J_{s_n} x - z\right) \right\rangle \geq 0.$$

As a result, we see that $\langle J_{s_n} x - x, J\left(J_{s_n} x - z\right)\rangle \leq 0$ for all $n \in \mathbb{N}$. Thus we have

$$\langle y - x, J(y - z)\rangle \leq 0$$

because $J_{s_n} x \to y$ and J is norm-to-weak* continuous. □

Let ℓ^∞ be the Banach space of bounded sequences in $\mathbb{R}$ with the supremum norm. It is known that there exists a bounded linear functional μ on ℓ^∞ such that the following conditions hold: $\|\mu\| = 1$; if $t_n = 1$ for every $n \in \mathbb{N}$, then $\mu(\{t_n\}) = 1$; $\mu(\{t_{n+1}\}) = \mu(\{t_n\})$ for all $\{t_n\} \in \ell^\infty$. Such a functional μ is said to be a *Banach limit* and $\mu(\{t_n\})$ is denoted by $\operatorname{Lim}_n t_n$ for $\{t_n\} \in \ell^\infty$. We know that

$$\liminf_n t_n \leq \operatorname*{Lim}_n t_n \leq \limsup_n t_n \text{ and } \left|\operatorname*{Lim}_n t_n\right| \leq \operatorname*{Lim}_n |t_n|$$

for all $\{t_n\} \in \ell^\infty$. We also know that if $\{s_n\}, \{t_n\} \in \ell^\infty$ and $s_n \leq t_n$ for all $n \in \mathbb{N}$, then $\operatorname{Lim}_n s_n \leq \operatorname{Lim}_n t_n$; see [26].

We need the following lemma. For the sake of completeness, we give the proof.

Lemma 2.4. *Let C be a nonempty convex subset of a Banach space E, $\{x_n\}$ a bounded sequence in E, and $g\colon C \to \mathbb{R}$ a function defined by $g(y) = \operatorname{Lim}_n \|x_n - y\|^2$ for $y \in C$. Then g is convex and continuous.*

Proof. Let $y, z \in C$ and $\lambda \in [0, 1]$. Since $\|\cdot\|^2$ is convex, we have

$$\left\|x_n - [\lambda y + (1 - \lambda) z]\right\|^2 \leq \lambda \|x_n - y\|^2 + (1 - \lambda) \|x_n - z\|^2,$$

and hence

$$g\big(\lambda y + (1 - \lambda) z\big) \leq \operatorname*{Lim}_n \big(\lambda \|x_n - y\|^2 + (1 - \lambda) \|x_n - z\|^2\big) = \lambda g(y) + (1 - \lambda) g(z).$$

Therefore, g is convex. We also have

$$\begin{aligned} \left|\|x_n - y\|^2 - \|x_n - z\|^2\right| &\leq (\|x_n - y\| + \|x_n - z\|) \left|\|x_n - y\| - \|x_n - z\|\right| \\ &\leq (2\|x_n - y\| + \|y - z\|) \|y - z\|. \end{aligned}$$

Thus it follows that

$$\begin{aligned} |g(y) - g(z)| &= \left|\operatorname*{Lim}_n \big(\|x_n - y\|^2 - \|x_n - z\|^2\big)\right| \\ &\leq \operatorname*{Lim}_n \left|\|x_n - y\|^2 - \|x_n - z\|^2\right| \leq \big(2g(y) + \|y - z\|\big) \|y - z\|. \end{aligned}$$

This shows that g is continuous. □

We also need the following lemma, which is a direct consequence of [26, Theorem 1.3.11]:

Lemma 2.5. *Let E be a reflexive Banach space, C a nonempty closed convex subset of E, and $g\colon C \to \mathbb{R}$ a convex continuous function. Suppose that $g(x_n) \to \infty$ whenever $\{x_n\}$ is a sequence in C such that $\|x_n\| \to \infty$. Then there exists $u \in C$ such that $g(u) = \inf g(C)$.*

3. Convergence theorems for resolvents of an accretive operator

In this section, we first prove a strong convergence theorem for resolvents of an accretive operator in a Banach space, which is a variant of [23, Theorem 1] and [28, Theorem 1]. Then we obtain convergence results for an m-accretive operator and a nonexpansive mapping as corollaries of the theorem.

