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# A WEAK CONVERGENCE THEOREM UNDER MANN'S ITERATION FOR GENERALIZED NONEXPANSIVE MAPPINGS IN A BANACH SPACE

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Abstract. In this paper, using the idea of Mann's iteration, we prove a weak convergence theorem for finding a common element of the fixed point sets of two generalized nonexpansive mappings and the zero point set of a maximal monotone operator in a Banach space. We apply this theorem to get well-known and new weak convergence theorems which are connected with generalized nonexpansive mappings and maximal monotone operators in Hilbert spaces and in Banach spaces.

## 1. Introduction

Let $E$ be a smooth Banach space and let $E^*$ be the dual space of $E$. Let $C$ be a nonempty subset of $E$ and let $T$ be a mapping of $C$ into $E$. We denote the set of fixed points of $T$ by $F(T)$. A mapping $T : C \to E$ is called generalized nonexpansive [6] if $F(T) \neq \emptyset$ and

$$\phi(Tx, z) \leq \phi(x, z), \quad \forall x \in C,\ z \in F(T),$$

where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in E$ and $J$ is the duality mapping of $E$.

In 1953, Mann [17] introduced the following iteration process. Let $C$ be a nonempty, closed and convex subset of a Banach space $E$ and let $T : C \to C$ be a nonexpansive mapping, that is, $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. For an initial guess $x_1 \in C$, an iteration process $\{x_n\}$ is defined recursively by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. Later, Reich [20] discussed Mann's iteration process in a uniformly convex Banach space with a Fréchet differentiable norm and he obtained that the sequence $\{x_n\}$ converges weakly to a fixed point of $T$ under some conditions. On the other hand, Ibaraki and Takahashi [7] proved a weak convergence theorem under Mann's iteration process for generalized nonexpansive mappings in a smooth and uniformly convex Banach space.

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In this paper, using the idea of Mann's iteration, we prove a weak convergence theorem for finding a common element of the fixed point sets of two generalized nonexpansive mappings and the zero point set of a maximal monotone operator in a Banach space. We apply this result to get well-known and new weak convergence theorems which are connected with generalized nonexpansive mappings and maximal monotone operators in Hilbert spaces and in Banach spaces.

## 2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. We have from [24] that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.1}$$

for all $x, y \in H$ and $\lambda \in \mathbb{R}$. Furthermore, we have that for $x, y, u, v \in H$,

$$2\langle x-y, u-v\rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \tag{2.2}$$

Let $C$ be a nonempty, closed and convex subset of a Hilbert space $H$. The nearest point projection of $H$ onto $C$ is denoted by $P_C$, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such $P_C$ is called the metric projection of $H$ onto $C$. We know that the metric projection $P_C$ is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle \tag{2.3}$$

for all $x, y \in H$. Furthermore $\langle x - P_C x, y - P_C x\rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [24].

Let $E$ be a real Banach space with norm $\|\cdot\|$ and let $E^*$ be the dual space of $E$. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in $E$, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus $\delta_E$ of convexity of $E$ is defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}$$

for every $\epsilon$ with $0 \leq \epsilon \leq 2$. A Banach space $E$ is said to be uniformly convex if $\delta_E(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. The duality mapping $J_E$ from $E$ into $2^{E^*}$ is defined by

$$J_E x = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. We also denote $J_E$ by $J$ simply. Let $U = \{x \in E : \|x\| = 1\}$. The norm of $E$ is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.4}$$

exists. In this case, $E$ is called smooth. We know that $E$ is smooth if and only if $J$ is a single-valued mapping of $E$ into $E^*$. The norm of $E$ is said to be Fréchet differentiable if for each $x \in U$, the limit (2.4) is attained uniformly for $y \in U$. The norm of $E$ is said to be uniformly smooth if the limit (2.4) is attained uniformly for $x, y \in U$. If $E$ is uniformly smooth, then $J$ is uniformly norm-to-norm continuous on each bounded subset of $E$. We also know that $E$ is reflexive if and only if $J$ is surjective, and $E$ is strictly convex if and only if $J$ is one-to-one. Therefore, if $E$ is a smooth, strictly convex and reflexive Banach space, then $J$ is a single-valued bijection and in this case, the inverse mapping $J^{-1}$ coincides with the duality mapping $J_*$ on $E^*$. For more details, see [22, 23]. We know the following result.

**Lemma 2.1** ([22])**.** *Let $E$ be a smooth Banach space and let $J$ be the duality mapping on $E$. Then, $\langle x-y, Jx-Jy\rangle \geq 0$ for all $x, y \in E$. Furthermore, if $E$ is strictly convex and $\langle x-y, Jx-Jy\rangle = 0$, then $x = y$.*

Let $E$ be a smooth Banach space. The function $\phi\colon E \times E \to (-\infty, \infty)$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \tag{2.5}$$

for $x, y \in E$, where $J$ is the duality mapping of $E$; see [1, 12]. We have from the definition of $\phi$ that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle \tag{2.6}$$

for all $x, y, z \in E$. From $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$, we can see that $\phi(x, y) \geq 0$. Furthermore, we can obtain the following equality:

$$2\langle x - y, Jz - Jw\rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w) \tag{2.7}$$

for $x, y, z, w \in E$. If $E$ is additionally assumed to be strictly convex, then from Lemma 2.1 we have

$$\phi(x, y) = 0 \Longleftrightarrow x = y. \tag{2.8}$$

Let $E$ be a smooth, strictly convex and reflexive Banach space. Let $\phi_*\colon E^* \times E^* \to (-\infty, \infty)$ be the function defined by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^*\rangle + \|y^*\|^2$$

for all $x^*, y^* \in E^*$, where $J$ is the duality mapping of $E$. It is easy to see that

$$\phi(x, y) = \phi_*(Jy, Jx) \tag{2.9}$$

for all $x, y \in E$. The following lemma which was by Kamimura and Takahashi [12] is well-known.

**Lemma 2.2** ([12])**.** *Let $E$ be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in $E$ such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

The following lemmas are in Xu [28] and Kamimura and Takahashi [12].

