Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 12, No. 1, pp. 101–118, January 2016

GENERALIZED SPLIT FEASIBILITY PROBLEMS AND STRONG CONVERGENCE THEOREMS IN HILBERT SPACES

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Abstract: In this paper, motivated by ideas of the split feasibility problem and the split common null point problem and results for solving the problems, we consider generalized split feasibility problems and then establish two strong convergence theorems which are related to the problems. As applications, we get new strong convergence theorems which are connected with fixed point problems, generalized split feasibility problems and equilibrium problems.

Key words: *maximal monotone operator, inverse strongly monotone mapping, fixed point, strong convergence theorem, equilibrium problem, split feasibility problem, strict pseudo-contraction.*

Mathematics Subject Classification: *47H05, 47H09, 47H20.*

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1 Introduction

Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $T : C \to H$ is called a strict pseudo-contraction [8] if there exists $k \in \mathbb{R}$ with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

We call such a mapping T a k-strict pseudo-contraction. If $k = 0$, then T is nonexpansive. A mapping $U : C \to H$ is called inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ux - Uy \rangle \geq \alpha\|Ux - Uy\|^2, \quad \forall x, y \in C.$$

Such a mapping U is called α-inverse strongly monotone. Let $T : C \to H$ be a k-strict pseudo-contraction. Putting $U = I - T$, we have that $U = I - T$ is $\frac{1-k}{2}$-inverse strongly monotone; see Section 4.

Censor and Elfving [10] introduced the split feasibility problem in Hilbert spaces. Let H_1 and H_2 be two real Hilbert spaces. Let D and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the *split feasibility problem* is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Byrne, Censor, Gibali and Reich [9] also considered the following problem: Given set-valued mappings $A_i : H_1 \to 2^{H_1}$, $1 \leq i \leq m$, and $B_j : H_2 \to 2^{H_2}$, $1 \leq j \leq n$, respectively, and bounded

*The third author was partially supported by Grant-in-Aid for Scientific Research No. 15K04906 from Japan Society for the Promotion of Science.

linear operators $T_j : H_1 \to H_2,\ 1 \le j \le n$, the *split common null point problem* [9] is to find $z \in H_1$ such that

$$z \in \big(\cap_{i=1}^{m} A_i^{-1}0 \big) \cap \big(\cap_{j=1}^{n} T_j^{-1}(B_j^{-1}0) \big),$$

where $A_i^{-1}0$ and $B_j^{-1}0$ are null point sets of A_i and B_j, respectively. Defining $U = A^*(I - P_Q)A$ in the split feasibility problem, we have that $U : H_1 \to H_1$ is an inverse strongly monotone operator [2], where A^* is the adjoint operator of A and P_Q is the metric projection of H_2 onto Q. Furthermore, if $D \cap A^{-1}Q$ is nonempty, then $z \in D \cap A^{-1}Q$ is equivalent to

$$z = P_D(I - \lambda A^*(I - P_Q)A)z, \tag{1.1}$$

where $\lambda > 0$ and P_D is the metric projection of H_1 onto D. Using such results regarding nonlinear operators and fixed points, many authors have studied the split feasibility problem and the split common null point problem; see, for instance, [9, 11, 19, 32].

In 1967, Halpern [15] introduced the following iteration process. Let C be a nonempty, closed and convex subset of a Hilbert space H and let $T : C \to C$ be a nonexpansive mapping. Take $x_0, x_1 \in C$ arbitrarily and define $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. There are many investigations of the iterative process for finding fixed points of nonexpansive mappings.

Recently, by using Halpern-type iteration and nonexpansive mappings, Akashi, Kimura and Takahashi [1] defined generalized split feasibility problems and then proved strong convergence theorems for the problems in Hilbert spaces.