Theorem 3.1. *Let E be a strictly convex reflexive Banach space whose norm is uniformly Gâteaux differentiable, A an accretive operator on E with a zero, and C a nonempty closed convex subset of E. Suppose that there exists $\eta > 0$ such that $J_\eta(C) \subset C$, where $J_\eta = (I + \eta A)^{-1}$, and that*

$$\mathrm{D}(A) \subset \mathrm{R}(I + \lambda A) \text{ and } C \subset \mathrm{R}(I + \lambda A) \tag{3.1}$$

for all $\lambda > 0$. Then

(1) *for each $x \in C$ the strong $\lim_{t\to\infty} J_t x$ exists and the limit is a point in $A^{-1}0 \cap C$, where $J_t = (I + tA)^{-1}$ for $t > 0$.*

Moreover, let $Q\colon C \to A^{-1}0 \cap C$ be a mapping defined by $Qx = \lim_{t\to\infty} J_t x$ for $x \in C$. Then

(2) *$\langle x - Qx, J(z - Qx)\rangle \le 0$ for all $x \in C$ and $z \in A^{-1}0 \cap C$;*
(3) *Q is a sunny nonexpansive retraction of C onto $A^{-1}0 \cap C$.*

To prove Theorem 3.1 above, we need the following lemmas:

Lemma 3.2. *Let E be a reflexive Banach space, C a nonempty closed convex subset of E, A an accretive operator on E with a zero, x a point in C, and $\{s_n\}$ a positive sequence such that $s_n \to \infty$. Suppose that there exists $\eta > 0$ such that $J_\eta(C) \subset C$, and that (3.1) holds for all $\lambda > 0$. Then*

(1) *$\{J_{s_n} x\}$ is bounded.*

Moreover, let $g\colon C \to \mathbb{R}$ be a function defined by $g(y) = \mathrm{Lim}_n \|J_{s_n} x - y\|^2$ for $y \in C$, $r_0 = \inf\{g(y)\colon y \in C\}$, and $K = \{y \in C\colon g(y) = r_0\}$, where Lim_n is a Banach limit. Then

(2) *K is a nonempty closed convex subset of E;*
(3) *$J_\eta(K) \subset K$.*

Proof. We first show (1). Let $u \in A^{-1}0$. Since $u = J_{s_n} u$ and J_{s_n} is nonexpansive, it follows that

$$\|J_{s_n} x\| \le \|J_{s_n} x - J_{s_n} u\| + \|J_{s_n} u\| \le \|x - u\| + \|u\|$$

for all $n \in \mathbb{N}$. Therefore, $\{J_{s_n} x\}$ is bounded.

We next show (2). By virtue of (1), the function g is well-defined. Lemma 2.4 shows that g is continuous and convex. Set $x_n = J_{s_n} x$ and let $\{z_m\}$ be a sequence in C such that $\|z_m\| \to \infty$. Then it follows that

$$\begin{aligned} g(z_m) &= \mathop{\mathrm{Lim}}_n \|x_n - z_m\|^2 \\ &\ge \mathop{\mathrm{Lim}}_n\bigl(-2\|x_n\|\|z_m\| + \|z_m\|^2\bigr) = -2\|z_m\| \mathop{\mathrm{Lim}}_n \|x_n\| + \|z_m\|^2 \to \infty \end{aligned}$$

as $m \to \infty$. Thus Lemma 2.5 implies that $K \neq \emptyset$. Since $K = \{y \in C \colon g(y) \leq r_0\}$, C is both closed and convex, and g is both lower semicontinuous and convex, we see that K is a closed convex subset of E.

Lastly, we show (3). Set $x_n = J_{s_n}x$ and $y_n = (x - x_n)/s_n$. Then (2.3) shows that $x_n + \eta y_n \in (I + \eta A)x_n$, and hence $J_\eta(x_n + \eta y_n) = x_n$. Since $x_n \in \mathrm{D}(A) \subset \mathrm{R}(I + \eta A) = \mathrm{D}(J_\eta)$ by assumption and J_η is nonexpansive, it follows from (1) that

$$\begin{aligned}\|x_n - J_\eta x_n\| &= \|J_\eta(x_n + \eta y_n) - J_\eta x_n\| \\ &\leq \|x_n + \eta y_n - x_n\| = \eta \left\| \frac{x - x_n}{s_n} \right\| \leq \frac{\eta}{s_n}\big(\|x\| + \|x_n\|\big) \to 0\end{aligned}$$

as $n \to \infty$. Let $z \in K$. Taking into account $J_\eta(C) \subset C$, we see that $J_\eta z \in C$. Thus we have

$$\begin{aligned}g(J_\eta z) &\leq \operatorname*{Lim}_n \big(\|x_n - J_\eta x_n\| + \|J_\eta x_n - J_\eta z\|\big)^2 \\ &\leq \operatorname*{Lim}_n \big[\|x_n - J_\eta x_n\| \big(\|x_n - J_\eta x_n\| + 2\|x_n - z\|\big) + \|x_n - z\|^2\big] \\ &= \operatorname*{Lim}_n \|x_n - z\|^2 = g(z) = r_0.\end{aligned}$$

