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STRONG CONVERGENCE THEOREMS BY SHRINKING PROJECTION METHODS FOR GENERALIZED SPLIT FEASIBILITY PROBLEMS IN HILBERT SPACES

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Abstract: In this paper, we consider new generalized split feasibility problems and then obtain two strong convergence theorems by shrinking projection methods in Hilbert spaces. As applications, we get new strong convergence theorems which are connected with the split feasibility problem and an equilibrium problem.

Key words: *maximal monotone operator, inverse strongly monotone mapping, fixed point, strong convergence theorem, hybrid method, equilibrium problem, split feasibility problem.*

Mathematics Subject Classification: *47H05, 47H09.*

1 Introduction

Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ is called inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ux - Uy\rangle \geq \alpha \|Ux - Uy\|^2, \quad \forall x, y \in C.$$

Such a mapping U is called α-inverse strongly monotone. Let H_1 and H_2 be two real Hilbert spaces. Let D and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the *split feasibility problem* [8] is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Recently, Byrne, Censor, Gibali and Reich [7] also considered the following problem: Given set-valued mappings $A_i : H_1 \to 2^{H_1}$, $1 \leq i \leq m$, and $B_j : H_2 \to 2^{H_2}$, $1 \leq j \leq n$, respectively, and bounded linear operators $T_j : H_1 \to H_2$, $1 \leq j \leq n$, the *split common null point problem* [7] is to find a point $z \in H_1$ such that

$$z \in \left(\bigcap_{i=1}^{m} A_i^{-1}0\right) \cap \left(\bigcap_{j=1}^{n} T_j^{-1}(B_j^{-1}0)\right),$$

where $A_i^{-1}0$ and $B_j^{-1}0$ are null point sets of A_i and B_j, respectively. Defining $U = A^*(I - P_Q)A$ in the split feasibility problem, we have that $U : H_1 \to H_1$ is an inverse strongly

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monotone operator [2], where A^* is the adjoint operator of A and P_Q is the metric projection of H_2 onto Q. Furthermore, if $D \cap A^{-1}Q$ is nonempty, then $z \in D \cap A^{-1}Q$ is equivalent to

$$z = P_D(I - \lambda A^*(I - P_Q)A)z, \tag{1.1}$$

where $\lambda > 0$ and P_D is the metric projection of H_1 onto D. Using such results regarding nonlinear operators and fixed points, many authors have studied the split feasibility problem and the split common null point problem; see, for instance, [1, 2, 7, 9, 19, 33].

On the other hand, in 2003, Nakajo and Takahashi [21] proved the following strong convergence theorem by using the hybrid method in mathematical programming. Let C be a nonempty, closed and convex subset of H. For a mapping $T : C \to C$, we denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

Theorem 1.1. Let C be a nonempty, closed and convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n \cap Q_n}$ is the metric projection from C onto $C_n \cap Q_n$ and $\{\alpha_n\} \subset [0, 1]$ is chosen so that $0 \leq \limsup_{n\to\infty} \alpha_n < 1$. Then $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the metric projection from H onto $F(T)$.

Takahashi, Takeuchi and Kubota [32] also obtained the following result by using the shrinking projection method:

Theorem 1.2. Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x \in H$. For $C_1 = C$ and $x_1 \in C$, define a sequence $\{x_n\}$ of C as follows:

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x, , \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 \leq \limsup_{n\to\infty} \alpha_n < 1$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $P_{F(T)}x$.

In this paper, motivated by these problems and results, we consider new generalized split feasibility problems and then obtain two strong convergence theorems by shrinking projection methods in Hilbert spaces. As applications, we get new strong convergence theorems which are connected with the split feasibility problem and an equilibrium problem.

2 Preliminaries

Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have from [29] that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle; \tag{2.1}$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.2}$$

Furthermore we have that for $x, y, u, v \in H$,

$$2\langle x-y, u-v\rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \tag{2.3}$$

Let C be a nonempty, closed and convex subset of a Hilbert space H. A mapping $T : C \to C$ is quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x-y\|$ for all $x \in C$ and $y \in F(T)$. A mapping $T : C \to C$ is firmly nonexpansive if $\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle$ for all $x, y \in C$. It is easily found that T is firmly nonexpansive if and only if $T = (I+V)/2$ for some nonexpansive mapping V; hence a firmly nonexpansive mapping must be nonexpansive. We also notice that if T is quasi-nonexpansive, then the fixed point set $F(T)$ of T is closed and convex; see [14]. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle \tag{2.4}$$

for all $x, y \in H$. Furthermore $\langle x - P_C x, y - P_C x\rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [27]. Let $\alpha > 0$ be a given constant and let $U \colon C \to H$ be α-inverse strongly monotone. Then $\|Ux - Uy\| \leq (1/\alpha)\|x - y\|$ for all $x, y \in C$, that is, U is continuous. Let B be a mapping of H into 2^H. The effective domain of B is denoted by $\mathrm{dom}(B)$, that is, $\mathrm{dom}(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping B on H is said to be monotone if $\langle x-y, u-v\rangle \geq 0$ for all $x, y \in \mathrm{dom}(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1} \colon H \to \mathrm{dom}(B)$, which is called the resolvent of B for r. Let B be a maximal monotone operator on H and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$. It is known that the resolvent J_r is firmly nonexpansive and $B^{-1}0 = F(J_r)$ for all $r > 0$, where $F(J_r)$ is the set of fixed points of J_r. It is also known that $\|J_\lambda x - J_\mu x\| \leq (|\lambda - \mu|/\lambda)\|x - J_\lambda x\|$ holds for all $\lambda, \mu > 0$ and $x \in H$; see [27, 12] for more details. As a matter of fact, we know the following lemma from Takahashi, Takahashi and Toyoda [26].