In this paper, motivated by the ideas of the split feasibility problem and the split common null point problem and the results of Akashi, Kimura and Takahashi [1], we consider generalized split feasibility problems with inverse strongly monotone mappings and then establish two strong convergence theorems which are related to the problems and generalize Akashi, Kimura and Takahashi's results. As applications, we get new strong convergence theorems which are connected with fixed point problems of strict pseudo-contractions, generalized split feasibility problems and equilibrium problems.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have from [28] that

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y \rangle; \tag{2.1}$$

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{2.2}$$

Furthermore, we have that for $x, y, u, v \in H$,

$$2\langle x - y, u - v \rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.3}$$

Let C be a nonempty, closed and convex subset of a Hilbert space H. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \le \|x - y\|$ for all $x \in H$ and $y \in C$. Such a mapping P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y \rangle \tag{2.4}$$

for all $x, y \in H$. Furthermore, $\langle x - P_C x, y - P_C x\rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [26]. Let $\alpha > 0$ and let $A\colon C \to H$ be an α-inverse strongly monotone mapping. Then we have that $\|Ax - Ay\| \leq (1/\alpha)\|x - y\|$ for all $x, y \in C$. Let B be a mapping of H into 2^H. The effective domain of B is denoted by $\mathrm{dom}(B)$, that is, $\mathrm{dom}(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping B on H is said to be monotone if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in \mathrm{dom}(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1}\colon H \to \mathrm{dom}(B)$, which is called the resolvent of B for r. Let B be a maximal monotone operator on H and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$. It is known that the resolvent J_r is firmly nonexpansive and $B^{-1}0 = F(J_r)$ for all $r > 0$. It is also known that $\|J_\lambda x - J_\mu x\| \leq (|\lambda - \mu|/\lambda)\|x - J_\lambda x\|$ holds for all $\lambda, \mu > 0$ and $x \in H$; see [26, 14] for more details. As a matter of fact, we know the following lemma [25].

Lemma 2.1 ([25])**.** Let H be a Hilbert space and let B be a maximal monotone operator on H. For $r > 0$ and $x \in H$, define the resolvent $J_r x$. Then

$$\frac{s - t}{s}\langle J_s x - J_t x, J_s x - x\rangle \geq \|J_s x - J_t x\|^2$$

for all $s, t > 0$ and $x \in H$.

We also know the following lemmas:

Lemma 2.2 ([4], [34])**.** Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$. Suppose that

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\gamma_n + \beta_n$$

for all $n = 1, 2, ...$ Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3 ([16])**.** Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n\geq n_0}$ of integers as follows:

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(n_0) \leq \tau(n_0 + 1) \leq \cdots$ and $\tau(n) \to \infty$;

(ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $\Gamma_n \leq \Gamma_{\tau(n)+1}$, $\forall n \geq n_0$.

Let H be a Hilbert space and let S be a firmly nonexpansive mapping of H into itself with $F(S) \neq \emptyset$. Then we have that

$$\langle x - Sx, Sx - y\rangle \geq 0 \tag{2.5}$$

for all $x \in H$ and $y \in F(S)$. In fact, we have that for all $x \in H$ and $y \in F(S)$,

$$\begin{aligned}\langle x - Sx, Sx - y\rangle &= \langle x - y + y - Sx, Sx - y\rangle \\ &= \langle x - y, Sx - y\rangle + \langle y - Sx, Sx - y\rangle \\ &\geq \|Sx - y\|^2 - \|Sx - y\|^2 \\ &= 0.\end{aligned}$$

We have the following lemma from Alsulami and Takahashi [2].

Lemma 2.4 ([2])**.** Let H_1 and H_2 be Hilbert spaces and let $\alpha > 0$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $U : H_2 \to H_2$ be an α-inverse strongly monotone mapping. Then a mapping $A^*UA : H_1 \to H_1$ is $\frac{\alpha}{\|A\|^2}$-inverse strongly monotone.

If T is a nonexpansive mapping, then $I - T$ is $\frac{1}{2}$-inverse strongly monotone. So we have the following result from Lemma 2.4.

Lemma 2.5. Let H_1 and H_2 be Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping. Then a mapping $A^*(I - T)A : H_1 \to H_1$ is $\frac{1}{2\|A\|^2}$-inverse strongly monotone.

The following lemma was proved in [32].