Therefore we conclude that $J_\eta z \in K$. □

Lemma 3.3. *Under the assumptions of Lemma 3.2, suppose that E is smooth and strictly convex, and let P be the metric projection of E onto K and $u \in A^{-1}0$. Then*

(1) *$Pu \in A^{-1}0 \cap C$;*
(2) *$\mathrm{Lim}_n \langle J_{s_n}x - x, J(J_{s_n}x - Pu)\rangle \leq 0$.*

Proof. We first show (1). It follows from Lemma 3.2 (2) that P is well-defined. Taking into account $Pu \in K$, we have $Pu \in C \subset \mathrm{R}(I + \eta A) = \mathrm{D}(J_\eta)$. Since $u \in A^{-1}0 = \mathrm{F}(J_\eta)$ and J_η is nonexpansive, we have

$$\|J_\eta Pu - u\| = \|J_\eta Pu - J_\eta u\| \leq \|Pu - u\|.$$

On the other hand, Lemma 3.2 (3) implies that $J_\eta Pu \in K$. Hence $J_\eta Pu = Pu$, that is, $Pu \in \mathrm{F}(J_\eta)$. This shows that $Pu \in A^{-1}0$, and thus $Pu \in A^{-1}0 \cap C$.

We next show (2). It follows from (1) that $Pu \in A^{-1}0$, and hence $0 \in APu$. Set $x_n = J_{s_n}x$. Since $(x - x_n)/s_n \in Ax_n$ by (2.3), $0 \in APu$, and A is accretive, we have $\langle (x - x_n)/s_n - 0, J(x_n - Pu)\rangle \geq 0$. As a result, it turns out that $\langle x_n - x, J(x_n - Pu)\rangle \leq 0$ for all $n \in \mathbb{N}$. Therefore we conclude that

$$\operatorname*{Lim}_n \langle x_n - x, J(x_n - Pu)\rangle \leq \limsup_n \langle x_n - x, J(x_n - Pu)\rangle \leq 0. \qquad \square$$

Using lemmas above and Lemma 2.3, we complete the proof of Theorem 3.1.

Proof of Theorem 3.1. We first show (1). Let $\{t_m\}$ be a positive sequence such that $t_m \to \infty$ and $x \in C$. It is enough to verify that $\{J_{t_m}x\}$ converges strongly to some point in $A^{-1}0 \cap C$ as $m \to \infty$. Now we confirm that for any subsequence $\{x_n\}$ of $\{J_{t_m}x\}$ there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges to a point in $A^{-1}0 \cap C$, and that the limit does not depend on $\{x_n\}$. Let $\{x_n\}$ be a subsequence of $\{J_{t_m}x\}$. Then we know that there exists a subsequence $\{s_n\}$ of $\{t_m\}$ such that

$x_n = J_{s_n}x$ for all $n \in \mathbb{N}$, and moreover, it is clear that $s_n \to \infty$. Let g and K be the same as in Lemma 3.2, P the metric projection of E onto K, and $u \in A^{-1}0$. Taking into account $Pu \in K$, we have $g(Pu) = \inf\{g(y) \colon y \in C\}$. Thus [28, Lemma 1] implies that

$$\operatorname*{Lim}_n \langle x - Pu, J(J_{s_n}x - Pu)\rangle \leq 0. \tag{3.2}$$

Since $x_n = J_{s_n}x$, it follows from Lemma 3.3 (2) and (3.2) that

$$\begin{aligned} 0 \leq g(Pu) &= \operatorname*{Lim}_n \langle x_n - Pu, J(x_n - Pu)\rangle \\ &= \operatorname*{Lim}_n \langle x_n - x, J(x_n - Pu)\rangle + \operatorname*{Lim}_n \langle x - Pu, J(x_n - Pu)\rangle \leq 0. \end{aligned}$$

Therefore, $g(Pu) = 0$, and hence

$$0 \leq \liminf_n \|x_n - Pu\|^2 \leq \operatorname*{Lim}_n \|x_n - Pu\|^2 = 0.$$

