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SPARSE OPTIMAL CONTROL OF CYBER-PHYSICAL SYSTEMS VIA PQA APPROACH*

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Abstract: In this study, we examine a linear time-invariant system with the influence of a static state feedback control mechanism, which takes the form of $Kx(t-\tau(||K||_0))$, where K represents the gain matrix and $||K||_0$ denotes the number of nonzero entries of the matrix K. The parameter $\tau(||K||_0)$ corresponds to a varying delay, which arises due to the time required for the transmission of the system state information and the subsequent computation of the control input. Governed by the linear time-invariant system, we minimize prescribed conventional cost functions as $J^0(K)$ to obtain the optimal feedback matrices K_1^*. Numerous computational methods are available for the achievement of this objective. Nevertheless, it is noteworthy that the resulting K_1^* matrix typically exhibit a dense structure. The primary objective of this paper is to minimize the l_0 norm of the feedback matrix K under conditions that satisfy the constraint $|J^0(K) - J^0(K_1^*)| \leq \varepsilon$. The l_0 norm acts as a quantifiable measure for assessing the degree of sparsity within the feedback matrix. The sparsity of the gain matrix is approximated through the application of a piecewise quadratic approximation (PQA) of the l_0-norm of the feedback matrix. Subsequently, we proceed to formulate an iterative algorithm designed to address the transformed problem, accompanied by a thorough analysis of its convergence properties. Finally, we undertake a numerical experiment employing the proposed algorithm, aiming to illustrate its practical utility and efficacy in solving the problem at hand.

Key words: *sparse optimal control, l_0 norm of matrix, sparse feedback matrix, time-delay systems, non-convex approximation, iterative thresholding algorithm*

Mathematics Subject Classification: *49J15, 49M37, 65K10, 90C30, 90C59, 90C90*

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1 Introduction

A cyber-physical system (CPS) is a multi-dimensional complex system that integrates computing, network, and physical environment [23]. Relying on the integrated collaboration capabilities of 3C (Computation, Communication, and Control) technology, real-time perception, control, and information services of large-scale engineering systems can be achieved.

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The CPS has a broad range of applications. In [1], the CPS is used for water management and governance, where the social and ecosystem are integrated into a CPS. The design is vital for the sustainable development of water management and governance. In [30], the study is concerned with the development of a simulation platform for lane optimization of connected autonomous vehicles based on CPS. Then, the usefulness of the platform is demonstrated by three typical dedicated lane scenarios. In [25], the focus is on the temporary security solutions based on artificial intelligence. For better accuracy, human activity can be identified and used as soft biometrics alongside raw biometrics. Traditionally, caregivers assist and care for patients with cognitive decline, but this places a financial and emotional burden on both caregivers and patients, affecting their quality of life. In [12], it considers the applications of CPS in smart buildings for occupants with cognitive decline. These applications can realize the integrated design of communication, computing, and physical systems, making the system more efficient, reliable, and real-time cooperation. In [21], CPS and artificial intelligence are being promoted and applied in the construction industry based on previous research. In conclusion, two characteristics of CPS are shown in [2]: (i) large-scale, complex physical, biological, and engineering orientated systems; (ii) network core consists of communication networks and computing facilities used to monitor, coordinate and control physical systems. CPS tightly integrates these two critical components so that analysis and design can be done within a joint framework.

In the past two decades, a great deal of research has been done on the control theory involving CPS. However, traditional CPS control designs are often result in dense feedback matrices, meaning that the optimal controller is formed using all the information in the feedback matrix. In large networks, implementation costs can be expensive, and computational burden involved in the communication between the controller and the dynamical system can be high [3]. Furthermore, there will be transmission delays and propagation delays in the network communication. In practice, these delays and the sparsity of the feedback matrix need to be taken into account in the design of the controller. In view of this, we propose a CPS system with a static state feedback controller $u(t) = -Kx(t-\tau)$, $K \in \mathbb{R}^{m\times n}$, where τ is the delay, which can be a constant delay or a varying delay, caused by the communication between the state x and the input u being computed [23]. The network control design of this paper aims to balance two key goals: (i) system performance (traditional cost function $J^0(K)$) and (ii) sparsity of the communication network. Therefore, in this paper, we aim to solve the following problem: given a CPS system, find a feedback matrix K that balances system performance and controller sparsity.

Sparsity means that the vast majority of components in a vector or matrix are zero. The sparsity of a vector or matrix is described by the l_0 norm of the vector or matrix. Sparsity plays an important role in large-scale optimization problems, such as compression mapping [8]. The use of sparsity not only saves storage space but also reduces transmission costs by compressing vectors. It simplifies a complex problem by leveraging only useful information from huge amounts of data. At present, sparse optimization has been widely used in areas such as signal and image processing [9], machine learning and pattern recognition [10], portfolio problems in economics [27], regression problems in statistics [11], and principal component analysis [5]. In [14], the sparsity function is approximated by a folded concave function. The pairwise separation yields a Z-type objective, and a linear-step parametric algorithm is proposed to optimize the problem. According to [29], the mathematical models of sparse optimization can be broadly grouped into two categories: (i) l_0-regularization optimization problems which penalize the traditional objective function by adding the l_0-norm to form a new objective function; and (ii) sparse constrained optimization problems which put l_0-norm in the constraints. However, both of these two problems are NP-hard. In the previous studies, the methods to solve the l_0-norm minimization problem can be divided into the model transformation method and the direct processing method. The common feature of the model transformation method is to approximate l_0-norm with l_1-norm, such as the famous lasso problem [17] and its variants [31]. In the aspect of algorithms, there are Iterative Hard-Thresholding Algorithm [7], Fast Iterative Shrinkage-Thresholding Algorithm [4], Augmented Lagrangian method [16], Alternating Direction Method of Multipliers [15], and others. In addition, there are other methods used to approximate the l_0-norm, such as the l_p-norm ($0 < p < 1$). In [13], the l_0-norm is written as the difference of two convex functions (DC method). The direct method usually starts with the optimality conditions of a sparse optimization problem and then deals with the original problem directly. In this way, there is no need to check the approximate effect model approximation. Furthermore, it tends to produce a sparser solution. Recent algorithms include Iterative Hard Thresholding (IHT) [7] and block coordinate descent (BCD). For the motion control task of manipulators, a novel motion planning scheme that takes into account sparsity is proposed in [19]. In [22], the study is concerned with a class of sparse optimization problems with l_1-norm regularization and convex constraints, in which the individual functions involved are differentiable except the l_1-regularization term. It also obtains the necessary and sufficient conditions. On this basis, a simple neural network with differential equation structure is proposed.