Lemma 2.6 ([32])**.** Let H_1 and H_2 be Hilbert spaces. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda = (I+\lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping and let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$. Let $\lambda, r > 0$ and $z \in H_1$. Then the following are equivalent:

(i) $z = J_\lambda(I - rA^*(I - T)A)z$;

(ii) $0 \in A^*(I - T)Az + Bz$;

(iii) $z \in B^{-1}0 \cap A^{-1}F(T)$.

Using Lemma 2.6, Plubtieng and Takahashi [21] proved the following lemma. This lemma is crucial for the proofs of our main results.

Lemma 2.7 ([21])**.** Let H_1 and H_2 be Hilbert spaces and let $\alpha > 0$. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $U : H_2 \to H_2$ be an α-inverse strongly monotone mapping and let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap A^{-1}(U^{-1}0) \neq \emptyset$. Let $\lambda, r > 0$ and $z \in H_1$. Then the following are equivalent:

(i) $z = J_\lambda(I - rA^*UA)z$;

(ii) $0 \in A^*UAz + Bz$;

(iii) $z \in B^{-1}0 \cap A^{-1}(U^{-1}0)$.

3 Main Results

In this section, we first prove a strong convergence theorem which generalizes Akashi, Kimura and Takahashi's theorem [1] in Hilbert spaces.

Theorem 3.1. Let H_1 and H_2 be Hilbert spaces and let $\alpha > 0$. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $U : H_2 \to H_2$ be an α-inverse strong monotone mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap A^{-1}(U^{-1}0) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H_1 such that $u_n \to u$. Let $x_1 = x \in H_1$ and let $\{x_n\} \subset H_1$ be a sequence generated by

$$x_{n+1} = \alpha_n u_n + (1 - \alpha_n)J_{\lambda_n}(I - \lambda_n A^*UA)x_n$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy

$$0 < a \leq \lambda_n \leq \frac{2\alpha}{\|A\|^2}, \quad \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty,$$

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \textit{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Then the sequence $\{x_n\}$ converges strongly to a point z_0 of $B^{-1}0 \cap A^{-1}(U^{-1}0)$, where $z_0 = P_{B^{-1}0\cap A^{-1}(U^{-1}0)}u$.

Proof. Put $y_n = J_{\lambda_n}(I-\lambda_n A^*UA)x_n$ and let $z \in B^{-1}0\cap A^{-1}(U^{-1}0)$. We have that $z = J_{\lambda_n}z$ and $UAz = 0$. Since J_{λ_n} is nonexpansive and U is α-inverse strongly monotone, we have that

$$\begin{aligned}
\|y_n - z\|^2 &= \|J_{\lambda_n}(I - \lambda_n A^*UA)x_n - J_{\lambda_n}z\|^2 \\
&\le \|x_n - \lambda_n A^*UAx_n - z\|^2 \\
&= \|x_n - z\|^2 - 2\lambda_n\langle x_n - z, A^*UAx_n\rangle + (\lambda_n)^2 \|A^*UAx_n\|^2 \qquad (3.1)\\
&= \|x_n - z\|^2 - 2\lambda_n\langle Ax_n - Az, UAx_n\rangle + (\lambda_n)^2 \|A^*UAx_n\|^2 \\
&\le \|x_n - z\|^2 - 2\alpha\lambda_n \|UAx_n\|^2 + (\lambda_n)^2 \|A\|^2 \|UAx_n\|^2 \\
&= \|x_n - z\|^2 + \lambda_n(\lambda_n \|A\|^2 - 2\alpha) \|UAx_n\|^2 .
\end{aligned}$$

From $0 < a \le \lambda_n \le \frac{2\alpha}{\|A\|^2}$ we have that $\|y_n - z\| \le \|x_n - z\|$ for all $n \in \mathbb{N}$. We also have from $x_{n+1} = \alpha_n u_n + (1-\alpha_n)y_n$ that

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n(u_n - z) + (1-\alpha_n)(y_n - z)\| \\
&\le \alpha_n \|u_n - z\| + (1-\alpha_n) \|x_n - z\| .
\end{aligned}$$