Thus we deduce from Lemma 3.3 (1) that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges strongly to some point in $A^{-1}0 \cap C$, that is, there exists a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ such that $\left\{J_{s_{n_i}}x\right\}$ converges strongly to some point $v \in A^{-1}0 \cap C$. Let $\{y_k\}$ be a subsequence of $\{J_{t_m}x\}$. Similarly, we know that there exists a subsequence $\{r_k\}$ of $\{t_m\}$ such that $y_k = J_{r_k}x$ for all $k \in \mathbb{N}$, and moreover, we also deduce that there exists a subsequence $\left\{r_{k_j}\right\}$ of $\{r_k\}$ such that $\left\{J_{r_{k_j}}x\right\}$ converges strongly to some point $w \in A^{-1}0 \cap C$. Hence Lemma 2.3 implies that

$$\|v - w\|^2 = \langle v - x, J(v - w)\rangle + \langle w - x, J(w - v)\rangle \leq 0.$$

Consequently, $v = w$. Therefore we conclude that $\{J_{t_m}x\}$ converges strongly to some point in $A^{-1}0 \cap C$.

We next show (2) and (3). Let $\{t_m\}$ be a positive sequence such that $t_m \to \infty$, $x \in C$, and $z \in A^{-1}0 \cap C$. Then, by assumption, $x \in \mathrm{R}(I + \lambda A)$ for all $\lambda > 0$. Since $J_{t_m}x \to Qx$ as $m \to \infty$, Lemma 2.3 implies that $\langle x - Qx, J(z - Qx)\rangle \leq 0$. Thus (2) holds, and (3) follows from Lemma 2.1. □

Remark 3.4. It is known that Q in Theorem 3.1 is the unique sunny nonexpansive retraction of C onto $A^{-1}0 \cap C$; see [15, Lemma 13.1].

As a direct consequence of Theorem 3.1, we obtain the following corollary:

Corollary 3.5. *Let E be the same as in Theorem 3.1, A an m-accretive operator on E with a zero, and $x \in E$. Then the strong $\lim_{t\to\infty} J_t x$ exists and the limit w is a point in $A^{-1}0$ such that $\langle x - w, J(z - w)\rangle \leq 0$ for all $z \in A^{-1}0$, where $J_t = (I + tA)^{-1}$ for $t > 0$.*

Proof. Set $C = E$. Then it is clear that C is a nonempty closed convex subset of E. Since A is m-accretive, $J_\lambda(C) \subset C$ and (3.1) hold for all $\lambda > 0$. Therefore Theorem 3.1 implies the conclusion. □

We also obtain the following corollary. Similar results can be found in [15, 19, 23, 26].

Corollary 3.6. *Let E be a strictly convex reflexive Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex subset of E, $T\colon C \to C$ a nonexpansive mapping with a fixed point, u a point in C, and z_s a point in C such that $z_s = su + (1-s)Tz_s$ for $s \in (0,1)$. Then $z_s \to Qu$ as $s \downarrow 0$, where Q is a sunny nonexpansive retraction of C onto $\mathrm{F}(T)$.*

Proof. Set $A = I - T$. Then Example 2.2 implies that A is an accretive operator and $C = \mathrm{D}(A) \subset \mathrm{R}\big(I + \lambda A\big)$ for all $\lambda > 0$. It is not hard to verify that $J_t u = z_s$ for all $s \in (0,1)$, where $t = 1/s - 1$ and $J_t = (I + tA)^{-1}$. Since $A^{-1}0 = \mathrm{F}(T)$ and $t \to \infty$ as $s \downarrow 0$, Theorem 3.1 implies the conclusion. □

4. Approximation of common fixed points of strongly nonexpansive sequences

In this section, we derive some strong convergence results from Corollary 3.6.

Before describing the results, we need some preliminaries. Recall that a Banach space E is said to have the *fixed point property for nonexpansive mappings* if every nonexpansive self-mapping of a bounded closed convex subset K of E has a fixed point in K; see [20,22]. Let C be a nonempty subset of a Banach space E, $T\colon C \to E$ a mapping, $\{S_n\}$ a sequence of mappings of C into E, and F the set of common fixed points of $\{S_n\}$, that is, $F = \bigcap_{n=1}^{\infty} \mathrm{F}(S_n)$. Recall that $\{S_n\}$ is said to be a *strongly nonexpansive sequence* [4, 7, 8] if each S_n is nonexpansive and $x_n - y_n - (S_n x_n - S_n y_n) \to 0$ whenever $\{x_n\}$ and $\{y_n\}$ are sequences in C, $\{x_n - y_n\}$ is bounded, and $\|x_n - y_n\| - \|S_n x_n - S_n y_n\| \to 0$; $\{S_n\}$ is said to satisfy the *NST condition (I) with T* [17, 27] if F is nonempty, $\mathrm{F}(T) \subset F$, and $x_n - Tx_n \to 0$ whenever $\{x_n\}$ is a bounded sequence in C and $x_n - S_n x_n \to 0$. We know that if $\{S_n\}$ satisfies the NST condition (I) with T, then $\mathrm{F}(T) = F$; see [12, Remark 2.4].