Lemma 2.1 ([26])**.** Let H be a Hilbert space and let B be a maximal monotone operator on H. For $r > 0$ and $x \in H$, define the resolvent $J_r x$. Then the following holds:

$$\frac{s-t}{s}\langle J_s x - J_t x, J_s x - x\rangle \geq \|J_s x - J_t x\|^2$$

for all $s, t > 0$ and $x \in H$.

Let H be a Hilbert space and let S be a firmly nonexpansive mapping of H into itself with $F(S) \neq \emptyset$. Then we have that

$$\langle x - Sx, Sx - y\rangle \geq 0 \tag{2.5}$$

for all $x \in H$ and $y \in F(S)$. In fact, we have that for all $x \in H$ and $y \in F(S)$

$$\begin{aligned}\langle x - Sx, Sx - y\rangle &= \langle x - y + y - Sx, Sx - y\rangle \\ &= \langle x - y, Sx - y\rangle + \langle y - Sx, Sx - y\rangle \\ &\geq \|Sx - y\|^2 - \|Sx - y\|^2 \\ &= 0.\end{aligned}$$

We have the following lemma from Alsulami and Takahashi [2].

Lemma 2.2 ([2]). Let H_1 and H_2 be Hilbert spaces and let $\alpha > 0$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $U : H_2 \to H_2$ be an α-inverse strongly monotone mapping. Then a mapping $A^*UA : H_1 \to H_1$ is $\frac{\alpha}{\|A\|^2}$-inverse strongly monotone.

Let $T : H_2 \to H_2$ be a nonexpansive mapping. Since $I-T$ is $\frac{1}{2}$-inverse strongly monotone, we have the following result from Lemma 2.2.

Lemma 2.3. Let H_1 and H_2 be Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping. Then a mapping $A^*(I-T)A : H_1 \to H_1$ is $\frac{1}{2\|A\|^2}$-inverse strongly monotone.

Using (2.5), Takahashi, Xu and Yao [34] proved the following lemma.

Lemma 2.4 ([34]). Let H_1 and H_2 be Hilbert spaces. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda = (I+\lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping and let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$. Let $\lambda, r > 0$ and $z \in H_1$. Then the following are equivalent:

(i) $z = J_\lambda(I - rA^*(I-T)A)z$;

(ii) $0 \in A^*(I-T)Az + Bz$;

(iii) $z \in B^{-1}0 \cap A^{-1}F(T)$.

Furthermore, using Lemma 2.4, Plubtieng and Takahashi [22] proved the following lemma. This lemma is crucial for the proofs of our main theorems.

Lemma 2.5 ([22]). Let H_1 and H_2 be Hilbert spaces. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda = (I+\lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $U : H_2 \to H_2$ be an inverse strongly monotone mapping and let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap A^{-1}(U^{-1}0) \neq \emptyset$. Let $\lambda, r > 0$ and $z \in H_1$. Then the following are equivalent:

(i) $z = J_\lambda(I - rA^*UA)z$;

(ii) $0 \in A^*UAz + Bz$;

(iii) $z \in B^{-1}0 \cap A^{-1}(U^{-1}0)$.

For a sequence $\{C_n\}$ of nonempty, closed and convex subsets of a Hilbert space H, define $\text{s-Li}_n C_n$ and $\text{w-Ls}_n C_n$ as follows: $x \in \text{s-Li}_n C_n$ if and only if there exists $\{x_n\} \subset H$ such that $\{x_n\}$ converges strongly to x and $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in \text{w-Ls}_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset H$ such that $\{y_i\}$ converges weakly to y and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If C_0 satisfies

$$C_0 = \underset{n}{\text{s-Li}}\, C_n = \underset{n}{\text{w-Ls}}\, C_n, \tag{2.6}$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [18] and we write $C_0 = \text{M-lim}_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\bigcap_{n=1}^\infty C_n$ in the sense of Mosco. For more details, see [18]. The following lemma is easily deduced from the theorem for a strictly convex reflexive Banach space with the Kadec-Klee property proved by Tsukada [35].

Lemma 2.6 ([35]). Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of a Hilbert space H. If $C_0 = \bigcap_{n=1}^\infty C_n$ is nonempty, then $P_{C_n}u \to P_{C_0}u$ for any $u \in H$.

Kocourek, Takahashi and Yao [15] defined a broad class of nonlinear mappings in a Hilbert space. Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $T : C \to C$ is called generalized hybrid [15] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{2.7}$$

for all $x, y \in C$. We call such a mapping (α, β)-generalized hybrid. Notice that this class of mappings covers several well-known classes of mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive. It is nonspreading [16, 17] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is also hybrid [30] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous. We here include such an example [13] of nonspreading mappings. Set $E = \{x \in H : \|x\| \leq 1\}$, $D = \{x \in H : \|x\| \leq 2\}$ and $C = \{x \in H : \|x\| \leq 3\}$. Define a mapping $S : C \to C$ by

$$Sx = \begin{cases} 0, & x \in D, \\ P_E x, & x \notin D. \end{cases}$$

Then S is a nonspreading mapping which is however not continuous. This implies that the class of nonexpansive mappings does not contain nonspreading mappings. From [15] we also know the following lemma for generalized hybrid mappings in a Hilbert space.

Lemma 2.7 ([15])**.** Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $T : C \to C$ be a generalized hybrid mapping. Suppose that $\{x_n\} \subset C$ is such that $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$. Then $z \in F(T)$.