Since $\{u_n\}$ is bounded, there exists $M > 0$ such that $\sup_{n\in\mathbb{N}} \|u_n - z\| \le M$. Putting $K = \max\{M, \|x_1 - z\|\}$, we have that $\|x_n - z\| \le K$ for all $n \in \mathbb{N}$. In fact, it is obvious that $\|x_1 - z\| \le K$. Suppose that $\|x_k - z\| \le K$ for some $k \in \mathbb{N}$. Then we have that

$$\begin{aligned}
\|x_{k+1} - z\| &\le \alpha_k\|u_k - z\| + (1-\alpha_k)\|x_k - z\| \\
&\le \alpha_k K + (1-\alpha_k)K \\
&= K.
\end{aligned}$$

By induction, we obtain that $\|x_n - z\| \le K$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is bounded. Furthermore, $\{Ax_n\}$ and $\{y_n\}$ are bounded. Since

$$\|UAx_n\|^2 \le \frac{1}{\alpha}\langle UAx_n, Ax_n - Az\rangle \le \frac{1}{\alpha}\|UAx_n\|\|Ax_n - Az\|,$$

$\{UAx_n\}$ is bounded. Then $\{A^*UAx_n\}$ is bounded. Putting $v_n = x_n - \lambda_n A^*UAx_n$, we have

$$\begin{aligned}
x_{n+2} - x_{n+1} &= (\alpha_{n+1} - \alpha_n)u_n + \alpha_{n+1}(u_{n+1} - u_n) \\
&\quad + (1-\alpha_{n+1})J_{\lambda_{n+1}}(x_{n+1} - \lambda_{n+1}A^*UAx_{n+1}) \\
&\quad - (1-\alpha_n)J_{\lambda_n}(x_n - \lambda_n A^*UAx_n) \\
&= (\alpha_{n+1} - \alpha_n)u_n + \alpha_{n+1}(u_{n+1} - u_n) \\
&\quad + (1-\alpha_{n+1})\{J_{\lambda_{n+1}}(x_{n+1} - \lambda_{n+1}A^*UAx_{n+1}) \\
&\quad - J_{\lambda_{n+1}}v_n + J_{\lambda_{n+1}}v_n - J_{\lambda_n}v_n + J_{\lambda_n}v_n\} - (1-\alpha_n)J_{\lambda_n}v_n.
\end{aligned}$$

Thus we have from Lemma 2.1 and the nonexpansiveness of $I-\lambda_{n+1}A^*UA$ that

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &\le |\alpha_{n+1}-\alpha_n|\ \|u_n\|+\alpha_{n+1}\|u_{n+1}-u_n\| \\
&\quad+(1-\alpha_{n+1})\|x_{n+1}-\lambda_{n+1}A^*UAx_{n+1}-(x_n-\lambda_nA^*UAx_n)\| \\
&\quad+(1-\alpha_{n+1})\|J_{\lambda_{n+1}}v_n-J_{\lambda_n}v_n\|+|\alpha_{n+1}-\alpha_n|\|J_{\lambda_n}v_n\| \\
&=|\alpha_{n+1}-\alpha_n|\ \|u_n\|+\alpha_{n+1}\|u_{n+1}-u_n\| \\
&\quad+(1-\alpha_{n+1})\|(I-\lambda_{n+1}A^*UA)x_{n+1}-(I-\lambda_{n+1}A^*UA)x_n \\
&\quad+(I-\lambda_{n+1}A^*UA)x_n-(x_n-\lambda_nA^*UAx_n)\| \\
&\quad+(1-\alpha_{n+1})\|J_{\lambda_{n+1}}v_n-J_{\lambda_n}v_n\|+|\alpha_{n+1}-\alpha_n|\|J_{\lambda_n}v_n\| \\
&\le|\alpha_{n+1}-\alpha_n|\ \|u_n\|+\alpha_{n+1}\|u_{n+1}-u_n\| \\
&\quad+(1-\alpha_{n+1})\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|A^*UAx_n\| \\
&\quad+|\alpha_{n+1}-\alpha_n|\|J_{\lambda_n}v_n\|+\|J_{\lambda_{n+1}}v_n-J_{\lambda_n}v_n\| \\
&\le|\alpha_{n+1}-\alpha_n|\ \|u_n\|+\alpha_{n+1}\|u_{n+1}-u_n\| \\
&\quad+(1-\alpha_{n+1})\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|A^*UAx_n\| \\
&\quad+|\alpha_{n+1}-\alpha_n|\|J_{\lambda_n}v_n\|+\frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}}\|J_{\lambda_{n+1}}v_n-v_n\| \\
&\le|\alpha_{n+1}-\alpha_n|\ \|u_n\|+\alpha_{n+1}\|u_{n+1}-u_n\| \\
&\quad+(1-\alpha_{n+1})\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|A^*UAx_n\| \\
&\quad+|\alpha_{n+1}-\alpha_n|\|J_{\lambda_n}v_n\|+\frac{|\lambda_{n+1}-\lambda_n|}{a}\|J_{\lambda_{n+1}}v_n-v_n\|.
\end{aligned}$$