Using Corollary 3.6 and [12, Lemma 3.3], we obtain the following strong convergence theorem for a strongly nonexpansive sequence:

Theorem 4.1. *Let E be a reflexive Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex subset of E, $\{S_n\}$ a strongly nonexpansive sequence of self-mappings of C, T a nonexpansive self-mapping of C, $\{\alpha_n\}$ a sequence in $(0,1]$, u a point in C, and $\{x_n\}$ a sequence defined by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) S_n x_n$$

for $n \in \mathbb{N}$. Suppose that $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\{S_n\}$ satisfies the NST condition (I) with T. If either

(1) *E is strictly convex or*
(2) *E has the fixed point property for nonexpansive mappings,*

then $\{x_n\}$ converges strongly to Qu, where Q is a sunny nonexpansive retraction of C onto $\mathrm{F}(T) = \bigcap_{n=1}^{\infty} \mathrm{F}(S_n)$.

Proof. In the case of (2), we directly obtain the conclusion from [12, Theorem 3.1]. Suppose that (1) holds and let z_s be a unique point in C such that $z_s = su + (1-s)Tz_s$ for $s \in (0,1)$. Then it follows from Corollary 3.6 that $z_s \to Qu$ as $s \downarrow 0$. Therefore [12, Lemma 3.3] implies the conclusion. □

Remark 4.2. For more information about strong nonexpansiveness for a sequence of mappings; see [1–5, 7–12].

It is known that a uniformly convex Banach space is strictly convex and reflexive. Thus the following corollary is a direct consequence of Theorem 4.1.

Corollary 4.3 ([12, Corollary 3.4])**.** *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable. Let C, $\{S_n\}$, T, $\{\alpha_n\}$, u, $\{x_n\}$, and Q be the same as in Theorem 4.1. Then $\{x_n\}$ converges strongly to Qu.*

Remark 4.4. We can apply Corollary 4.3 to explicit iterative methods for

- the common fixed point problem of a sequence of nonexpansive mappings;
- the zero point problem of an accretive operator;

see [11, Theorem 3.1] and [12, Theorems 4.1 and 4.5] for more details.

Lastly, we deal with a strong convergence result for a strongly nonexpansive mapping. Let C be a nonempty subset of a Banach space E and $T: C \to E$ a mapping. Recall that T is said to be *strongly nonexpansive* [13] if T is nonexpansive and $x_n - y_n - (Tx_n - Ty_n) \to 0$ whenever $\{x_n\}$ and $\{y_n\}$ are sequences in C, $\{x_n - y_n\}$ is bounded, and $\|x_n - y_n\| - \|Tx_n - Ty_n\| \to 0$. It is clear that if T is a strongly nonexpansive mapping with a fixed point and $S_n = T$ for all $n \in \mathbb{N}$, then $\{S_n\}$ is a strongly nonexpansive sequence and $\{S_n\}$ satisfies the NST condition (I) with T. Therefore Theorem 4.1 leads to the following:

Corollary 4.5. *Let E, C, $\{\alpha_n\}$, and u be the same as in Theorem 4.1, $T: C \to C$ a strongly nonexpansive mapping with a fixed point, and $\{x_n\}$ a sequence defined by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to Qu, where Q is a sunny nonexpansive retraction of C onto $\mathrm{F}(T)$.

Remark 4.6. Corollary 4.5 is a generalization of [24, Theorem 4]; see [12, Remark 3.7].

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K. Aoyama
Aoyama Mathematical Laboratory, Inage-ku, Chiba, Chiba 263-0043, Japan
E-mail address: `aoyama@bm.skr.jp`

M. Toyoda
Department of Information Science, Toho University, Miyama, Funabashi, Chiba 274-8510, Japan
E-mail address: `mss-toyoda@is.sci.toho-u.ac.jp`