**Lemma 2.3** ([28])**.** *Let $E$ be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

*for all $x, y \in B_r$ and $\lambda$ with $0 \leq \lambda \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$.*

**Lemma 2.4** ([12]). *Let $E$ be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \leq \phi(x, y)$$

*for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \leq r\}$.*

Let $E$ be a smooth Banach space and let $C$ be a nonempty subset of $E$. Then a mapping $T : C \to E$ is called *generalized nonexpansive* [6] if $F(T) \neq \emptyset$ and

$$\phi(Tx, y) \leq \phi(x, y)$$

for all $x \in C$ and $y \in F(T)$. Let $D$ be a nonempty subset of a Banach space $E$. A mapping $R : E \to D$ is said to be *sunny* [19] if

$$R(Rx + t(x - Rx)) = Rx$$

for all $x \in E$ and $t \geq 0$. A mapping $R : E \to D$ is said to be a *retraction* or a *projection* if $Rx = x$ for all $x \in D$. A nonempty subset $D$ of a smooth Banach space $E$ is said to be a *generalized nonexpansive retract* (resp. *sunny generalized nonexpansive retract*) of $E$ if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) $R$ from $E$ onto $D$; see [5, 6] for more details. The following results are in Ibaraki and Takahashi [6].

**Lemma 2.5** ([6]). *Let $C$ be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space $E$. Then the sunny generalized nonexpansive retraction from $E$ onto $C$ is uniquely determined.*

**Lemma 2.6** ([6]). *Let $C$ be a nonempty and closed subset of a smooth and strictly convex Banach space $E$ such that there exists a sunny generalized nonexpansive retraction $R$ from $E$ onto $C$ and let $(x, z) \in E \times C$. Then the following hold:*

(i) *$z = Rx$ if and only if $\langle x - z, Jy - Jz \rangle \leq 0$ for all $y \in C$;*
(ii) *$\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.*

In 2007, Kohsaka and Takahashi [14] proved the following results:

**Lemma 2.7** ([14]). *Let $E$ be a smooth, strictly convex and reflexive Banach space and let $C$ be a nonempty and closed subset of $E$. Then the following are equivalent:*

(a) *$C$ is a sunny generalized nonexpansive retract of $E$;*
(b) *$C$ is a generalized nonexpansive retract of $E$;*
(c) *$JC$ is closed and convex.*

**Lemma 2.8** ([14]). *Let $E$ be a smooth, strictly convex and reflexive Banach space and let $C$ be a nonempty closed sunny generalized nonexpansive retract of $E$. Let $R$ be the sunny generalized nonexpansive retraction from $E$ onto $C$ and let $(x, z) \in E \times C$. Then the following are equivalent:*

(i) *$z = Rx$;*
(ii) *$\phi(x, z) = \min_{y \in C} \phi(x, y)$.*

Using Lemma 2.7 , we also have the following result.

**Lemma 2.9.** *Let $E$ be a smooth, strictly convex and reflexive Banach space and let $\{C_\alpha\}$ be a family of sunny generalized nonexpansive retracts of $E$. Then $\cap_\alpha C_\alpha$ is a sunny generalized nonexpansive retract of $E$.*

Let $E$ be a Banach space and let $B$ be a mapping of of $E$ into $2^{E^*}$. A multi-valued mapping $B$ on $E$ is said to be monotone if $\langle x-y, u^*-v^*\rangle \geq 0$ for all $u^* \in Bx$, and $v^* \in By$. A monotone operator $B$ on $E$ is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on $E$. The following theorem is due to Browder [2]; see also [23, Theorem 3.5.4].

**Theorem 2.10** ([2])**.** *Let $E$ be a smooth, strictly convex and reflexive Banach space and let $J$ be the duality mapping of $E$ into $E^*$. Let $B$ be a monotone operator of $E$ into $2^{E^*}$. Then $B$ is maximal if and only if for any $r>0$,*

$$R(J+rB)=E^*,$$

*where $R(J+rB)$ is the range of $J+rB$.*

Let $E$ be a smooth, strictly convex and reflexive Banach space and let $B$ be a maximal monotone operator of $E$ into $2^{E^*}$. The set of null points of a maximal monotone operator $B$ is defined by $B^{-1}0=\{z\in E: 0\in Bz\}$. We know that $B^{-1}0$ is closed and convex; see [23]. Let $B \subset E^* \times E$ be a maximal monotone operator. For all $x\in E$ and $r>0$, we consider the following equation

$$x \in x_r + rBJx_r.$$

This equation has a unique solution $x_r$. We define $J_r$ by $x_r = J_r x$. Such $J_r, r>0$ is called the generalized nonexpansive resolvent [6, 8] of $B$.

From [6], we have thr following result for generalized nonexpansive resolvents in a Banach space.

**Lemma 2.11** ([6])**.** *Let $E$ be a smooth, strictly convex and reflexive Banach space and let $B \subset E^* \times E$ be a maximal monotone operator. Let $r>0$ and let $J_r$ be the generalized nonexpansive resolvent of $B$. Then the following hold:*

(i) $F(J_r)=(BJ)^{-1}0$;
(ii) $\phi(x, J_r x)+\phi(J_r x, p) \leq \phi(x,p)$. $\forall x\in E,\ p\in (BJ)^{-1}0$.

Furthermore, we can prove the following result for generalized nonexpansive resolvents in a Banach space.

**Lemma 2.12.** *Let $E$ be a smooth, strictly convex and reflexive Banach space and let $B \subset E^* \times E$ be a maximal monotone operator. Let $r>0$ and let $J_r$ be the generalized nonexpansive resolvent of $B$. Then*

$$\langle x - J_r x - (y - J_r y), JJ_r x - JJ_r y\rangle \geq 0 \tag{2.10}$$

*for all $x,y\in E$ and $r>0$,*

*Proof.* Let $x,y\in E$ and $r>0$. Put $x_r = J_r x$ and $y_r = J_r y$. Then we have that

$$x \in x_r + rBJx_r \text{ and } y \in y_r + rBJy_r.$$

Therefore, we get that

$$\frac{x-x_r}{r} \in BJx_r \text{ and } \frac{y-y_r}{r} \in BJy_r.$$

From the definition of $B$, we have that

$$\left\langle \frac{x-x_r}{r} - \frac{y-y_r}{r}, Jx_r - Jy_r \right\rangle \geq 0.$$

Since $r > 0$, we get that

$$\langle x - x_r - (y - y_r), Jx_r - Jy_r \rangle \geq 0.$$

This completes the proof. □

## 3. Weak convergence theorem

In this section, we prove a weak convergence theorem of Mann's type iteration for generalized nonexpansive mappings and maximal monotone operators in a Banach space. Let $E$ be a smooth Banach space and let $J_E$ be the duality mapping of $E$. Let $D$ be a nonempty, closed and convex subset of $E$. A mapping $U : D \to E$ is called generalised demiclosed if for a sequence $\{x_n\}$ in $D$ such that $J_E x_n \rightharpoonup J_E p$ and $J_E x_n - J_E U x_n \to 0$, it holds that $p = Up$. The following lemma was proved by Matsushita and Takahashi [18].