3 Main Results

In this section, we prove a strong convergence theorem by the shrinking projection method, which was first proposed by Takahashi, Takeuchi, and Kubota [32].

Theorem 3.1. Let H_1 and H_2 be Hilbert spaces and let $\alpha > 0$. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $U : H_2 \to H_2$ be an α-inverse strongly monotone mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap A^{-1}(U^{-1}0) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H_1 such that $u_n \to u$. Let $x_1 \in H_1$, $C_1 = H_1$, and $\{x_n\}$ be a sequence generated by

$$\begin{cases} y_n = J_{\lambda_n}(I - \lambda_n A^* UA)x_n, \\ C_{n+1} = \{z \in H_1 : \|y_n - z\| \leq \|x_n - z\|\} \cap C_n, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\lambda_n\} \subset (0, \infty)$ satisfies

$$0 < \lambda_n \leq \frac{2\alpha}{\|A\|^2} \quad \text{and} \quad \limsup_{n\to\infty} \lambda_n > 0.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in B^{-1}0 \cap A^{-1}(U^{-1}0)$, where $z_0 = P_{B^{-1}0 \cap A^{-1}(U^{-1}0)} u$.

Proof. We first show that the sequence $\{x_n\}$ is well defined. Let $x_1 \in H_1$ and $y_n = J_{\lambda_n}(I - \lambda_n A^*UA)x_n$ with $0 < \lambda_n \leq \frac{2\alpha}{\|A\|^2}$. For $z \in B^{-1}0 \cap A^{-1}(U^{-1}0)$, we have that

$$\begin{aligned}
\|y_n - z\|^2 &= \|J_{\lambda_n}(I - \lambda_n A^*UA)x_n - J_{\lambda_n} z\|^2 \\
&\leq \|x_n - \lambda_n A^*UAx_n - z\|^2 \\
&= \|x_n - z\|^2 - 2\lambda_n \langle x_n - z, A^*UAx_n\rangle + (\lambda_n)^2 \|A^*UAx_n\|^2 \\
&= \|x_n - z\|^2 - 2\lambda_n \langle Ax_n - Az, UAx_n\rangle + (\lambda_n)^2 \|A^*UAx_n\|^2 \\
&\leq \|x_n - z\|^2 - 2\alpha\lambda_n \|UAx_n\|^2 + (\lambda_n)^2 \|A^*\|^2 \|A^*UAx_n\|^2 \\
&= \|x_n - z\|^2 + \lambda_n(\lambda_n \|A\|^2 - 2\alpha) \|UAx_n\|^2 \\
&\leq \|x_n - z\|^2 .
\end{aligned}$$

Moreover, since

$$\begin{aligned}
\{z \in H_1 : \|y_n - z\| \leq \|x_n - z\|\} &= \{z \in H_1 : \|y_n - z\|^2 \leq \|x_n - z\|^2\} \\
&= \{z \in H_1 : \|y_n\|^2 - \|x_n\|^2 \leq 2\langle y_n - x_n, z\rangle\},
\end{aligned}$$

it is closed and convex. Applying these facts inductively, we obtain that C_n is nonempty, closed, and convex for every $n \in \mathbb{N}$, and hence $\{x_n\}$ is well defined.

Let $C_0 = \bigcap_{n=1}^{\infty} C_n$. Then since $C_0 \supset B^{-1}0 \cap A^{-1}(U^{-1}0) \neq \emptyset$, C_0 is also nonempty. Let $z_n = P_{C_n} u$ for every $n \in \mathbb{N}$. Then, by Lemma 2.6, we have $z_n \to z_0 = P_{C_0} u$. Since a metric projection is nonexpansive, it follows that

$$\begin{aligned}
\|x_n - z_0\| &\leq \|x_n - z_n\| + \|z_n - z_0\| \\
&= \|P_{C_n} u_n - P_{C_n} u\| + \|z_n - z_0\| \\
&\leq \|u_n - u\| + \|z_n - z_0\| \\
&\to 0,
\end{aligned}$$

and hence $x_n \to z_0$.

Since $z_0 \in C_0 \subset C_{n+1}$, we have $\|y_n - z_0\| \leq \|x_n - z_0\|$ for all $n \in \mathbb{N}$. Tending $n \to \infty$, we get that $y_n \to z_0$. By the assumption of $\{\lambda_n\}$, there exists a subsequence $\{\lambda_{n_i}\}$ of $\{\lambda_n\}$ converging to λ_0. From $\lambda_n \leq \frac{2\alpha}{\|A\|^2}$ and $\limsup_{n\to\infty} \lambda_n > 0$, we have that $0 < \lambda_0 \leq \frac{2\alpha}{\|A\|^2}$. Put $v_n = x_n - \lambda_n A^*UAx_n$. We have from Lemma 2.1 that

$$\begin{aligned}
&\|J_{\lambda_0}(I - \lambda_0 A^*UA)x_{n_i} - y_{n_i}\| \\
&\quad = \left\|J_{\lambda_0}(I - \lambda_0 A^*UA)x_{n_i} - J_{\lambda_{n_i}}(I - \lambda_{n_i} A^*UA)x_{n_i}\right\| \\
&\quad = \|J_{\lambda_0}(I - \lambda_0 A^*UA)x_{n_i} - J_{\lambda_0}(I - \lambda_{n_i} A^*UA)x_{n_i} \\
&\qquad + J_{\lambda_0}(I - \lambda_{n_i} A^*UA)x_{n_i} - J_{\lambda_{n_i}}(I - \lambda_{n_i} A^*UA)x_{n_i}\| \\
&\quad \leq \|(I - \lambda_0 A^*UA)x_{n_i} - (I - \lambda_{n_i} A^*UA)x_{n_i}\| + \|J_{\lambda_0} v_{n_i} - J_{\lambda_{n_i}} v_{n_i}\| \\
&\quad \leq |\lambda_0 - \lambda_{n_i}| \|A^*UAx_{n_i}\| + \frac{|\lambda_0 - \lambda_{n_i}|}{\lambda_0} \|J_{\lambda_0} v_{n_i} - v_{n_i}\| \to 0.
\end{aligned}$$