Using Lemma 2.2, we obtain that

$$\|x_{n+2}-x_{n+1}\|\to 0. \tag{3.2}$$

We also have from (2.2) that

$$\begin{aligned}
\|x_{n+1}-x_n\|^2 &= \|\alpha_n(u_n-x_n)+(1-\alpha_n)(y_n-x_n)\|^2 \\
&=\alpha_n\|u_n-x_n\|^2+(1-\alpha_n)\|y_n-x_n\|^2-\alpha_n(1-\alpha_n)\|u_n-y_n\|^2
\end{aligned}$$

and hence

$$(1-\alpha_n)\|y_n-x_n\|^2=\|x_{n+1}-x_n\|^2-\alpha_n\|u_n-x_n\|^2+\alpha_n(1-\alpha_n)\|u_n-y_n\|^2.$$

From $\alpha_n\to 0$ and $\|x_{n+1}-x_n\|\to 0$, we get that

$$y_n-x_n\to 0. \tag{3.3}$$

From $\sum_{n=1}^{\infty}|\lambda_n-\lambda_{n+1}|<\infty$, we have that $\{\lambda_n\}$ is a Cauchy sequence. Then we have that $\lambda_n\to\lambda_0$ and $\lambda_0\in[a,\frac{2\alpha}{\|A\|^2}]$. Using $v_n=x_n-\lambda_nA^*UAx_n$ and $y_n=J_{\lambda_n}(I-\lambda_nA^*UA)x_n$, we have from Lemma 2.1 that

$$\begin{aligned}
\|J_{\lambda_0}(I-\lambda_0A^*UA)x_n-y_n\| &= \|J_{\lambda_0}(I-\lambda_0A^*UA)x_n-J_{\lambda_n}(I-\lambda_nA^*UA)x_n\| \\
&=\|J_{\lambda_0}(I-\lambda_0A^*UA)x_n-J_{\lambda_0}(I-\lambda_nA^*UA)x_n \\
&\quad+J_{\lambda_0}(I-\lambda_nA^*UA)x_n-J_{\lambda_n}(I-\lambda_nA^*UA)x_n\| \\
&\le\|(I-\lambda_0A^*UA)x_n-(I-\lambda_nA^*UA)x_n\|+\|J_{\lambda_0}v_n-J_{\lambda_n}v_n\| \\
&\le|\lambda_0-\lambda_n|\|A^*UAx_n\|+\frac{|\lambda_0-\lambda_n|}{\lambda_0}\|J_{\lambda_0}v_n-v_n\|\to 0.
\end{aligned} \tag{3.4}$$

We also have from (3.3) and (3.4) that

$$\|x_n - J_{\lambda_0}(I - \lambda_0 A^* U A)x_n\| \leq \|x_n - y_n\| + \|y_n - J_{\lambda_0}(I - \lambda_0 A^* U A)x_n\| \to 0. \tag{3.5}$$

We will use (3.4) and (3.5) later.