**Lemma 3.1** ([18])**.** *Let $E$ be a smooth, strictly convex and reflexive Banach space and let $C$ be a nonempty, closed and convex subset of $E$. Let $T : C \to E$ be a mapping with $F(T) \neq \emptyset$ satisfying the following;*

$$\phi(z, Tx) \leq \phi(z, x), \quad \forall x \in C,\ z \in F(T). \tag{3.1}$$

*Then $F(T)$ is closed and convex.*

Using this result, we can prove the following lemma.

**Lemma 3.2** ([9, 11])**.** *Let $E$ be a smooth, strictly convex and reflexive Banach space and let $C$ be a nonempty, closed and convex subset of $E$ such that $J_E C$ is closed and convex. Let $T : C \to C$ be a generalized nonexpansive mapping with $F(T) \neq \emptyset$, that is;*

$$\phi(Tx, z) \leq \phi(x, z), \quad \forall x \in C,\ z \in F(T). \tag{3.2}$$

*Then $F(T)$ is a sunny generalized nonexpansive retract of $E$.*

*Proof.* Define the duality mapping $T^*$ of $T$ by $T^+ = J_E T J_E^{-1}$. Then we have that $T$ is a mapping of $J_E C$ into itself; see [4, 26]. We prove $J_E F(T) = F(T^*)$. In fact, we have that

$$\begin{aligned} z^* \in & J_E F(T) \Longleftrightarrow z^* = J_E z,\ z \in F(T) \\ & \Longleftrightarrow z^* = J_E T z = J_E T J_E^{-1} J_E z = T^* J_E z = T^* z^. \\ & \Longleftrightarrow z^* \in F(T^*). \end{aligned}$$

Furthermore, we have that, for $z^* = J_E z \in F(T^*)$ and $x^* = J_E x \in J_E C$,

$$\phi_*(z^*, T^* x^*) = \phi_*(J_E z, J_E T J_E^{-1} J_E x)$$

$$
\begin{aligned}
&= \phi(TJ_E^{-1}J_E x, z) = \phi(Tx, z) \\
&\leq \phi(x, z) = \phi_*(J_E z, J_E x) \\
&= \phi_*(z^*, x^*)
\end{aligned}
$$

and hence $T^+$ satisfies (3.2). From Lemma 3.1, we have that $F(T^*) = J_E F(T)$ is closed and convex and hence $F(T)$ is a sunny generalized nonexpansive retract of $E$. $\square$

The following is our main result.

**Theorem 3.3.** *Let $E$ be a uniformly convex and uniformly smooth Banach space which the duality mapping $J_E$ is weakly sequentially continuous. Let $C$ be a nonempty, closed and convex subset of E such that $J_E C$ is closed and convex. Let $A \subset E^* \times E$ be a maximal monotone operator satisfying $D(A) \subset J_E C$ and let $J_\mu$ be the generalized nonexpansive resolvent of $A$, i.e., $J_\mu = (I + \mu A J_E)^{-1}$ for all $\mu > 0$. Let $T$ and $U$ be generalized nonexpansive mappings of $C$ inti itself such that $T$ and $U$ are generalized demiclosed. Suppose that*

$$\Omega = F(T) \cap F(U) \cap (AJ_E)^{-1}0 \neq \emptyset.$$

*For any $x_1 = x \in C$, define $\{x_n\}$ as follows:*

$$
\begin{cases}
y_n = (1 - r_n)x_n + r_n U J_{\mu_n} x_n, \\
x_{n+1} = (1 - \beta_n)x_n + \beta_n T y_n, \quad \forall n \in \mathbb{N},
\end{cases}
$$

*where $\{\mu_n\} \subset (0, \infty)$, $\{\beta_n\} \subset (0, 1)$, $a, b, \delta, \gamma \in \mathbb{R}$ and $\{r_n\} \subset (0, 1)$ satisfy the following:*

$$0 < a \leq \beta_n \leq b < 1, \ 0 < \delta \leq r_n \leq \gamma < 1 \quad \text{and} \quad 0 < c \leq \mu_n, \quad \forall n \in \mathbb{N}.$$

*Then the sequence $\{x_n\}$ converges weakly to an elementt $z_0 \in \Omega$, where $z_0 = \lim_{n\to\infty} R_\Omega x_n$ anf $R_\Omega$ is the sunny generalized nonexpansive retraction of $E$ onto $\Omega$.*

*Proof.* Since $T$ and $U$ are generalized nonexpansive, we have that from Lemma 3.2 that $F(T)$ and $F(U)$ are sunny generalized nonexpansive retracts of $E$. From Lemma 2.11, we have that $(AJ_E)^{-1}0$ is a sunny generalized nonexpansive retract of $E$. Then, from Lemma 2.9,

$$\Omega = F(T) \cap F(U) \cap (AJ_E)^{-1}0$$

is a sunny generalized nonexpansive retract of $E$ and hence there exists a unique sunny generalized nonexpansive retraction of $E$. We define by $R_\Omega$ this retraction.

Let $z \in \Omega$. Then we have that $z = Tz$, $z = Uz$ and $J_{\mu_n} z = z$ for all $n \in \mathbb{N}$. Put $y_n = (1 - r_n)x_n + r_n U J_{\mu_n} x_n$ and $z_n = J_{\mu_n} x_n$ for all $n \in \mathbb{N}$. We have that, for $z \in \Omega$,

$$
\begin{aligned}
\phi(y_n, z) &= \phi((1 - r_n)x_n + r_n U z_n, z) \\
&= \|(1 - r_n)x_n + r_n U z_n\|^2 \\
&\quad - 2\langle (1 - r_n)x_n + r_n U z_n, J_E z\rangle + \|z\|^2 \\
&\leq (1 - r_n)\|x_n\|^2 + r_n \|U z_n\|^2 \qquad (3.3)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - 2(1-r_n)\langle x_n, J_E z\rangle - 2r_n\langle Uz_n, J_E z\rangle + \|z\|^2 \\
&= (1-r_n)\phi(x_n, z) + r_n\phi(Uz_n, z) \\
&\le (1-r_n)\phi(x_n, z) + r_n\phi(x_n, z) \\
&= \phi(z, x_n).
\end{aligned}
$$