The network control design in this paper aims to balance the two key measures: (i) the system performance (the traditional cost function solvable by many available methods. For example, the control parametrization used in conjunction with the time scaling transform in [26]. However, the feedback matrix thus obtained tends to be dense); and (ii) the sparsity of communication networks. Therefore, in this paper, we aim to solve the following sparse optimal control problem (SOCP): given the CPS system, find a feedback matrix K to balance the system performance and the sparsity of the controller. The phase diagram analysis suggests that the piecewise quadratic approximation (PQA) performs better than l_1 and $l_{1/2}$ regularization. It contains one smooth non-convex term and one non-smooth convex term [19]. The PQA model works best on $[-e, e]$, where $e = (1, 1, \ldots, 1)^\top$ [18]. In this paper, the PQA model approximation method is extended to sparse matrix optimization. By dividing the matrix into columns, and the sparse optimization of each column vector completes the sparsity of the feedback matrix. The SOCP, the l_0-norm of the feedback matrix is approximated by the PQA model. An iterative algorithm with the support of the convergence analysis is developed to solve the approximate problem. Finally, several numerical examples are solved so as to demonstrate the effectiveness of the proposed algorithm.

The rest of the paper is organized as follows. We first describe the formulation of the problem in Section 2, where $\|K\|_0$ is approximated through the use of PQA model. In Section 3, after the model approximation, we develop an iterative algorithm to solve the problem and then carry out the convergence analysis of the algorithm. Finally, several numerical examples are considered and solved to evaluate the performance of the proposed algorithm in finding sparse controllers.

2 Problem Formulation

2.1 CPS modeling

2.1.1 LTI system

We consider the following linear time-invariant (LTI) system [23]:

$$\dot{x}(t) = Ax(t) + Bu(t) + B_\omega w(t), \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and $w \in \mathbb{R}^p$ is the exogenous input, $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ and $B_w \in \mathbb{R}^{n\times p}$ are the respective matrices. Assume that (A, B) is controllable.

2.1.2 Feedback control system with varying delay $\tau(\|K\|_0)$

After the sparsity level of matrix $K \in \mathbb{R}^{m\times n}$ is achieved, the bandwidth c is equally redistributed among the remaining links. Assume that the communication network follows frequency division multiplexing. Then, delay τ can be defined by [6]:

$$\tau(\|K\|_0) = \tau_t + \tau_p = \mathcal{Z}(\|K\|_0, c, \tau_p) := \kappa\left(\|K\|_0/c\right) + \tau_p, \tag{2.2}$$

where $\|K\|_0$ denotes the number of non-zero elements in K, and $\kappa : \mathbb{R} \to \mathbb{R}$ is a positive function. Eq.(2.2) implies that τ will change as $\|K\|_0$ changes. This change is captured by the function $\mathcal{Z}(\cdot) : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$. The transmission of state x_j for the computation of input u_i is anticipated to encounter a delay denoted as τ_{ij} (expressed in seconds), $i \in I_m, j \in I_n$. This delay comprises two distinct components: $\tau_{ij} = \tau_{p_{ij}} + \tau_{t_{ij}}$, where $\tau_{p_{ij}}$ denotes the propagation delay, and $\tau_{t_{ij}}$ represents the transmission delay. The parameter $\tau_{p_{ij}}$ is characterized as the quotient of the link length divided by the speed of light, assumed to possess a uniform value denoted as τ_p across all pairs i, j. Our assumption posits an equal allocation of bandwidth for the communication link connecting any j_{th} sensor to any i_{th} actuator. Consequently, this implies that $\tau_{t_{ij}}$ maintains a uniform value across all i, j pairs [6]. Henceforth, we denote τ_{ij} uniformly as τ across all pairs i, j. In practice, potential deviations of τ_{ij} from the designated τ due to variations in traffic and uncertainties within the network are acknowledged. However, our design remains robust as long as such deviations remain within the stability radius of the plant (see Proposition 1.14 and Theorem 1.16 in [24]).

This controller will be deployed in a distributed manner utilizing a communication network, as depicted in Figure 1. This figure presents the CPS representation, illustrating the closed-loop system architecture.

The control input is written as $u(t) = -Kx(t-\tau(\|K\|_0))$, and accordingly the closed-loop system is written as:

$$\begin{cases} \dot{x}(t) = Ax(t) - BKx(t - \tau(\|K\|_0)) + B_\omega \omega(t), \\ x(t) = \nu, t \le 0, \end{cases} \tag{2.3}$$

where $\nu \in \mathbb{R}^n$ is a given vector, each of its elements is assumed, without loss of generality, to be 0.5. Let $x(\cdot|K)$ be the solution of system (2.3) corresponding to the feedback matrix $K \in \mathbb{R}^{m\times n}$.