Put $z_0 = P_{B^{-1}0 \cap A^{-1}(U^{-1}0)}u$. Let us show that $\limsup_{n\to\infty} \langle u - z_0, y_n - z_0\rangle \leq 0$. Put $l = \limsup_{n\to\infty} \langle u - z_0, y_n - z_0\rangle$. Then without loss of generality, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $l = \lim_{i\to\infty} \langle u - z_0, y_{n_i} - z_0\rangle$ and $\{y_{n_i}\}$ converges weakly to some point $w \in H_1$. From $\|x_n - y_n\| \to 0$, we also have that $\{x_{n_i}\}$ converges weakly to $w \in H_1$. On the other hand, from $\lambda_n \to \lambda_0 \in [a, \frac{2\alpha}{\|A\|^2}]$, we have $\lambda_{n_i} \to \lambda_0 \in [a, \frac{2\alpha}{\|A\|^2}]$. Using (3.4), we have that

$$\|J_{\lambda_0}(I - \lambda_0 A^* U A)x_{n_i} - y_{n_i}\| \to 0.$$

Furthermore, using (3.5), we have that

$$\|x_{n_i} - J_{\lambda_0}(I - \lambda_0 A^* U A)x_{n_i}\| \to 0.$$

Since $J_{\lambda_0}(I - \lambda_0 A^* U A)$ is nonexpansive, we have $w = J_{\lambda_0}(I - \lambda_0 A^* U A)w$ from [28, p.114]. From Lemma 2.7 we have that $w \in B^{-1}0 \cap A^{-1}(U^{-1}0)$. Then we have

$$l = \lim_{i\to\infty} \langle u - z_0, y_{n_i} - z_0\rangle = \langle u - z_0, w - z_0\rangle \leq 0.$$

Since $x_{n+1} - z_0 = \alpha_n(u_n - z_0) + (1 - \alpha_n)(y_n - z_0)$, we have from (2.1) that

$$\begin{aligned} \|x_{n+1} - z_0\|^2 &\leq (1 - \alpha_n)^2 \|y_n - z_0\|^2 + 2\langle \alpha_n(u_n - z_0), x_{n+1} - z_0\rangle \\ &\leq (1 - \alpha_n)\|x_n - z_0\|^2 + 2\alpha_n\langle u_n - u, x_{n+1} - z_0\rangle \\ &\quad + 2\alpha_n\langle u - z_0, x_{n+1} - x_n + x_n - y_n + y_n - z_0\rangle. \end{aligned} \tag{3.6}$$

Putting $\gamma_n = 2\langle u_n - u, x_{n+1} - z_0\rangle + 2\langle u - z_0, x_{n+1} - x_n + x_n - y_n + y_n - z_0\rangle$, $s_n = \|x_n - z_0\|^2$ and $\beta_n = 0$ in Lemma 2.2, from $\sum_{n=1}^{\infty} \alpha_n = \infty$ and (3.6) we have that $x_n \to z_0$. This completes the proof. □

Next, we prove another strong convergence theorem which is obtained by using Maingé lemma (Lemma 2.3).

Theorem 3.2. *Let H_1 and H_2 be Hilbert spaces and let $\alpha > 0$. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $U : H_2 \to H_2$ be an α-inverse strong monotone mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap A^{-1}(U^{-1}0) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H_1 such that $u_n \to u$. Let $x_1 = x \in H_1$ and let $\{x_n\} \subset H_1$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u_n + (1 - \alpha_n)J_{\lambda_n}(I - \lambda_n A^* U A)x_n)$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, \infty)$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy

$$0 < a \leq \lambda_n \leq \frac{2\alpha}{\|A\|^2}, \quad 0 < c \leq \beta_n \leq d < 1,$$

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then $\{x_n\}$ converges strongly to a point z_0 of $B^{-1}0 \cap A^{-1}(U^{-1}0)$, where $z_0 = P_{B^{-1}0 \cap A^{-1}(U^{-1}0)}u$.