Similarly, we have that

$$
\begin{aligned}
\phi(x_{n+1}, z) &= \phi((1-\beta_n)x_n + \beta_n Ty_n, z) \\
&\le (1-\beta_n)\phi(x_n, z) + \beta_n\phi(Ty_n, z) \\
&\le (1-\beta_n)\phi(x_n, z) + \beta_n\phi(y_n, z) \\
&\le (1-\beta_n)\phi(x_n, z) + \beta_n\phi(x_n, z) \\
&= \phi(x_n, z).
\end{aligned} \tag{3.4}
$$

Then $\lim_{n\to\infty}\phi(x_n, z)$ exists. Thus $\{x_n\}$, $\{Uz_n\}$, $\{y_n\}$ and $\{Ty_n\}$ are bounded. Putting

$$
r = \max\left\{\sup_{n\in\mathbb{N}}\|x_n\|,\ \sup_{n\in\mathbb{N}}\|Uz_n\|,\ \sup_{n\in\mathbb{N}}\|Ty_n\|\right\},
$$

we have from Lemma 2.3 that there exists a strictly increasing, continuous and convex function $g : [0,\infty) \to [0,\infty)$ such that $g(0) = 0$ and

$$
\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|)
$$

for all $x, y \in B_r$ and $\lambda$ with $0 \le \lambda \le 1$, where $B_r = \{z \in E : \|z\| \le r\}$. Using this, we have that for $n \in \mathbb{N}$ and $z \in \Omega$

$$
\begin{aligned}
\phi(y_n, z) &= \phi((1-r_n)x_n + r_n Uz_n, z) \\
&= \|(1-r_n)x_n + r_n Uz_n\|^2 \\
&\quad - 2\langle(1-r_n)x_n + r_n Uz_n, J_E z\rangle + \|z\|^2 \\
&\le (1-r_n)\|x_n\|^2 + r_n\|Uz_n\|^2 - r_n(1-r_n)g(\|x_n - Uz_n\|) \\
&\quad - 2\langle(1-r_n)x_n + r_n Uz_n, J_E z\rangle + \|z\|^2 \\
&= (1-r_n)\phi(x_n, z) + r_n\phi(Uz_n, z) \\
&\quad - r_n(1-r_n)g(\|x_n - Uz_n\|) \\
&\le (1-r_n)\phi(x_n, z) + r_n\phi(x_n, z) \\
&\quad - r_n(1-r_n)g(\|x_n - Uz_n\|) \\
&= \phi(x_n, z) - r_n(1-r_n)g(\|x_n - Uz_n\|).
\end{aligned}
$$

Similarly, we have that

$$
\begin{aligned}
\phi(x_{n+1}, z) &= \phi((1-\beta_n)x_n + \beta_n Ty_n, z) \\
&= \|(1-\beta_n)x_n + \beta_n Ty_n\|^2 \\
&\quad - 2\langle(1-\beta_n)x_n + \beta_n Ty_n, J_E z\rangle + \|z\|^2 \\
&\le (1-\beta_n)\|x_n\|^2 + \beta_n\|Ty_n\|^2 - \beta_n(1-\beta_n)g(\|x_n - Ty_n\|) \\
&\quad - 2\langle(1-\beta_n)x_n + \beta_n Ty_n, J_E z\rangle + \|z\|^2
\end{aligned}
$$

$$
\begin{aligned}
&= (1-\beta_n)\phi(x_n, z) + \beta_n\phi(Ty_n, z) \\
&\quad - \beta_n(1-\beta_n)g(\|x_n - Ty_n\|) \\
&\leq (1-\beta_n)\phi(x_n, z) + \beta_n\phi(y_n, z) \\
&\quad - \beta_n(1-\beta_n)g(\|x_n - Ty_n\|) \\
&\leq (1-\beta_n)\phi(x_n, z) \\
&\quad + \beta_n\big(\phi(x_n, z) - r_n(1-r_n)g(\|x_n - Uz_n\|)\big) \\
&\quad - \beta_n(1-\beta_n)g(\|x_n - Ty_n\|) \\
&= \phi(x_n, z) - \beta_n r_n(1-r_n)g(\|x_n - Uz_n\|)\big) \\
&\quad - \beta_n(1-\beta_n)g(\|x_n - Ty_n\|).
\end{aligned}
$$

Therefore, we have that

$$\beta_n(1-\beta_n)g(\|x_n - Ty_n\|). \leq \phi(x_n, z) - \phi(x_{n+1}, z)$$

and

$$\beta_n r_n(1-r_n)g(\|x_n - Uz_n\|) \leq \phi(x_n, z) - \phi(x_{n+1}, z).$$

We have from $0 < a \leq \beta_n \leq b < 1$ and $0 < \delta \leq r_n \leq \gamma < 1$ that

$$\lim_{n\to\infty} g(\|x_n - Ty_n\|) = 0 \text{ and } \lim_{n\to\infty} g(\|x_n - Uz_n\|) = 0. \tag{3.5}$$

From the properties of $g$, we have that

$$\lim_{n\to\infty} \|x_n - Ty_n\| = 0. \text{ and } \lim_{n\to\infty} \|x_n - Uz_n\| = 0. \tag{3.6}$$

From the definition of $y_n$, we also have that

$$\|x_n - y_n\| \leq r_n\|x_n - Uz_n\|.$$

Since $\lim_{n\to\infty} \|x_n - Uz_n\| = 0$, we have that $\|x_n - y_n\| \to 0$ and hence $\|y_n - Ty_n\| \to 0$. Since $E$ is uniformly smooth, we have that

$$\|J_E y_n - J_E Ty_n\| \to 0 \quad \text{and } \|J_E x_n - J_E Uz_n\| \to 0. \tag{3.7}$$

Using $z_n = J_{\mu_n} x_n$ and Lemma 2.11, we have that, for $z \in \Omega$,

$$\phi(x_n, z_n) = \phi(x_n, J_{\mu_n} x_n) \leq \phi(x_n, z) - \phi(J_{\mu_n} x_n, z) = \phi(x_n, z) - \phi(z_n, z).$$