On the other hand, by the continuity of $J_{\lambda_0}(I - \lambda_0 A^*UA)$, we have

$$\|J_{\lambda_0}(I - \lambda_0 A^*UA)x_{n_i} - J_{\lambda_0}(I - \lambda_0 A^*UA)z_0\| \to 0.$$

Hence we have

$$\begin{aligned}\|z_0 - J_{\lambda_0}(I - \lambda_0 A^*UA)z_0\| &\le \|z_0 - y_{n_i}\| + \|y_{n_i} - J_{\lambda_0}(I - \lambda_0 A^*UA)x_{n_i}\| \\ &\quad + \|J_{\lambda_0}(I - \lambda_0 A^*UA)x_{n_i} - J_{\lambda_0}(I - \lambda_0 A^*UA)z_0\| \\ &\to 0.\end{aligned}$$

This implies $z_0 \in B^{-1}0 \cap A^{-1}(U^{-1}0)$ by Lemma 2.4. Since $z_0 = P_{C_0}u \in B^{-1}0 \cap A^{-1}(U^{-1}0)$ and $B^{-1}0 \cap A^{-1}(U^{-1}0) \subset C_0$, we have $z_0 = P_{B^{-1}0\cap A^{-1}(U^{-1}0)}u$, which completes the proof. □

Next, we prove a strong convergence theorem for generalized split feasibility problems which are governed by generalized hybrid mappings in Hilbert spaces.

Theorem 3.2. Let H_1 and H_2 be Hilbert spaces and let C be a nonempty, closed and convex subset of H_1. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping such that the domain of B is included in C and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let S be a generalized hybrid mapping from C into C. Let $U : H_2 \to H_2$ be an α-inverse strongly monotone mapping with $\alpha > 0$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Suppose that $F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H_1 such that $u_n \to u$. Let $C_1 = H_1$ and let $\{x_n\}$ be a sequence in H_1 generated by $x_1 = x \in H_1$ and

$$\begin{cases} z_n = J_{\lambda_n}(I - \lambda_n A^*UA)x_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n)Sz_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ is the metric projection of H_1 onto C_{n+1}, and $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset (0, \infty)$ are sequences such that

$$\liminf_{n\to\infty} \alpha_n < 1 \quad \text{and} \quad 0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < \frac{2\alpha}{\|A\|^2}.$$

Then the sequence $\{x_n\}$ converges strongly to $w_0 = P_{F(S)\cap B^{-1}0\cap A^{-1}(U^{-1}0)}u$, where $P_{F(S)\cap B^{-1}0\cap A^{-1}(U^{-1}0)}$ is the metric projection of H onto $F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0)$.

Proof. Since S is a generalized hybrid mapping from C into C with $F(S) \neq \emptyset$, S is quasi-nonexpansive. Then $F(S)$ is closed and convex. Since $B^{-1}0$ and $U^{-1}0$ are closed and convex [26], $B^{-1}0 \cap A^{-1}(U^{-1}0)$ is closed and convex. Then $F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0)$ is closed and convex. Thus there exists the mertic projection of H onto $F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0)$. We show that C_n are closed and convex for all $n \in \mathbb{N}$. It is obvious from assumption that $C_1 = H_1$ is closed and convex. Suppose that C_k is closed and convex. We know that for $z \in C_k$,

$$\begin{aligned}&\|y_k - z\|^2 \le \|x_k - z\|^2 \\ \Longleftrightarrow &\|y_k\|^2 - \|x_k\|^2 - 2\langle y_k - x_k, z\rangle \le 0.\end{aligned}$$

Then C_{k+1} is closed and convex. By induction, C_n are closed and convex for all $n \in \mathbb{N}$. Next we show that $F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \subset C_n$ for all $n \in \mathbb{N}$. It is obvious from assumption that $F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \subset C_1 = H_1$. Suppose that $F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \subset C_k$ for some $k \in \mathbb{N}$. Put $z_k = J_{\lambda_k}(I - \lambda_k A^*UA)x_k$ and take $z \in F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \subset C_k$.

From $z = J_{\lambda_n}(I - \lambda_n A^*UA)z$ and $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < \frac{2\alpha}{\|A\|^2}$, we have that

$$\begin{aligned}
\|z_k - z\|^2 &= \|J_{\lambda_k}(I - \lambda_k A^*UA)x_k - J_{\lambda_k}(I - \lambda_k A^*UA)z\|^2 \\
&\le \|x_k - \lambda_k A^*UAx_k - z\|^2 \\
&= \|x_k - z\|^2 - 2\lambda_k\langle x_k - z, A^*UAx_k\rangle + \lambda_k^2 \|A^*UAx_k\|^2 \\
&= \|x_k - z\|^2 - 2\lambda_k\langle Ax_k - Az, UAx_k\rangle + (\lambda_k)^2 \|A^*UAx_k\|^2 \\
&\le \|x_k - z\|^2 - 2\lambda_k\alpha \|UAx_k\|^2 + (\lambda_k)^2\|A^*\|^2\|UAx_k\|^2 \\
&= \|x_k - z\|^2 + \lambda_k(\lambda_k \|A\|^2 - 2\alpha) \|UAx_k\|^2 \\
&\le \|x_k - z\|^2 .
\end{aligned} \tag{3.1}$$