2.2 Problem statement

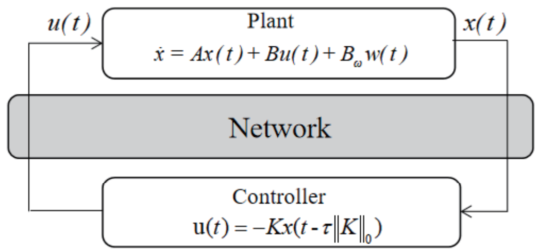


Figure 1: Closed-loop CPS representation [23]. The cyber network layer is to receive an input signal, denoted as $x(t)$, which is then transmitted through the network to generate an output signal, represented as $u(t)$. Clearly, this process will admit a time delay, denoted as $\tau(\|K\|_0)$. After the computation of $u(t)$, the resultant signal is transmitted to the actuators for further action.

2.2.1 Traditional optimal control problem

With reference to the delay mentioned above, we introduce the corresponding cost functions as given below [20]:

$$J^0(K) = (x(T|K))^\top Sx(T|K) + \int_{t_0}^{T} [(x(t|K))^\top Qx(t|K) + (u(t))^\top Ru(t)]\mathrm{d}t, \tag{2.4}$$

where t_0 is the initial time, T is the final time, the matrix R is symmetric positive definite, and the matrices S and Q are symmetric positive semidefinite.

We now present the feedback optimal control problem as follows.

$$\begin{aligned}\text{Problem } \mathbf{P_1}: \quad \min_{K\in\mathbb{R}^{m\times n}} \quad & J^0(K)\\ s.t. \quad & \dot{x}(t) = Ax(t) - BKx(t-\tau(\|K\|_0)) + B_\omega\omega(t),\\ & x(t) = \nu, t \leq 0,\end{aligned}$$

where $J^0(K)$ is given by (2.4).

2.2.2 Sparse optimal control problem

Gradient-based optimization methods [26] can be used to solve Problem $\mathbf{P_1}$. Let $K_1^* \in \mathbb{R}^{m\times n}$ be the optimal feedback matrices for Problem $\mathbf{P_1}$. However, these matrices tend to be rather dense, and for large networks, the implementation cost will be expensive. Furthermore, the computation burden of the controller will be high because the state information is required to be transmitted through the communication network. Thus, we introduce the following Problem $\mathbf{P_2}$ given by

$$\begin{aligned}\text{Problem } \mathbf{P_2}: \quad \min_{K\in\mathbb{R}^{n\times m}} \quad & \|K\|_0\\ s.t. \quad & \dot{x}(t) = Ax(t) - BKx(t-\tau(\|K\|_0)) + B_\omega\omega(t),\\ & x(t) = \nu, t \leq 0,\\ & |J^0(K) - J^0(K_1^*)| \leq \varepsilon,\end{aligned} \tag{2.5}$$

where $\|K\|_0$ denotes the number of nonzero entries of the matrix $K \in \mathbb{R}^{m\times n}$, $J^0(K)$ is given by (2.4), and $J^0(K_1^*)$ is the benchmark optimal cost index obtained through solving the traditional optimal control Problem $\mathbf{P_1}$. ε is a small number that is used to ensure that the system performance is not greatly affected during the sparsity process of $K \in \mathbb{R}^{m\times n}$.

Obviously, Problem $\mathbf{P_2}$ balances system performance and the sparse level of the controller $K \in \mathbb{R}^{m\times n}$.

2.3 Preconditioning algorithm : the approximation of $\|\mathrm{K}\|_0$.

The matrix K is decomposed as n column vectors, i.e., $K = (K^1, K^2, \ldots, K^n) \in \mathbb{R}^{m\times n}$. Note that $\|K^l\|_0, l \in I_n$, regularization is NP-hard. Thus, it is difficult to solve. In the past two decade, many approximation methods, such as $\|K^l\|_1$, and $\|K^l\|_q^q, (0 < q < 1)$, have been proposed to approximate it. In [18], the l_0-norm of vector is approximated by a piecewise quadratic approximation (PQA) method. In this paper, we shall extend PQA to spare the feedback matrix K. We use the following piecewise quadratic function [18] to approximate $\|\mathrm{K}^l\|_0$ over $[-e, e]$.

$$P(K^l) = -(K^l)^\top K^l + 2\|\mathrm{K}^l\|_1,\ l \in I_n,\ K^l \in \mathbb{R}^m.$$

Remark 2.1. We shall illustrate that $P(K^l), l \in I_n$, performs better than other common approximations of $\|K^l\|_0, l \in I_n$, on $[-e, e]$, $e = \{1, 1, \ldots, 1\} \in \mathbb{R}^m$. For $l \in I_n$, Figure 2

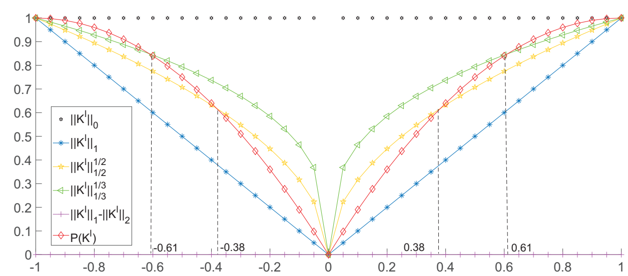


Figure 2: Various approximations for the one-dimensional case in [-1,1] [18].

shows the approximation effects of $\|K^l\|_1$, $\|K^l\|_{1/2}^{1/2}$, $\|K^l\|_{1/3}^{1/3}$, $\|K^l\|_1 - \|K^l\|_2$ and $P(K^l)$ for one-dimensional case in [-1,1] [18]. Obviously, for $l \in I_n$, $P(K^l)$ is superior to $\|K^l\|_1$ on approximating the l_0-norm when $|K_i^l| \le 1, i \in I_m$. For $l \in I_n$, when $0.38 \le |K_i^l| \le 1, i \in I_m$, $P(K^l)$ gives a better approximation to $\|K^l\|_0$, and when $0.61 \le |K_i^l| \le 1, i \in I_m$, $P(K^l)$ is better than $\|K^l\|_{1/3}^{1/3}$. Also, for $l \in I_n$, $P(K^l)$ is superior than $\|K^l\|_1 - \|K^l\|_2$, which is identically equal to 0 and has a large gap with $\|K^l\|_0$ in [-1,1].