It follows from (3.3) that

$$
\begin{aligned}
\phi(x_n, z_n) &\leq \phi(x_n, z) - \phi(z_n, z) \\
&\leq \phi(x_n, z) - \frac{1}{r_n}\big(\phi(y_n, z) - (1-r_n)\phi(x_n, z)\big) \\
&= \frac{1}{r_n}\big(\phi(x_n, z) - \phi(y_n, z)\big) \\
&= \frac{1}{r_n}\big(\|x_n\|^2 - \|y_n\|^2 - 2\langle x_n - y_n, J_E z\rangle\big) \\
&\leq \frac{1}{r_n}\big(|\|x_n\|^2 - \|y_n\|^2| + 2|\langle x_n - y_n, J_E z\rangle|\big) \\
&\leq \frac{1}{r_n}\big(|\|x_n\| - \|y_n\||(\|x_n\| + \|y_n\|) + 2\|z\|\|x_n - y_n\|\big)
\end{aligned}
$$

$$\leq \frac{1}{r_n}\big(\|x_n - y_n\|(\|x_n\| + \|y_n\|) + 2\|z\|\|x_n - y_n\|\big).$$

Since $\|x_n - y_n\| \to 0$, we have that $\lim_{n\to\infty} \phi(z_n, x_n) = 0$. Since $E$ is uniformly convex and smooth, we have from Lemma 2.2 that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.8}$$

Since

$$\|z_n - Uz_n\| \leq \|z_n - x_n\| + \|x_n - Uz_n\|,$$

we obtain that

$$\lim_{n\to\infty} \|z_n - Uz_n\| = 0. \tag{3.9}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to $w$. Since the duality mapping $J_E$ is weakly sequentially continuous, $\{J_E x_{n_i}\}$ convers weakly to $J_E w$ Using $\lim_{n\to\infty} \|J_E x_n - J_E U x_n\| = 0$ and $U$ is generalized demiclosed, we have that $w = Uw$ and hence $w \in F(U)$. Since $T$ is generalized demiclosed, we have from $J_E y_{n_i} \rightharpoonup J_E w$.and $\|J_E y_n - J_E T y_n\| \to 0$ that $w \in F(T)$. This implies that $w \in F(T) \cap F(U)$. Next, we show $w \in (AJ_E)^{-1}0$. Since $\lim_{n\to\infty} \|x_n - z_n\| = 0$. we have from $\mu_n \geq c$ that

$$\lim_{n\to\infty} \frac{1}{\mu_n}\|x_n - z_n\| = 0.$$

Since the duality mapping $J_E$ is weakly sequentially continuous, we have from $\|x_n - z_n\| \to 0$ that $J_E z_{n_i} \rightharpoonup J_E w$. We also have from $z_n = J_{\mu_n} x_n$ that

$$\frac{x_n - z_n}{\mu_n} \in AJ_E z_n.$$

For $(p^*, p) \in A$, from the monotonicity of $A$, we have $\langle p - \frac{x_n - z_n}{\mu_n}, p^* - J_E z_n\rangle \geq 0$ for all $n \geq 0$. Replacing $n$ by $n_i$ and letting $i \to \infty$, we get $\langle p, p^* - J_E w\rangle \geq 0$. From the maximallity of $A$, we have $0 \in AJ_E w$ and hence $w \in (AJ_E)^{-1}0$, Therefore, $w \in \Omega$.

We next show that if $x_{n_i} \rightharpoonup u$ and $x_{n_j} \rightharpoonup v$, then $u = v$. In fact, we have that $u, v \in \Omega$. Put $a = \lim_{n\to\infty}(\phi(x_n, u) - \phi(x_n, v)$. Since

$$\phi(x_n, u) - \phi(x_n, v) = 2\langle x_n, J_E v - J_E u\rangle + \|u\|^2 - \|v\|^2,$$

we have $a = 2\langle u, J_E v - J_E u\rangle + \|u\|^2 - \|v\|^2$ and $a = 2\langle v, J_E v - J_E u\rangle + \|u\|^2 - \|v\|^2$. From these equalities, we obtain $2\langle u - v, Jv - Ju\rangle = 0$ and hence $\langle u - v, Ju - Jv\rangle = 0$. From Lemma 2.1, it follows that $u = v$. Therefore, $\{x_n\}$ converges weakly to an element $z_0 \in \Omega$

Putting $R = R_\Omega$, we hava from Lemma 2.6 and (3.4) that

$$\begin{aligned} \phi(x_{n+1}, Rx_{n+1}) &\leq \phi(x_{n+1}, Rx_{n+1}) + \phi(Rx_{n+1}, Rx_n) \\ &\leq \phi(x_{n+1}, Rx_n) \\ &\leq \phi(x_n, Rx_n) \end{aligned}$$

for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} \phi(x_n, Rx_n)$ exists. It follows from Lemma 2.6 that, for $k \in \mathbb{N}$,

$$\begin{aligned}\phi(x_{n+k}, Rx_n) &= \phi(x_{n+k}, Rx_{n+k}) + \phi(Rx_{n+k}, Rx_n)\\ &\quad + 2\langle x_{n+k} - Rx_{n+k}, J_E Rx_{n+k} - J_E Rx_n\rangle\\ &\geq \phi(x_{n+k}, Rx_{n+k}) + \phi(Rx_{n+k}, Rx_n)\end{aligned}$$

and hence

$$\phi(Rx_{n+k}, Rx_n) \leq \phi(x_{n+k}, Rx_n) - \phi(x_{n+k}, Rx_{n+k}) \leq \phi(x_n, Rx_n) - \phi(x_{n+k}, Rx_{n+k}).$$

We also have from Lemma 2.6 that, for $p \in \Omega$,

$$\phi(Rx_n, p) \leq \phi(Rx_n, p) + \phi(x_n, Rx_n) \leq \phi(x_n, p) \leq \phi(x, p)$$

and hence $\{Rx_n\}$ is bounded. Using Lemma 2.4, we have that, for $m, n \in \mathbb{N}$ with $m > n$,

$$g'(\|Rx_n - Rx_m\|) \leq \phi(Rx_n, Rx_m) \leq \phi(x_n, Rx_n) - \phi(x_m, Rx_m),$$

where $g'$ is a strictly increasing, continuous and convex function such that $g'(0) = 0$. The the properties of $g'$ yieeld that $\{Rx_n\}$ ia a Cauchy sequence. Since $E$ is complete, $\{Rx_n\}$ converges strongly to a point $u \in \Omega$. Furthermore, we have from Lemma 2.6 that

$$\langle x_n - Rx_n, J_E Rx_n - J_E z_0\rangle \geq 0.$$

Since $x_n \rightharpoonup z_0$, we have that

$$\langle z_0 - u, J_E u - J_E z_0\rangle \geq 0$$

and hence $\phi(z_0, u) + \phi((u, z_0) \leq 0$. This implies that $\phi(z_0, u) = \phi(u, z_0) = 0$ and hence $u = z_0$. Therefore, $z_0 = \lim_{n\to\infty} Rx_n = \lim_{n\to\infty} R_\Omega x_n$. This completes the proof. □

## 4. Applications

In this section, using Theorem 3.3, we get well-known and new weak convergence theorems of Mann's type iteration which are connected with generalized nonexpansive mappings and maximal monotone operators in Hilbert spaces and in Banach spaces. We first prove a weak convergence theorem for finding a zero point of a maximal monotone operator in a Banach space.