Since S is quasi-nonexpansive, we have from (3.1) that

$$\begin{aligned}
\|y_k - z\|^2 &= \|\alpha_k x_k + (1 - \alpha_k)Sz_k - z\|^2 \\
&\le \alpha_k\|x_k - z\|^2 + (1 - \alpha_k)\|Sz_k - z\|^2 \\
&\le \alpha_k\|x_k - z\|^2 + (1 - \alpha_k)\|z_k - z\|^2 \\
&\le \alpha_k\|x_k - z\|^2 + (1 - \alpha_k)\|x_k - z\|^2 \\
&\le \|x_k - z\|^2.
\end{aligned}$$

Hence we have $z \in C_{k+1}$. By induction, we have that

$$F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \subset C_n$$

for all $n \in \mathbb{N}$. Since C_n is nonempty, closed and convex, there exists the metric projection P_{C_n} of H onto C_n. Thus $\{x_n\}$ is well-defined.

Since $\{C_n\}$ is a nonincreasing sequence of nonempty, closed and convex subsets of H with respect to inclusion, it follows that

$$\emptyset \ne F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \subset \text{M-}\lim_{n\to\infty} C_n = \bigcap_{n=1}^{\infty} C_n. \tag{3.2}$$

Put $C_0 = \bigcap_{n=1}^{\infty} C_n$. Then, by Theorem 2.6, we have that $\{P_{C_n}u\}$ converges strongly to $w_0 = P_{C_0}u$, i.e.,

$$w_n = P_{C_n}u \to w_0.$$

To complete the proof, it is sufficient to show that $w_0 = P_{F(S)\cap B^{-1}0\cap A^{-1}(U^{-1}0)}u$.

Since a metric projection is nonexpansive, it follows that

$$\begin{aligned}
\|x_n - w_0\| &\le \|x_n - w_n\| + \|w_n - w_0\| \\
&= \|P_{C_n}u_n - P_{C_n}u\| + \|w_n - w_0\| \\
&\le \|u_n - u\| + \|w_n - w_0\|
\end{aligned}$$

and hence $x_n \to w_0$. Thus we have that

$$\|x_n - x_{n+1}\| \to 0. \tag{3.3}$$

From $x_{n+1} \in C_{n+1}$, we also have that $\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$. Then we get from (3.3) that $\|y_n - x_{n+1}\| \to 0$. Using this, we have

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{3.4}$$

From $0 \leq \liminf_{n\to\infty} \alpha_n < 1$, we have a subsequence $\{\alpha_{n_i}\}$ of $\{\alpha_n\}$ such that $\alpha_{n_i} \to \gamma$ and $0 \leq \gamma < 1$. From

$$\|x_n - y_n\| = \|x_n - \alpha_n x_n - (1-\alpha_n)Sz_n\| = (1-\alpha_n)\|x_n - Sz_n\|,$$

we have that

$$\|Sz_{n_i} - x_{n_i}\| \to 0. \tag{3.5}$$

Let us show $\|Sz_{n_i} - z_{n_i}\| \to 0$ by using (3.5). We have from (3.1) that for any $z \in F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0)$,

$$\begin{aligned}
\|y_n - z\|^2 &= \|\alpha_n x_n + (1-\alpha_n)Sz_n - z\|^2 \\
&\leq \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|z_n - z\|^2 \\
&\leq \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 \\
&\quad + (1-\alpha_n)\lambda_n(\lambda_n \|A\|^2 - 2\alpha)\|UAx_n\|^2 \\
&\leq \|x_n - z\|^2 + (1-\alpha_n)\lambda_n(\lambda_n \|A\|^2 - 2\alpha)\|UAx_n\|^2 .
\end{aligned}$$

Thus we have

$$\begin{aligned}
(1-\alpha_n)\lambda_n(2\alpha - \lambda_n \|A\|^2)\|UAx_n\|^2 &\leq \|x_n - z\|^2 - \|y_n - z\|^2 \\
&= (\|x_n - z\| + \|y_n - z\|)(\|x_n - z\| - \|y_n - z\|) \\
&\leq (\|x_n - z\| + \|y_n - z\|)\|x_n - y_n\| .
\end{aligned}$$

From $\|y_n - x_n\| \to 0$ and $\alpha_{n_i} \to \gamma$, we have that

$$\lim_{i\to\infty} \|UAx_{n_i}\| = 0. \tag{3.6}$$

Since J_{λ_n} is firmly nonexpansive, we have that

$$\begin{aligned}
2\|z_n - z\|^2 &= 2\|J_{\lambda_n}(I - \lambda_n A^*UA)x_n - J_{\lambda_n}(I - \lambda_n A^*UA)z\|^2 \\
&\leq 2\langle z_n - z, (I - \lambda_n A^*UA)x_n - z\rangle \\
&= \|z_n - z\|^2 + \|(I - \lambda_n A^*UA)x_n - z\|^2 \\
&\quad - \|z_n - (I - \lambda_n A^*UA)x_n\|^2 \\
&\leq \|z_n - z\|^2 + \|x_n - z\|^2 - \|z_n - (I - \lambda_n A^*UA)x_n\|^2 \\
&= \|z_n - z\|^2 + \|x_n - z\|^2 - \|z_n - x_n + \lambda_n A^*UAx_n\|^2 \\
&\leq \|z_n - z\|^2 + \|x_n - z\|^2 - \|z_n - x_n\|^2 \\
&\quad - 2\lambda_n\langle z_n - x_n, A^*UAx_n\rangle - \lambda_n^2\|A^*UAx_n\|^2
\end{aligned}$$