On this basis, if we choose $f(K^l) = -\|\mathrm{K}^l\|_2^2$ and $g(K^l) = 2\|\mathrm{K}^l\|_1$, then

$$F(K) = \sum_{l=1}^{n} P(K^l) = \sum_{l=1}^{n} \Big[f(K^l) + g(K^l)\Big], \tag{2.6}$$

where g is a proper closed convex and possibly non-smooth function; f is a smooth non-convex function of the type $C_{L_f}^{1,1}(\mathbb{R}^n)$, i.e., continuously differentiable with Lipschitz continuous gradient

$$\|\nabla f(K^l) - \nabla f(y^l)\| \le L_f\|K^l - y^l\|,\quad K^l \in \mathbb{R}^m, y^l \in \mathbb{R}^m, l \in I_n,$$

with $L_f > 0$ denoting the Lipschitz constant of ∇f.

Based on the piecewise quadratic approximation [18], Problem $\mathbf{P_2}$ can be approximated as given below:

$$\begin{array}{lll} \text{Problem } \mathbf{P_3}: & \min_{K\in\mathbb{R}^{m\times n}} & F(K) \\ & s.t. & \dot{x}(t) = Ax(t) - BKx(t-\tau(\|K\|_0)) + B_\omega\omega(t), \\ & & x(t) = \nu, t \leq 0, \\ & & |J^0(K) - J^0(K_1^*)| \leq \varepsilon, \end{array}$$

where $J^0(K)$, $J^0(K_1^*)$, and ε are the same as Problem $\mathbf{P_2}$.

3 Computational Approach

In this section, we present an iterative algorithm for solving Problem $\mathbf{P_3}$. The basic idea of this kind of method is to give the initial point, calculate the new point through the given iteration formula at each step, and continue the iterative process until the termination condition is met. In this paper, the iterative formula of each step is a regularization of the linearized differentiable part $f(K^l)$ of the objective function in Problem $\mathbf{P_3}$ [4, 18]. Then, the unique minimizer at each step can be obtain with constant stepsize $1/L_f$. This process is repeated until we get the minimum value.

3.1 The iterative algorithm

We first present a simple minimization problem and obtain its iterative format to solve it based on some accepted facts of the gradient-based methods, then we extend it and obtain the iterative format of our optimization problem in this paper.

3.1.1 the iterative format of a simple optimization problem

Firstly, we consider the following simple minimization Problem $\mathbf{O_1}$:

$$\min_{K^l\in\mathbb{R}^m} f(K^l),$$

where $K^l \in \mathbb{R}^m$ is the optimization variable, and the objective function f is smooth and continuously differentiable. If we use k as the number of iteration steps, then it is well known that the gradient iterative form can simply obtain a sequence $(K^l)^k$ via

$$(K^l)^k = ((K)^l)^{k-1} - \frac{1}{L_f}\nabla f((K^l)^{k-1}), \tag{3.1}$$

where $1/L_f$ denotes the suitable stepsize.

Based on [4], this gradient iterative form (3.1) can be viewed as a proximal regularization of the linearized function f at $(K^l)^{k-1}$, and written equivalently as following iteration:

$$(K^l)^k = arg \min_{K^l\in\mathbb{R}^m} \Big\{ f((K^l)^{k-1}) + < K^l - (K^l)^{k-1}, \nabla f((K^l)^{k-1}) > + \frac{L_f}{2}\|K^l - (K^l)^{k-1}\|^2 \Big\}. \tag{3.2}$$

Since we have the iterative formula (3.2) from step $k-1$ to step k, we can easily solve Problem $\mathbf{O_1}$.

3.1.2 the iterative format of the optimization problem in this paper

With the basic results written in above section, we consider the following minimization Problem $\mathbf{O_2}$:

$$\min_{K^l \in \mathbb{R}^m} f(K^l) + g(K^l),$$

where g is a proper closed convex and possibly non-smooth function. It is exactly the form of the optimization problem to be solved in this paper.

To obtain the iterative format, we first define the following quadratic approximation $Q(K^l, y^l)$ to approximate $f(K^l) + g(K^l)$ at a given point $y^l \in \mathbb{R}^m$ [4]:

$$Q_{L_f}(K^l, y^l) := f(y^l) + < K^l - y^l, \bigtriangledown f(y^l) > + \frac{L_f}{2} \parallel K^l - y^l \parallel^2 + g(K^l),\ l \in I_n,$$

and define its unique minimizer as $P_{L_f}(y^l)$. Then

$$\begin{aligned} P_{L_f}(y^l) &:= arg \min_{K^l \in \mathbb{R}^m} \left\{ Q_{L_f}(K^l, y^l) : K^l \in \mathbb{R}^m \right\} \\ &= arg \min_{K^l \in \mathbb{R}^m} \left\{ f(y^l) + < K^l - y^l, \bigtriangledown f(y^l) > + \frac{L_f}{2} \parallel K^l - y^l \parallel^2 + g(K^l) \right\}, y^l \in \mathbb{R}^m. \end{aligned}$$