**Theorem 4.1.** *Let $E$ be a uniformly convex and uniformly smooth Banach space which the duality mapping $J_E$ is weakly sequentially continuous. Let $A \subset E^* \times E$ be a maximal monotone operator and let $J_\mu$ be a generalized nonexpansive resolvent of $A$, i.e., $J_\mu = (I + \mu A J_E)^{-1}$ for all $\mu > 0$. Suppose that $(AJ_E)^{-1}0 \neq \emptyset$. For any $x_1 = x \in E$, define $\{x_n\}$ as follows:*

$$x_{n+1} = J_E^{-1}\big((1 - r_n)J_E x_n + r_n J_E J_{\mu_n} x_n\big)$$

*for all $n \in \mathbb{N}$, where $\{\mu_n\} \subset (0, \infty)$, $\delta, \gamma \in \mathbb{R}$ and $\{r_n\} \subset (0, 1)$ satisfy the following:*

$$0 < \delta \leq r_n \leq \gamma < 1 \quad and \quad 0 < c \leq \mu_n, \quad \forall n \in \mathbb{N}.$$

*Then the sequence* $\{x_n\}$ *converges weakly to an elementt* $z_0 \in (AJ_E)^{-1}0$*, where* $z_0 = \lim_{n\to\infty} R_{(AJ_E)^{-1}0}x_n$.

*Proof.* Putting $C = E$ and $T = U = I$ in Theorem 3.3, we obtain the desired result from Theorem 3.3. □

Let $E$ be a Banach space and let $f : E \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex function. Define the subdifferential of $f$ as follows:

$$\partial f(x) = \{x^* \in E^* : f(y) \geq \langle y - x, x^*\rangle + f(x),\ \forall y \in E\}$$

for all $x \in E$. Then we know that $\partial f$ is a maximal monotone operator; see [21] for more details. Let $E$ be a smooth, strictly convex and reflexive Banach space. Let $C$ be a nonempty and closed subset of $E$ such that $J_E C$ is closed and convex. We have from Lemma 2.7 that there exists the sunny generalized nonexpansive retraction $R_C$ of $E$ onto $C$. We also have that, for the indicator function $i_{J_E C}$, that is,

$$i_{J_E C}x^* = \begin{cases} 0, & x^* \in J_E C, \\ \infty, & x^* \notin J_E C, \end{cases}$$

the subdifferential $\partial i_{J_E C} \subset E^* \times E$ is a maximal monotone operator and the generalized nonexpansive resolvent $J_r = R_C$ of $\partial i_{J_E C}$ for every $r > 0$. In fact, for any $x \in E$ and $r > 0$, we have from Lemma 2.8 that

$$\begin{aligned} z = J_r x &\Leftrightarrow z + r\partial i_{J_E C} J_E(z) \ni x \\ &\Leftrightarrow x - z \in r\partial i_{J_E C} J_E(z) \\ &\Leftrightarrow i_{J_E C}(y) \geq \left\langle J_E y - J_E z, \frac{x - z}{r}\right\rangle + i_{J_E C}(z),\ \forall y \in E \\ &\Leftrightarrow 0 \geq \langle J_E y - J_E z, x - z\rangle,\ \forall y \in C \\ &\Leftrightarrow z = \arg\min_{y\in C} \phi(x, y) \\ &\Leftrightarrow z = R_C x. \end{aligned} \tag{4.1}$$

Using (4.1) and Theorem 3.3, we get the following weak convergence theorem for two generalized nonexpansive mappings in a Banach space.

**Theorem 4.2.** *Let* $E$ *be a uniformly convex and uniformly smooth Banach space which the duality mapping* $J_E$ *is weakly sequentially continuous. Let* $C$ *be a nonempty, closed and convex subset of* $E$ *such that* $J_E C$ *is closed and convex. Let* $T$ *and* $U$ *be generalized nonexpansive mappings of* $C$ *into itself such that*

$$\Omega = F(T) \cap F(U) \neq \emptyset.$$

*For any* $x_1 = x \in C$*, define* $\{x_n\}$ *as follows:*

$$\begin{cases} y_n = J_E^{-1}\big((1 - r_n)J_E x_n + r_n J_E U x_n\big), \\ x_{n+1} = J_E^{-1}\big((1 - \beta_n)J_E x_n + \beta_n J_E T y_n\big), \quad \forall n \in \mathbb{N}, \end{cases}$$

*where* $\{\beta_n\} \subset (0, 1)$*,* $a, b, \delta, \gamma \in \mathbb{R}$ *and* $\{r_n\} \subset (0, 1)$ *satisfy the following:*

$$0 < a \leq \beta_n \leq b < 1 \quad \textit{and} \quad 0 < \delta \leq r_n \leq \gamma < 1, \quad \forall n \in \mathbb{N}.$$

*Then the sequence* $\{x_n\}$ *converges weakly to an element* $z_0 \in \Omega$*, where* $z_0 = \lim_{n\to\infty} R_\Omega x_n$*.*

*Proof.* Putting $A = \partial i_{J_E C}$ in Theorem 3.3, we obtain that $J_{\mu_n} = R_C$ for all $\mu_n > 0$. Therefore, we obtain the desired result from Theorem 3.3. □

Let $H$ be a Hilbert space and let $C$ be a nonempty, closed and convex subset of $H$. A mapping $U : C \to H$ is called generalized hybrid [13] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Ux - Uy\|^2 + (1-\alpha)\|x - Uy\|^2 \le \beta\|Ux - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$. Such a mapping $U$ is called ($\alpha$, $\beta$)-generalized hybrid. Notice that the class of ($\alpha$, $\beta$)-generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive. It is nonspreading [15, 16] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Ux - Uy\|^2 \le \|Ux - y\|^2 + \|Uy - x\|^2, \quad \forall x, y \in C.$$