Proof. Let $z \in B^{-1}0 \cap A^{-1}(U^{-1}0)$. As in the proof of Theorem 3.1, we obtain that

$$\begin{aligned} \|J_{\lambda_n}(x_n - \lambda_n A^* UAx_n) - z\|^2 &\le \|x_n - z\|^2 + \lambda_n(\lambda_n \|A\|^2 - 2\alpha) \|UAx_n\|^2 \\ &\le \|x_n - z\|^2 . \end{aligned} \tag{3.7}$$

Let $y_n = \alpha_n u_n + (1-\alpha_n)J_{\lambda_n}(x_n - \lambda_n A^* UAx_n)$. We have that

$$\begin{aligned} \|y_n - z\| &= \|\alpha_n(u_n - z) + (1-\alpha_n)(J_{\lambda_n}(x_n - \lambda_n A^* UAx_n) - z)\| \\ &\le \alpha_n \|u_n - z\| + (1-\alpha_n) \|x_n - z\| . \end{aligned}$$

Using this, we get

$$\begin{aligned} \|x_{n+1} - z\| &= \|\beta_n(x_n - z) + (1-\beta_n)(y_n - z)\| \\ &\le \beta_n \|x_n - z\| + (1-\beta_n) \|y_n - z\| \\ &\le \beta_n \|x_n - z\| + (1-\beta_n)(\alpha_n \|u_n - z\| + (1-\alpha_n) \|x_n - z\|) \\ &\quad (1 - \alpha_n(1-\beta_n))\|x_n - z\| + \alpha_n(1-\beta_n)\|u_n - z\|. \end{aligned}$$

Since $\{u_n\}$ is bounded, there exists $M > 0$ such that $\sup_{n\in\mathbb{N}} \|u_n - z\| \le M$. Putting $K = \max\{\|x_1 - z\|, M\}$, we have that $\|x_n - z\| \le K$ for all $n \in \mathbb{N}$. In fact, it is obvious that $\|x_1 - z\| \le K$. Suppose that $\|x_k - z\| \le K$ for some $k \in \mathbb{N}$. Then we have that

$$\begin{aligned} \|x_{k+1} - z\| &\le (1 - \alpha_k(1-\beta_k))\|x_k - z\| + \alpha_k(1-\beta_k)\|u_k - z\| \\ &\le (1 - \alpha_k(1-\beta_k))K + \alpha_k(1-\beta_k)K = K. \end{aligned}$$

By induction, we obtain that $\|x_n - z\| \le K$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is bounded. Furthermore, $\{Ax_n\}$, $\{y_n\}$ and $\{J_{\lambda_n}(x_n - \lambda_n A^* UAx_n)\}$ are bounded. Take $z_0 = P_{B^{-1}0 \cap A^{-1}(U^{-1}0)}u$. Putting $z_n = J_{\lambda_n}(I - \lambda_n A^* UA)x_n$, from the definition of $\{x_n\}$ we have that

$$x_{n+1} - x_n = \beta_n x_n + (1-\beta_n)\{\alpha_n u_n + (1-\alpha_n)z_n\} - x_n$$

and hence

$$\begin{aligned} x_{n+1} - x_n - (1-\beta_n)\alpha_n u_n &= \beta_n x_n + (1-\beta_n)(1-\alpha_n)z_n - x_n \\ &= (1-\beta_n)\{(1-\alpha_n)z_n - x_n\} \\ &= (1-\beta_n)\{z_n - x_n - \alpha_n z_n\}. \end{aligned}$$

Thus we have that

$$\begin{aligned} \langle x_{n+1} - x_n - (1-\beta_n)\alpha_n u_n, x_n - z_0\rangle &= (1-\beta_n)\langle z_n - x_n, x_n - z_0\rangle \\ &\quad - (1-\beta_n)\langle \alpha_n z_n, x_n - z_0\rangle \\ &= -(1-\beta_n)\langle x_n - z_n, x_n - z_0\rangle \\ &\quad - (1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle. \end{aligned} \tag{3.8}$$

From (2.3) and (3.7), we have that

$$\begin{aligned} 2\langle x_n - z_n, x_n - z_0\rangle &= \|x_n - z_0\|^2 + \|z_n - x_n\|^2 - \|z_n - z_0\|^2 \\ &\ge \|x_n - z_0\|^2 + \|z_n - x_n\|^2 - \|x_n - z_0\|^2 \\ &= \|z_n - x_n\|^2. \end{aligned} \tag{3.9}$$