Since the constant term has no effect on the result, we remove $f(y^l)$ and add $(1/2L_f)\nabla\|f(y^l)\|^2$, then

$$\begin{aligned} P_{L_f}(y^l) &:= arg \min_{K^l \in \mathbb{R}^m} \left\{ \frac{L_f}{2} \left[K^l - \left(y^l - \frac{1}{L_f} \nabla f(y^l) \right) \right]^2 + g(K^l) \right\} \\ &= arg \min_{K^l \in \mathbb{R}^m} \left\{ \frac{L_f}{2} \left\| K^l - \left(y^l - \frac{1}{L_f} \nabla f(y^l) \right) \right\|^2 + g(K^l) \right\}. \end{aligned}$$

According to (3.2), and let $y^l = (K^l)^{k-1}$, then we have iterative formula $(K^l)^k = P_{L_f}((K^l)^{k-1})$ from step $k-1$ to step k, we can easily solve Problem $\mathbf{O_2}$.

If we write the proximal operator for each column of $K = (K^1, K^2, \ldots, K^n)$, then we get the following algorithm to minimize the following approximating of the $\|K\|_0$:

$$F(K) = \sum_{l=1}^{n} \left[f(K^l) + g(K^l) \right],\ l \in I_n.$$

Algorithm 1: The iterative algorithm (IA)

1:**Input:** L_f(The Lipschitz constant of $\bigtriangledown f$);
2:**Step 0**: Take $(K)^0 \in \mathbb{R}^{m \times n}$;
3:**Step k**: $(k \geq 1)$ Compute

$$(K^l)^k = P_{L_f}((K^l)^{k-1}), \tag{3.3}$$

where $K^k = ((K^1)^k, (K^2)^k, \ldots, (K^n)^k)$, $l \in I_n$; $(K_i^l)^k \in [-1, 1]$, $i \in I_m$.

The convergence of Algorithm 1 and its convergence rate are stated in the next subsection.

3.2 Convergence analysis

To begin with, we need to present a key result (Lemma 3.3). It is required for the convergence analysis. Some fundamental properties for a smooth function in the class $C^{1,1}_{L_f}(\mathbb{R}^n)$ are stated in the following.

Lemma 3.1 ([18, 28]). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function with Lipschitz continuous gradient and Lipschitz constant L_f. Then, for any $K \in \mathbb{R}^{m\times n}, y^l \in \mathbb{R}^m$, it holds that*

$$-\frac{L_f}{2} \parallel K^l - y \parallel^2 \leq f(K^l) - f(y^l) - < K^l - y^l, \bigtriangledown f(y^l) > \leq \frac{L_f}{2} \parallel K^l - y^l \parallel^2, l \in I_n.$$

This statement makes clear the geometric interpretation of functions from $C^{1,1}_{L_f}(\mathbb{R}^n)$.

Lemma 3.2 ([18, 28]). *For any $y^l \in \mathbb{R}^m$, it holds that $z^l = P_{L_f}(y^l)$ if and only if there exists $h(y^l) \in \partial g(z^l)$, the subdifferential of $g(\cdot)$, such that*

$$\bigtriangledown f(y^l) + L_f(z^l - y^l) + h(y^l) = 0, \; l \in I_n.$$

This result characterizes the optimality of $P_{L_f(\cdot)}$. Now, we can present the required key result.

Lemma 3.3. *Let $y := (y^1, ..., y^n) \in \mathbb{R}^{m\times n}$. Then, for any $K \in \mathbb{R}^{m\times n}$,*

$$\bigtriangledown F(K) - F(P_{L_f}(y)) \geq \sum_{l=1}^{n} \Big(\frac{L_f}{2} \|P_{L_f}(y^l) - K^l\|^2 - L_f \|K^l - y^l\|^2 \Big).$$

Proof. From Lemma 3.1, we obtain

$$F(P_{L_f}(y)) = \sum_{l=1}^{n} [f(P_{L_f}(y^l)) + g(P_{L_f}(y^l))] \leq \sum_{l=1}^{n} Q(P_{L_f}(y^l), y^l).$$

Thus,

$$F(K) - F(P_{L_f}(y)) \geq F(K) - \sum_{l=1}^{n} Q(P_{L_f}(y^l), y^l). \tag{3.4}$$

Since Lemma 3.1 holds and $g(K^l)$ is convex, we have

$$\begin{aligned} f(K^l) &\geq f(y^l) + \langle K^l - y^l, \nabla f(y^l)\rangle - \frac{L_f}{2}\|K^l - y^l\|^2, \\ g(K^l) &\geq g(P_{L_f}(y^l)) + \langle K^l - P_{L_f}(y^l), h(y^l)\rangle. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{l=1}^{n} f(K^l) &\geq \sum_{l=1}^{n} f(y^l) + \langle \sum_{l=1}^{n} (K^l - y^l), \nabla f(y^l)\rangle - \frac{L_f}{2} \sum_{l=1}^{n} \|K^l - y^l\|^2, \\ \sum_{l=1}^{n} g(K^l) &\geq \sum_{l=1}^{n} g(P_{L_f}(y^l)) + \sum_{l=1}^{n} \Big(\langle K^l - P_{L_f}(y^l), h(y^l)\rangle \Big). \end{aligned}$$