It is also hybrid [25] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Ux - Uy\|^2 \le \|x - y\|^2 + \|Ux - y\|^2 + \|Uy - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [10]. Let $k$ be a real number with $0 \le k < 1$. A mapping $S : C \to H$ is called a $k$-strict pseudo-contraction [3] if

$$\|Sx - Sy\|^2 \le \|x - y\|^2 + k\|x - Sx - (y - Sy)\|^2$$

for all $x, y \in C$. Putting $U = kI + (1-k)S$, we have that $U$ is nonexpansive. In fact, we have from (2.1) that, for $x, y \in C$ and $k \in \mathbb{R}$ with $0 \le k < 1$,

$$\begin{aligned}
\|Ux - Uy\|^2 &= \|(kI + (1-k)S)x - (kI + (1-k)S)y\|^2 \\
&= \|k(x - y) + (1-k)(Sx - Sy)\|^2 \\
&= k\|x - y\|^2 + (1-k)\|Sx - Sy\|^2 - k(1-k)\|x - y - (Sx - Sy)\|^2 \\
&\le k\|x - y\|^2 + (1-k)\|Sx - Sy\|^2 + (1-k)(\|x - y\|^2 - \|Sx - Sy\|^2) \\
&= k\|x - y\|^2 + (1-k)\|x - y\|^2 \\
&= \|x - y\|^2.
\end{aligned}$$

This implies that $U$ is nonexpansive. We also know the following result obtained by Kocourek, Takahashi and Yao [13]; see also [27].

**Lemma 4.3** ([13, 27])**.** *Let* $H$ *be a Hilbert space, let* $C$ *be a nonempty, closed and convex subset of* $H$ *and let* $U : C \to H$ *be generalized hybrid. If* $x_n \rightharpoonup z$ *and* $x_n - Ux_n \to 0$*, then* $z \in F(U)$*.*

The following are two weak convergence theorems for two nonlinear mappings in Hilbert spaces.

**Theorem 4.4.** *Let $H$ bea Hilbert space and let $C$ be a nonempty, closed and convex subset of $H$. Let $A \subset H \times H$ be a maximal monotone operator satisfying $D(A) \subset C$ and let $J_\mu$ be the resolvent of $A$, i.e., $J_\mu = (I + \mu A)^{-1}$ for all $\mu > 0$. Let $T : C \to C$ be a nonspreading mapping and let $U : C \to C$ be a nonepansive mapping. Suppose that*

$$\Omega = F(T) \cap F(U) \cap A^{-1}0 \neq \emptyset.$$

*For any $x_1 = x \in C$, define $\{x_n\}$ as follows:*

$$\begin{cases} y_n = (1 - r_n)x_n + r_n U J_{\mu_n} x_n, \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n T y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

*where $\{\beta_n\} \subset (0, 1)$, $a, b, \delta, \gamma \in \mathbb{R}$ and $\{r_n\} \subset (0, 1)$ satisfy the following:*

$$0 < a \leq \beta_n \leq b < 1, \; 0 < \delta \leq r_n \leq \gamma < 1 \;\; \text{and} \;\; 0 < c \leq \mu_n, \quad \forall n \in \mathbb{N}.$$

*Then the sequence $\{x_n\}$ converges weakly to an element $z_0 \in \Omega$, where $z_0 = \lim_{n\to\infty} P_\Omega x_n$ anf $P_\Omega$ is the metric projection of $H$ onto $\Omega$.*

*Proof.* Since $T$ is nonspreading of $C$ into $C$, it satisfies the following:

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Putting $y = p$ for $p \in F(T)$, we have that

$$2\|Tx - p\|^2 \leq \|Tx - p\|^2 + \|p - x\|^2, \quad \forall x \in C$$

and hence

$$\|Tx - p\|^2 \leq \|p - x\|^2, \quad \forall x \in C.$$

This implies that $T$ is quasi-nonexpansive. Furthermore, we have from Lemma 4.3 that $T$ is demiclosed. On the other hand, since $U$ is a nonexpansive mapping of $C$ into $C$ such that $F(U) \neq \emptyset$, $U$ is quasi-nonexpansive. Furthermore, from Lemma 4.3, $U$ is demiclosed. Therefore, we have the desired result from Theorem 3.3. $\square$

**Theorem 4.5.** *Let $H$ be a Hilbert space and let $C$ b4 a nonempty, closed and convex subset of $H$. Let $A \subset H \times H$ be a maximal monotone operator satisfying $D(A) \subset C$ and let $J_\mu$ be the resolvent of $A$, i.e., $J_\mu = (I + \mu A)^{-1}$ for all $\mu > 0$. Let $k$ be a real number with $0 \leq k < 1$. Let $T : C \to C$ be a hybrid mapping and let $S : C \to C$ be a $k$-strict pseudo-contraction. Suppose that*

$$\Omega = F(T) \cap F(S) \cap A^{-1}0 \neq \emptyset.$$

*For any $x_1 = x \in C$, define $\{x_n\}$ as follows:*

$$\begin{cases} y_n = (1 - r_n)x_n + r_n(kI + (1 - k)S)J_{\mu_n} x_n, \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n T y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

*where $\{\beta_n\} \subset (0, 1)$, $a, b, \delta, \gamma \in \mathbb{R}$ and $\{r_n\} \subset (0, 1)$ satisfy the following:*

$$0 < a \leq \beta_n \leq b < 1, \; 0 < \delta \leq r_n \leq \gamma < 1 \;\; \text{and} \;\; 0 < c \leq \mu_n, \quad \forall n \in \mathbb{N}.$$

*Then the sequence $\{x_n\}$ converges weakly to an elementt $z_0 \in \Omega$, where $z_0 = \lim_{n\to\infty} P_\Omega x_n$ anf $P_\Omega$ is the metric projection of $H$ onto $\Omega$.*

*Proof.* Since $T$ is a hybrid mapping of $C$ into $C$ such that $F(T) \neq \emptyset$, it satisfies the following:

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Putting $y = p$ for $p \in F(T)$, we have that

$$3\|Tx - p\|^2 \leq \|x - p\|^2 + \|Tx - p\|^2 + \|p - x\|^2, \quad \forall x \in C$$

and hence

$$\|Tx - p\|^2 \leq \|p - x\|^2, \quad \forall x \in C.$$

This implies that $T$ is quasi-nonexpansive. Furthermore, we have from Lemma 4.3 that $T$ is demiclosed. Putting $U = kI + (1-k)S$, we have that $U$ is nonexpansive and demiclosed. Furthermore, we have $F(S) = F(U)$. Therefore, we have the desired result from Theorem 3.3. □

The following is a weak convergence theorems for finding a common element of three sets of a Banach space.