and hence

$$\begin{aligned}
\|z_n - z\|^2 \leq \|x_n - z\|^2 &- \|z_n - x_n\|^2 \\
&- 2\lambda_n\langle z_n - x_n, A^*UAx_n\rangle - \lambda_n^2\|A^*UAx_n\|^2.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\|y_n - z\|^2 \le& \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|Sz_n - z\|^2 \\ \le& \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|z_n - z\|^2 \\ \le& \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\{\|x_n - z\|^2 - \|z_n - x_n\|^2 \\ &- 2\lambda_n\langle z_n - x_n, A^*UAx_n\rangle - {\lambda_n}^2\|A^*UAx_n\|^2\} \\ \le& \|x_n - z\|^2 - (1-\alpha_n)\|z_n - x_n\|^2 - {\lambda_n}^2(1-\alpha_n)\|A^*UAx_n\|^2 \\ &- 2\lambda_n(1-\alpha_n)\langle z_n - x_n, A^*UAx_n\rangle.\end{aligned}$$

This means that

$$\begin{aligned}(1-\alpha_n)\|z_n - x_n\|^2 \le& \|x_n - z\|^2 - \|y_n - z\|^2 \\ &+ \|A^*UAx_n\|\{2\lambda_n\|z_n - x_n\| + \lambda_n^2\|A^*UAx_n\|\} \\ \le& (\|x_n - z\| + \|y_n - z\|)\|x_n - y_n\| \\ &+ \|A^*UAx_n\|\{2\lambda_n\|z_n - x_n\| + \lambda_n^2\|A^*UAx_n\|\}.\end{aligned}$$

Since $\lim_{i\to\infty}\|UAx_{n_i}\| = 0$, $\lim_{n\to\infty}\|x_n - y_n\| = 0$, $\alpha_{n_i} \to \gamma < 1$ and $\{y_n\}$, $\{z_n\}$ and $\{x_n\}$ are bounded, we have

$$\lim_{n\to\infty}\|z_{n_i} - x_{n_i}\| = 0. \tag{3.7}$$

Since $y_n = \alpha_n x_n + (1-\alpha_n)Sz_n$, we have $y_n - Sz_n = \alpha_n(x_n - Sz_n)$. From (3.5) we have

$$\|y_{n_i} - Sz_{n_i}\| = \alpha_{n_i}\|x_{n_i} - Sz_{n_i}\| \to 0. \tag{3.8}$$

Since $\|z_{n_i} - Sz_{n_i}\| \le \|z_{n_i} - x_{n_i}\| + \|x_{n_i} - y_{n_i}\| + \|y_{n_i} - Sz_{n_i}\|$, from (3.4), (3.7) and (3.8) we have

$$\|z_{n_i} - Sz_{n_i}\| \to 0. \tag{3.9}$$

Since $x_{n_i} = P_{C_{n_i}}u_{n_i} \to w_0$, we have from (3.7) that $z_{n_i} \to w_0$. Then we have $z_{n_i} \rightharpoonup w_0$. From (3.9) and Lemma 2.7, we have that $w_0 \in F(S)$. Next, let us show that $w_0 \in B^{-1}0 \cap A^{-1}(U^{-1}0)$. From the definition of J_{λ_n}, we have that

$$\begin{aligned}z_n &= J_{\lambda_n}(I - \lambda_n A^*UA)x_n \\ &\Leftrightarrow (I - \lambda_n A^*UA)x_n \in (I + \lambda_n B)z_n = z_n + \lambda_n Bz_n \\ &\Leftrightarrow x_n - z_n - \lambda_n A^*UAx_n \in \lambda_n Bz_n \\ &\Leftrightarrow \frac{1}{\lambda_n}(x_n - z_n - \lambda_n A^*UAx_n) \in Bz_n.\end{aligned}$$

Since B is monotone, we have that for $(s,t) \in B$,

$$\left\langle z_n - s, \frac{x_n - z_n}{\lambda_n} - A^*UAx_n - t\right\rangle \ge 0. \tag{3.10}$$

From $z_{n_i} \rightharpoonup w_0, \|x_{n_i} - z_{n_i}\| \to 0$ and $A^*UAx_{n_i} \to 0$, we have $\langle w_0 - s, -t\rangle \ge 0$. Since B is maximal, we have $0 \in Bw_0$. Furthermore, since U is α-inverse strongly monotone,

$$\langle Ax_{n_i} - Aw_0, UAx_{n_i} - UAw_0\rangle \ge \alpha\|UAx_{n_i} - UAw_0\|^2.$$

From $x_{n_i} \to w_0$ and $UAx_{n_i} \to 0$, we have $UAw_0 = 0$. This implies $Aw_0 \in U^{-1}0$. Therefore, $w_0 \in B^{-1}0 \cap A^{-1}(U^{-1}0)$. Thus we have $w_0 \in F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0)$. Put $z_0 =$

$P_{F(S)\cap B^{-1}0\cap A^{-1}(U^{-1}0)}u$. Since $z_0 = P_{F(S)\cap B^{-1}0\cap A^{-1}(U^{-1}0)}x \in C_n$ and $w_n = P_{C_n}u$, we have that

$$\|u - w_n\|^2 \leq \|u - z_0\|^2. \tag{3.11}$$

Thus we have that

$$\|u - w_0\|^2 = \lim_{n\to\infty} \|u - w_n\|^2 \leq \|u - z_0\|^2.$$

Then we get $z_0 = w_0$. This completes the proof. □

We do not know whether such theorems (Theorems 3.1 and 3.2) hold or not for the hybrid method of Nakajo and Takahashi (Theorem 1.1).