Summing the two inequalities up yields

$$F(K) = \sum_{l=1}^{n}[f(K^l) + g(K^l)] \geq \sum_{l=1}^{n} f(y^l) + \sum_{l=1}^{n} g(P_{L_f}(y^l)) + \langle \sum_{l=1}^{n}(K^l - y^l), \bigtriangledown f(y^l)\rangle \quad (3.5)$$
$$+ \sum_{l=1}^{n}\langle K^l - P_{L_f}(y^l), h(y^l)\rangle - \frac{L_f}{2}\sum_{l=1}^{n}\|K^l - y^l\|^2.$$

By the definition of $P_{L_f}(y^l)$, we get

$$Q(P_{L_f}(y^l), y^l) = f(y^l) + \langle P_{L_f}(y^l) - y^l, \bigtriangledown f(y^l)\rangle + \frac{L_f}{2}\|P_{L_f}(y^l) - y^l\|^2 + g(P_{L_f}(y^l)),$$
$$\sum_{l=1}^{n} Q(P_{L_f}(y^l), y^l) = \sum_{l=1}^{n} f(y^l) + \sum_{l=1}^{n}\Big[\langle P_{L_f}(y^l) - y^l, \bigtriangledown f(y^l)\rangle + \frac{L_f}{2}\|P_{L_f}(y^l) - y^l\|^2 + g(P_{L_f}(y^l))\Big]. \quad (3.6)$$

Therefore, substituting (3.5) and (3.6) into (3.4) gives

$$\begin{aligned}
&F(K) - F(P_{L_f}(y))\\
&\geq \sum_{l=1}^{n}\langle K^l - P_{L_f}(y^l), \nabla f(y^l) + h(y^l)\rangle\\
&\quad - \frac{L_f}{2}\sum_{l=1}^{n}\|K^l - y^l\|^2 - \frac{L_f}{2}\sum_{l=1}^{n}\|P_{L_f}(y^l) - y^l\|^2\\
&= \sum_{l=1}^{n}\Big[L_f\langle K^l - P_{L_f}(y^l), y^l - P_{L_f}(y^l)\rangle - \frac{L_f}{2}\|K^l - y^l\|^2 - \frac{L_f}{2}\|P_{L_f}(y^l) - y^l\|^2\Big]\\
&= \sum_{l=1}^{n}\Big(\frac{L_f}{2}\|P_{L_f}(y^l) - K^l\|^2 - L_f\|K^l - y^l\|^2\Big),
\end{aligned}$$

where the first equality above comes from Lemma 3.2. □

Lemma 3.3 is essential for the establishment of the convergence of Algorithm 1. To give the stopping criterion for Algorithm 1, we need the following lemma.

Lemma 3.4 ([18])**.** *Let K^k be the sequence generated by Algorithm 1. If $\|L_f((K^l)^k - (K^l)^{k-1})\|^2 < \epsilon$ after k iterations, then there exists $h((K^l)^{k-1}) \in \partial g((K^l)^k)$, such that*

$$\| \bigtriangledown f((K^l)^k) + h((K^l)^{k-1})\|^2 \leq 4\epsilon.$$

Lemma 3.4 shows that as $\|L_f(K^k - K^{k-1})\|^2$ decreases, iterations sequence $\{K^k\}$ generated by Algorithm 1 converges to a stationary point of $F(K)$. Now, we discuss the convergence and convergence rate of Algorithm 1.

Theorem 3.5. *Let $\{K^k\}$ be the sequence generated by Algorithm 1. Suppose that $\{K^k\}$ is bounded. Then, the following statements hold:*

(i) $\sum_{k=0}^{\infty}\sum_{l=0}^{n}\|(K^l)^{k+1} - (K^l)^k\|^2 < \infty$;

(ii) *Any accumulation point of $\{K^k\}$ is a stationary point of F.*

Proof. Invoking Lemma 3.3 with $K = y = K^k$, we obtain

$$F(K^k) - F(K^{k+1}) \geq \sum_{l=1}^{n} \Big(\frac{L_f}{2} \sum_{l=0}^{n} \|(K^l)^{k+1} - (K^l)^k\|^2 - L_f \|(K^l)^k - (K^l)^k\| \Big), \quad \forall k \geq 0.$$

Thus, we have

$$F(K^k) - F(K^{k+1}) \geq \frac{L_f}{2} \sum_{l=0}^{n} \|(K^l)^{k+1} - (K^l)^k\|^2, \quad \forall k \geq 0, \tag{3.7}$$

which implies that the sequence $\{F(x^k)\}$ is monotonically decreasing. Summing both sides of (3.7) from 0 to N, we have

$$\frac{L_f}{2} \sum_{k=0}^{N} \sum_{l=1}^{n} \|(K^l)^{k+1} - (K^l)^k\|^2 \leq F(K^0) - F(K^{N+1}). \tag{3.8}$$

Since $\{K^k\}$ is bounded, together with the fact that $\{F(K^k)\}$ is monotonically decreasing, $\{F(K^k)\}$ is convergent. Let $N \to \infty$ in (3.8). Then, we have

$$\frac{L_f}{2} \sum_{k=0}^{\infty} \sum_{l=0}^{n} \|(K^l)^{k+1} - (K^l)^k\|^2 < \infty.$$

This proves (i).

We now prove (ii). Let $K^* = ((K^l)^*, (K^2)^*, \cdots, (K^n)^*)$ be an accumulation point of the sequence $\{K^k\}$. Then, there exists a subsequence $\{(K^l)^{k_i}\}$ such that $\lim_{i\to\infty}(K^l)^{k_i} = (K^l)^*$. Using Lemma 3.2, we obtain

$$-L_f((K^l)^{k_i} - (K^l)^{k_i}) \in \nabla f((K^l)^{k_i}) + \partial g((K^l)^{k_i+1}). \tag{3.9}$$

Invoking $\|(K^l)^{k_i} - (K^l)^{k_i}\| \to 0$ from (i), together with the continuity of ∇f and the closeness of ∂g, passing to the limit in (3.9), we have

$$0 \in \nabla f((K^l)^*) + \partial g((K^l)^*).$$

This means that $(K^l)^*$ is a stationary point of F. Thus, the proof is complete. □

By virtue of Theorem 3.5, we see that $\| K^{k+1} - K^k\|^2$ measures the convergence of the iterative sequence $\{K^k\}$ to a stationary point. We shall use it to establish its convergence rate.