**Theorem 4.6.** *Let $E$ be a uniformly convex and uniformly smooth Banach space which the duality mapping $J_E$ of $E$ is weakly sequentially continuous. Let $C$, $D$ and $F$ be nonempty, closed and convex subsets of $E$ . Let $R_C$, $R_D$ and $R_F$ be the sunny generalized nonexpansive retractions of $E$ onto $C$, $D$ and $F$, respectively. Suppose that $C \cap D \cap F \neq \emptyset$. For any $x_1 = x \in E$, define $\{x_n\}$ as follows:*

$$\begin{cases} y_n = (1 - r_n)x_n + r_n R_D R_F x_n, \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n R_C y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

*where $\{\beta_n\} \subset (0, 1)$, $a, b, \delta, \gamma \in \mathbb{R}$ and $\{r_n\} \subset (0, 1)$ satisfy the following:*

$$0 < a \leq \beta_n \leq b < 1 \quad \textit{and} \quad 0 < \delta \leq r_n \leq \gamma < 1, \quad \forall n \in \mathbb{N}.$$

*Then the sequence $\{x_n\}$ converges weakly to an elementt $z_0 \in C \cap D \cap F$, where $z_0 = \lim_{n\to\infty} R_{C\cap D\cap F} x_n$ anf $R_{C\cap D\cap F}$ is the sunny generalized nonexpansive retraction of $E$ onto $C \cap D \cap F$.*

*Proof.* Take $A = \partial i_{J_E F}$ in Theorem 3.3. Then we have that $J_{\mu_n} = R_F$ for all $n \in \mathbb{N}$. Furthermore, since $R_C$ is the sunny generalized nonexpansive retraction of $E$ onto $C$, we have from Lemma 2.6 that

$$\phi(R_C xz) \leq \phi(x, z), \quad \forall x \in E, \ z \in C.$$

We show that $R_C$ is generalized demiclosed. In fact, assume that $J_E x_n \rightharpoonup J_E p$ a nd $J_E x_n - J_E R_C x_n \to 0$. It is clear that $J_E R_C x_n \rightharpoonup J_E p$. Since $E^*$ is uniformly smooth, we have that $\|x_n - R_C x_n\| \to 0$. Since $R_C$ is the sunny generalized nonexpansive retraction of $E$ onto $C$, we have from Lemma 2.12 and (4.1) that

$$\langle x_n - R_C x_n - (p - R_C p), J_E R_C x_n - J_E R_C p \rangle \geq 0.$$

Therefore, $\langle -(p - R_C p), J_E p - J_E R_C p \rangle \geq 0$ and hence $\phi(p, R_C p) + \phi(R_C p, p) \leq 0$. This implies that $p = R_C p$ and hence $R_C$ is generalized demiclosed.

Similarly, $R_D$ is generalized nonexpansive and $R_D$ is generalized demiclosed. Therefore, we have the desired result from Theorem 3.3. □

The following is a weak convergence theorems for finding a common point of zero point sets of three maximal monotone operators of a Banach space.

**Theorem 4.7.** *Let $E$ be a uniformly convex and uniformly smooth Banach space which the duality mapping $J_E$ of $E$ is weakly sequentially continuous. Let $A$, $B$ and $G$ be maximal monotone operators of $E^*$ into $E$. Let $J_r^A$ be the generalized nonexpansive resolvent of $A$ for $r > 0$, let $J_\mu^B$ be the generalized nonexpansive resolvent of $B$ for $\mu > 0$ and let $J_\lambda^G$ be the generalized nonexpansive resolvent of $G$ for $\lambda > 0$, respectively. Suppose that*

$$\Omega = (AJ_E)^{-1}0 \cap (BJ_E)^{-1}0 \cap (GJ_E)^{-1}0 \neq \emptyset.$$

*For any $x_1 = x \in E$, define $\{x_n\}$ as follows:*

$$\begin{cases} y_n = (1-r_n)x_n + r_n J_\lambda^G J_r^A x_n, \\ x_{n+1} = (1-\beta_n)x_n + \beta_n J_\mu^B y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

*where $\{\beta_n\} \subset (0,1)$, $a, b, \delta, \gamma \in \mathbb{R}$ and $\{r_n\} \subset (0,1)$ satisfy the following:*

$$0 < a \leq \beta_n \leq b < 1 \quad \text{and} \quad 0 < \delta \leq r_n \leq \gamma < 1, \quad \forall n \in \mathbb{N}.$$

*Then the sequence $\{x_n\}$ converges weakly to an element $z_0 \in \Omega$, where $z_0 = \lim_{n\to\infty} R_\Omega x_n$ anf $R_\Omega$ is the sunny generalized nonexpansive retraction of $E$ onto $\Omega$.*

*Proof.* Take $\mu_n = r$ for $r > 0$ in Theorem 3.3. Then we have that $J_{\mu_n}^A = J_r^A$ for all $n \in \mathbb{N}$. Furthermore, since $J_\mu^B$ is the generalized nonexpansive resolvent of $B$, we have from Lemma 2.11 that

$$\phi(J_\mu^B x, z) \leq \phi(x, z), \quad \forall x \in E,\ z \in (BJ_E)^{-1}0.$$

Next, we show that $J_\mu^B$ is generalized demiclosed. In fact, assume that $J_E x_n \rightharpoonup J_E p$ and $J_E x_n - J_E J_\mu^B x_n \to 0$. It is clear that $J_E J_\mu^B x_n \rightharpoonup J_E p$. Since $E^*$ is unifrmly smooth, we have that $\|x_n - J_\mu^B x_n\| \to 0$. Since $J_\mu^B$ is the generalized nonexpansive resolvent of $B$, we have from Lemma 2.12 that

$$\langle x_n - J_\mu^B x_n - (p - J_\mu^B p), J_E J_\mu^B x_n - J_E J_\mu^B p \rangle \geq 0.$$

Therefore, $\langle -(p - J_\mu^B p), J_F p - J_E J_\mu^B p \rangle \geq 0$ and hence $\phi(p, J_\mu^B p) + \phi(J_\mu^B p, p) \leq 0$. This implies that $p = J_\mu^B p$ and hence $J_\mu^B$ is generalized demiclosed.

Similarly $J_\lambda^G$ is generalized nonexpansive and generalized demiclosed. Therefore, we have the desired result from Theorem 3.3. □

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