4 Applications

Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $T : C \to H$ is called a strict pseudo-contraction [6] if there exists $k \in \mathbb{R}$ with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

We call such T a k-strict pseudo-contraction. If $k = 0$, then T is nonexpansive. Putting $U = I - T$, where T is a k-strict pseudo-contraction, we have that

$$\|(I - U)x - (I - U)y\|^2 \leq \|x - y\|^2 + k\|Ux - Uy\|^2, \quad \forall x, y \in C.$$

Thus we have that

$$\|x - y\|^2 + \|Ux - Uy\|^2 - 2\langle x - y, Ux - Uy\rangle \leq \|x - y\|^2 + k\|Ux - Uy\|^2.$$

Then

$$\frac{1-k}{2}\|Ux - Uy\|^2 \leq \langle x - y, Ux - Uy\rangle.$$

Therefore, $U = I - T$ is $\frac{1-k}{2}$-inverse strongly monotone.

Let H be a Hilbert space and let f be a proper, lower semicontinuous and convex function of H into $(-\infty, \infty]$. Then the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{z \in H : f(x) + \langle z, y - x\rangle \leq f(y), \quad \forall y \in H\}$$

for all $x \in H$. By Rockafellar [23], it is shown that ∂f is maximal monotone. Let C be a nonempty, closed and convex subset of H and let i_C be the indicator function of C, i.e.,

$$i_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ \infty, & \text{if } x \notin C. \end{cases}$$

Then $i_C : H \to (-\infty, \infty]$ is a proper, lower semicontinuous and convex function on H and hence ∂i_C is a maximal monotone operator. Thus we can define the resolvent J_λ of ∂i_C for $\lambda > 0$ as follows:

$$J_\lambda x = (I + \lambda \partial i_C)^{-1}x, \quad \forall x \in H,\ \lambda > 0.$$

On the other hand, for any $u \in C$, we also define the normal cone $N_C(u)$ of C at u as follows:

$$N_C(u) = \{z \in H : \langle z, y - u\rangle \leq 0, \quad \forall y \in C\}.$$

Then we have that for any $x \in C$

$$\begin{aligned}\partial i_C(x) &= \{z \in H : i_C(x) + \langle z, y - x\rangle \le i_C(y), \quad \forall y \in H\}\\ &= \{z \in H : \langle z, y - x\rangle \le 0, \quad \forall y \in C\}\\ &= N_C(x).\end{aligned}$$

Thus we have that

$$\begin{aligned}u = J_\lambda x &\Leftrightarrow (I + \lambda \partial i_C)^{-1} x = u \Leftrightarrow x \in u + \lambda \partial i_C(u)\\ &\Leftrightarrow x \in u + \lambda N_C(u) \Leftrightarrow x - u \in \lambda N_C(u)\\ &\Leftrightarrow \langle x - u, y - u\rangle \le 0, \quad \forall y \in C\\ &\Leftrightarrow P_C(x) = u,\end{aligned}$$

that is, $J_\lambda = P_C$. Using these results and Theorems 3.1 and 3.2, we can obtain the following strong convergence theorems in Hilbert spaces.

Theorem 4.1. Let H_1 and H_2 be Hilbert spaces and let C be a nonempty, closed and convex subset of H_1. Let $T : H_2 \to H_2$ be a k-strict pseudo-contraction with $0 \le k < 1$. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $C \cap A^{-1}F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H_1 such that $u_n \to u$. Let $x_1 \in H_1$, $C_1 = H_1$, and $\{x_n\}$ be a sequence generated by

$$\begin{cases} y_n = P_C(I - \lambda_n A^*(I - T)A)x_n,\\ C_{n+1} = \{z \in H_1 : \|y_n - z\| \le \|x_n - z\|\} \cap C_n,\\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N},\end{cases}$$

where $\{\lambda_n\} \subset (0, \infty)$ satisfies

$$0 < \lambda_n \le \frac{1-k}{\|A\|^2} \quad \text{and} \quad \limsup_{n\to\infty} \lambda_n > 0.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in C \cap A^{-1}F(T)$, where $z_0 = P_{C\cap A^{-1}F(T)}u$.

Proof. Define $U = I - T$ in Theorem 3.1. Then U is $\frac{1-k}{2}$-inverse strongly monotone. Thus we have the desired result from Theorem 3.1. □

Theorem 4.2. Let H_1 and H_2 be Hilbert spaces and let C be a nonempty, closed and convex subset of H_1. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping such that the domain of B is included in C and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let S be a generalized hybrid mapping from C into C. Let $T : H_2 \to H_2$ be a k-strict pseudo-contraction with $0 \le k < 1$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Suppose that $F(S) \cap B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H_1 such that $u_n \to u$. Let $C_1 = H_1$ and let $\{x_n\}$ be a sequence in H_1 generated by $x_1 = x \in H_1$ and

$$\begin{cases} z_n = J_{\lambda_n}(I - \lambda_n A^*(I - T)A)x_n,\\ y_n = \alpha_n x_n + (1 - \alpha_n)S z_n,\\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\},\\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N},\end{cases}$$

where $P_{C_{n+1}}$ is the metric projection of H_1 onto C_{n+1}, and $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$ are sequences such that

$$\liminf_{n\to\infty} \alpha_n < 1 \quad \text{and} \quad 0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{1-k}{\|A\|^2}.$$

Then the sequence $\{x_n\}$ converges strongly to $w_0 = P_{F(S)\cap B^{-1}0\cap A^{-1}F(T)}u$, where $P_{F(S)\cap B^{-1}0\cap A^{-1}F(T)}$ is the metric projection of H onto $F(S) \cap B^{-1}0 \cap A^{-1}F(T)$.