Theorem 3.6. *Let $\{K^k\}$ be the sequence generated by Algorithm 1. Assume that $\{K^k\}$ is bounded. Then, for any $N \geq 1$, there exists a constant M such that*

$$\min_{k=0,\cdots,N-1} \sum_{l=1}^{n} \| L_f((K^l)^{k+1} - (K^l)^k) \|^2 \leq \frac{2(N+1)nML_f^2}{N(N-1)}. \tag{3.10}$$

Proof. Invoking Lemma 3.3 with $y = K^k, K = K^* := ((K^*)^1, ..., (K^*)^n)$, where K^* is an accumulation point of K^k, we obtain

$$F(K^*) - F(K^{k+1}) \geq \frac{L_f}{2} \sum_{l=1}^{n} \Big(\|(K^l)^{k+1} - (K^*)^l\|^2 - 2\|(K^l)^k - (K^*)^l\|^2 \Big). \tag{3.11}$$

Since $F(K^k)$ is monotonically decreasing and K^k is bounded, we have $\lim_{k\to\infty} F(K^k) = F(K^*)$ and $F(K^k) \geq F(K^*)$, $k \geq 0$. Summing the inequality (3.11) over $k \in \{0, 1, \cdots, N-1\}$ gives

$$\begin{aligned} 0 &\geq NF(K^*) - \sum_{k=0}^{N-1} F(K^{k+1}) \\ &\geq \frac{L_f}{2} \sum_{l=1}^{n} (\, \|(K^l)^N - (K^*)^l\|^2 - \|(K^l)^0 - (K^*)^l\|^2 - \sum_{k=0}^{N-1} \|(K^l)^k - (K^*)^l\|^2 \,). \end{aligned} \tag{3.12}$$

Invoking Lemma 3.3 one more time with $K = y = K^k$ yields

$$F(K^k) - F(K^{k+1}) \geq \frac{L_f}{2} \sum_{l=1}^{n} \|(K^l)^{k+1} - (K^l)^k\|^2.$$

Multiplying the last inequality by k and summing over $k \in \{0, 1, \cdots, N-1\}$, we obtain

$$\sum_{k=0}^{N-1} (\, kF(K^k) - kF(K^{k+1}) \,) \geq \frac{L_f}{2} \sum_{l=1}^{n} \sum_{k=0}^{N-1} k\|(K^l)^{k+1} - (K^l)^k\|^2,$$

which can be simplified to

$$-NF(K^N) + \sum_{k=0}^{N-1} \geq \frac{L_f}{2} \sum_{l=1}^{n} \sum_{k=0}^{N-1} k\|(K^l)^{k+1} - (K^l)^k\|^2. \tag{3.13}$$

Adding (3.12) and (3.13), we get

$$\begin{aligned} 0 \;\;&\geq\;\; NF(K^*) - NF(K^N), \\ &\geq\;\; \frac{L_f}{2} \sum_{l=1}^{n} \|(K^l)^N - (K^*)^l\|^2 - \frac{L_f}{2} \sum_{l=1}^{n} \|(K^l)^0 - (K^*)^l\|^2 - \frac{L_f}{2} \sum_{l=1}^{n} \|(K^l)^N - (K^*)^l\|^2 \\ &\quad + \frac{L_f}{2} \sum_{l=1}^{n} \sum_{k=0}^{N-1} k\|(K^l)^{k+1} - (K^l)^k\|^2, \end{aligned}$$

By the assumption that K^k is bounded, there exists a series of constants $(M^1,\ M^2, \cdots, M^l)$ such that

$$\|(K^l)^{k+1} - (K^*)^l\|^2 \leq M^l.$$

Define $M = \min_{l} M^l$. Then, we have

$$\begin{aligned} \|(K^l)^{k+1} - (K^*)^l\|^2 \;\;&\leq\;\; M, \;\; \forall k \geq 0, \;\; l \in I_n, \\ \sum_{l=1}^{n} \sum_{k=0}^{N-1} k\|(K^l)^{k+1} - (K^l)^k\|^2 \;\;&\leq\;\; (N+1)nM. \end{aligned}$$

Thus, (3.10) holds. This completes the proof. □

From (3.10), we see that a stationary point $(K^l)^k$ satisfying $\| \, L_f(K^k - K^{k-1}) \, \|^2 \leq \epsilon$, i.e. $\| \, \nabla f(K^k) + h(K^{k-1}) \, \|^2 \leq 4\epsilon, h(K^{k-1}) \in \partial g(K^k)$ can be obtained after running Algorithm 1 for at most $\mathcal{O}(1/\epsilon)$ iterations.

4 Numerical Results

4.1 Experiment Design

In this section, all computations are carried out in MATLAB R2021a on a computer with 3.70 GHz Intel Core i9-10900K CPU243 and 32.0GB RAM. The Euler method is used to solve system (2.3) with a step size of 1/10, and the initial time and terminal time are 0 and 1, respectively. We consider 10-th order, and set τ changes with $\|K\|_0$ according to (2.2).

We use the gradient-based methods described in [26] for solving Problem $\mathbf{P_1}$ to obtain the optimal dense feedback matrix denoted by K_1^*, respectively. Then, through the application of Algorithm 1, we solve Problem $\mathbf{P_3}$ to obtain the optimal sparse feedback matrix K_2^*. To indicate the sparse level of the feedback matrix K, we define the following indicators:

$$r = \frac{\text{Number of non} - \text{zero elements in } K}{\text{Number of elements in } K},$$

it represents the proportion of non-zero elements in the matrix. Obviously, a smaller value of r means a better sparse level of K.

4.2 Parameters Setting

We consider a 10-th order LTI system with a randomly generated state matrix

$$A = \begin{bmatrix} -8 & -10 & 0 & -1 & -9 & -2 & -8 & -4 & -4 & -6 \\ -10 & -4 & -9 & -7 & -5 & -6 & -4 & -3 & -1 & 0 \\ -4 & -6 & -4 & -7 & -3 & -4 & -4 & -8 & -1 & -9 \\ -6 & -9 & -7 & 0 & -4 & -6 & -5 & -7 & -5 & -1 \\ -2 & -1 & -5 & -9 & 0 & -2 & -5 & -5 & -10 & -7 \\ -5 & -5 & -10 & -2 & -6 & -8 & -5 & -3 & -3 & 0 \\ -4 & -6 & 0 & -9 & -9 & -4 & -5 & -10 & -9 & -9 \\ -3 & -3 & -4 & -2 & -3 & -6 & -6 & -7 & -10 & -5 \\ -9 & -7 & -8 & -6 & -9 & -4 & -10 & -2 & -7 & -9 \\ 0 & -5 & -9 & -3 & 0 & 0 & -1 & -4 & -8 & -2 \end{bmatrix}.$$

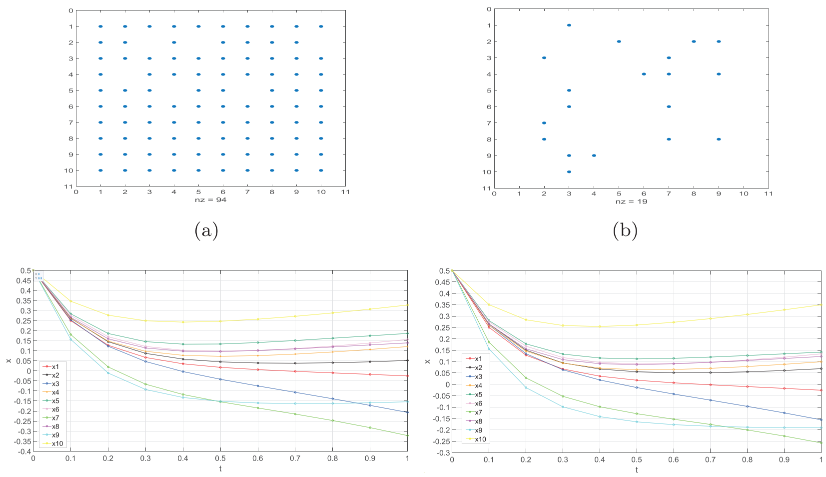
Assume that $B^1 = B^1_\omega = R = Q = E_{10}$, $S = \mathbf{0} \in \mathbb{R}^{10\times 10}$, $\varepsilon = 0.1$, $\tau_p = 9.83ms$, $c = 956$ and $\kappa = 0.01$, ω is a 10-dimensional column vector with all 1 elements. The cost function is given by [20]:

$$J^0(K) = \int_0^1 [(x(t|K))^\top x(t|K) + u(t)^\top u(t)]\mathrm{d}t.$$

4.3 Results and Analysis

Table 1: The optimal feedback matrix, the optimal cost function and sparsity indicators r.

The optimal feedback matrix	The optimal cost function	r
K_1^*	$J^0(K_1^*) = 0.4188$	0.94
K_2^*	$J^0(K_2^*) = 0.4406$	0.19



(a) (b) (c) (d)

Figure 3: (a)Distribution of nonzero components in K_1^*; (b)Distribution of nonzero components in K_2^*; (c)Changes in state values x of K_1^*; (d)Changes in state values x of K_2^*.

The distributions of nonzero components in the feedback matrices K_1^*, K_2^* and their corresponding state at each moment are displayed in Figure 3, where nz means the number of non-zero elements. Their corresponding optimal cost and sparsity levels are given in Table 1. Figure 3 indicates that the feedback matrix K_1^* exhibits a high degree of density, while the feedback matrix K_2^* displays a notable level of sparsity. Furthermore, from Table 1, it is pertinent to note that the value of the cost function, specifically $J^0(K_2^*)$, slightly exceeds that of $J^0(K_1^*)$. We can see that the number of zero components in K_2^* increases rapidly initially with only a small increase in cost. So we can conclude that the Algorithm 1 proposed in this paper can produce a better quality solution which balances the system performance and sparsity.

5 Discussions

Traditional CPS control designs are often result in dense feedback matrices, meaning that the optimal controller is formed using all the information in the feedback matrix. However, in large networks, these controllers can be expensive and computational burdensome. In the paper, we develop an algorithm to design sparse optimal controllers for an LTI system in the presence of feedback delay. The design of control balances system performance and the l_0 norm of feedback matrix. We use the piecewise quadratic approximation to approximate the $\|K\|_0$, and propose an iterative algorithm to solve the sparsity problem. Finally, we use a numerical experiment to evaluate the effectiveness of the proposed algorithm in balance the performance of LTI system and the sparsity of feedback matrix.

Future research can focus on the weight of system performance and sparsity to make them better balanced. It is also possible to consider other types of systems and perform numerical experiments with larger dimensions.

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