Let C be a nonempty, closed and convex subset of a Hilbert space H, let $f : C \times C \to \mathbb{R}$ be a bifunction. Then we consider the following equilibrium problem: Find $z \in C$ such that

$$f(z,y) \geq 0, \quad \forall y \in C. \tag{4.1}$$

The set of such $z \in C$ is denoted by $EP(f)$, i.e.,

$$EP(f) = \{z \in C : f(z,y) \geq 0, \ \forall y \in C\}.$$

For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions:

(A1) $f(x,x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e., $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$;

(A3) $\limsup_{t\to 0} f(tz + (1-t)x, y) \leq f(x,y)$ for all $x, y, z \in C$;

(A4) $f(x,\cdot)$ is convex and lower semicontinuous for all $x \in C$.

We know the following lemmas; see, for instance, [5] and [10].

Lemma 4.3 ([5])**.** Let C be a nonempty, closed and convex subset of H, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$f(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0$$

for all $y \in C$.

Lemma 4.4 ([10])**.** Define the resolvent $T_r : H \to C$ of f for $r > 0$ as follows:

$$T_r x = \left\{ z \in C : f(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C \right\}$$

for all $x \in H$. Then, the following hold:

(i) T_r is single-valued;

(ii) T_r is firmly nonexpansive, i.e., for all $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(iii) $F(T_r) = EP(f)$;

(iv) $EP(f)$ is closed and convex.

Takahashi, Takahashi and Toyoda [26] showed the following. See [3] for a more general result.

Lemma 4.5 ([26])**.** Let C be a nonempty, closed and convex subset of a Hilbert space H and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (A1)–(A4). Define A_f as follows:

$$A_f(x) = \begin{cases} \{z \in H : f(x, y) \geq \langle y - x, z\rangle, \ \forall y \in C\}, & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

Then $EP(f) = A_f^{-1}(0)$ and A_f is maximal monotone with the domain in C. Furthermore,

$$T_r(x) = (I + rA_f)^{-1}(x), \quad \forall x \in H, \ r > 0.$$

Using Theorems 3.1 and 3.2 and Lemma 4.5, we have the following theorems.

Theorem 4.6. Let H_1 and H_2 be Hilbert spaces and let C be a nonempty, closed and convex subset of H_1. Let $f : C \times C \to \mathbb{R}$ satisfy the conditions (A1)–(A4) and let T_{λ_n} be the resolvent of A_f for $\lambda_n > 0$ in Lemma 4.5. Let $T : H_2 \to H_2$ be a k-strict pseudo-contraction with $0 \leq k < 1$. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $EP(f) \cap A^{-1}F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H_1 such that $u_n \to u$. Let $x_1 \in H_1$, $C_1 = H_1$, and $\{x_n\}$ be a sequence generated by

$$\begin{cases} y_n = T_{\lambda_n}(I - \lambda_n A^*(I - T)A)x_n, \\ C_{n+1} = \{z \in H_1 : \|y_n - z\| \leq \|x_n - z\|\} \cap C_n, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\lambda_n\} \subset (0, \infty)$ satisfies

$$0 < \lambda_n \leq \frac{1 - k}{\|A\|^2} \quad \text{and} \quad \limsup_{n\to\infty} \lambda_n > 0.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in EP(f) \cap A^{-1}F(T)$, where $z_0 = P_{EP(f)\cap A^{-1}F(T)} u$.

Proof. Define A_f for the bifunction f and set $B = A_f$ in Theorem 3.1. Thus we have the desired result from Theorem 3.1. □

Theorem 4.7. Let H_1 and H_2 be Hilbert spaces. Let C be a nonempty, closed and convex subset of a real Hilbert space H_1. Let $f : C \times C \to \mathbb{R}$ satisfy the conditions (A1)–(A4) and let T_{λ_n} be the resolvent of A_f for $\lambda_n > 0$ in Lemma 4.5. Let S be a generalized hybrid mapping from C into C. Let $T : H_2 \to H_2$ be a k-strict pseudo-contraction with $0 \leq k < 1$. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $F(S) \cap EP(f) \cap A^{-1}F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H_1 such that $u_n \to u$. Let $C_1 = H_1$ and let $\{x_n\}$ be a sequence in H_1 generated by $x_1 = x \in H_1$ and

$$\begin{cases} z_n = J_{\lambda_n}(I - \lambda_n A^*(I - T)A)x_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n)Sz_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ is the metric projection of H_1 onto C_{n+1}, and $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$ are sequences such that

$$\liminf_{n\to\infty} \alpha_n < 1 \quad \text{and} \quad 0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{1-k}{\|A\|^2}.$$

Then the sequence $\{x_n\}$ converges strongly to $w_0 = P_{F(S)\cap EP(f)\cap A^{-1}F(T)}u$, where $P_{F(S)\cap EP(f)\cap A^{-1}F(T)}$ is the metric projection of H onto $F(S) \cap EP(f) \cap A^{-1}F(T)$.

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