From (3.8) and (3.9), we have that

$$\begin{aligned} -2\langle x_n - x_{n+1}, x_n - z_0\rangle &= 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle \\ &\quad - 2(1-\beta_n)\langle x_n - z_n, x_n - z_0\rangle - 2(1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle \\ &\le 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle \\ &\quad - (1-\beta_n)\|z_n - x_n\|^2 - 2(1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle. \end{aligned} \tag{3.10}$$

Furthermore, using (2.3) and (3.10), we have that

$$\begin{aligned} \|x_{n+1} - z_0\|^2 - \|x_n - x_{n+1}\|^2 - \|x_n - z_0\|^2 &\le 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle \\ &\quad - (1-\beta_n)\|z_n - x_n\|^2 - 2(1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle. \end{aligned}$$

Setting $\Gamma_n = \|x_n - z_0\|^2$, we have that

$$\begin{aligned} \Gamma_{n+1} - \Gamma_n - \|x_n - x_{n+1}\|^2 &\le 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle \\ &\quad - (1-\beta_n)\|z_n - x_n\|^2 - 2(1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle. \end{aligned} \tag{3.11}$$

Noting that

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|(1-\beta_n)\alpha_n(u_n - z_n) + (1-\beta_n)(z_n - x_n)\| \\ &\le (1-\beta_n)\big(\|z_n - x_n\| + \alpha_n\|u_n - z_n\|\big) \end{aligned} \tag{3.12}$$

and hence

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &\le (1-\beta_n)^2\big(\|z_n - x_n\| + \alpha_n\|u_n - z_n\|\big)^2 \\ &= (1-\beta_n)^2\|z_n - x_n\|^2 \\ &\quad + (1-\beta_n)^2\big(2\alpha_n\|z_n - x_n\|\|u_n - z_n\| + \alpha_n^2\|u_n - z_n\|^2\big). \end{aligned} \tag{3.13}$$

Thus we have from (3.11) and (3.13) that

$$\begin{aligned} \Gamma_{n+1} - \Gamma_n &\le \|x_n - x_{n+1}\|^2 + 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle \\ &\quad - (1-\beta_n)\|z_n - x_n\|^2 - 2(1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle \\ &\le (1-\beta_n)^2\|z_n - x_n\|^2 \\ &\quad + (1-\beta_n)^2\big(2\alpha_n\|z_n - x_n\|\|u_n - z_n\| + \alpha_n^2\|u_n - z_n\|^2\big) \\ &\quad + 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle - (1-\beta_n)\|z_n - x_n\|^2 \\ &\quad - 2(1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle \end{aligned}$$

and hence

$$\begin{aligned} &\Gamma_{n+1} - \Gamma_n + \beta_n(1-\beta_n)\|z_n - x_n\|^2 \\ &\quad \le (1-\beta_n)^2\big(2\alpha_n\|z_n - x_n\|\|u_n - z_n\| + \alpha_n^2\|u_n - z_n\|^2\big) \\ &\qquad + 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle - 2(1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle. \end{aligned} \tag{3.14}$$

We will divide the proof into two cases.

Case 1: Suppose that there exists a natural number N such that $\Gamma_{n+1} \le \Gamma_n$ for all $n \ge N$. In this case, $\lim_{n\to\infty}\Gamma_n$ exists and then $\lim_{n\to\infty}(\Gamma_{n+1} - \Gamma_n) = 0$. Using $\lim_{n\to\infty}\alpha_n = 0$ and $0 < c \le \beta_n \le d < 1$, we have from (3.14) that

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.15}$$

From (3.12) we have that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.16}$$

We also have that

$$\begin{aligned} \|y_n - z_n\| &= \|\alpha_n u_n + (1-\alpha_n) z_n - z_n\| \\ &= \alpha_n \|u_n - z_n\| \to 0. \end{aligned} \tag{3.17}$$

Furthermore, from $\|y_n - x_n\| \le \|y_n - z_n\| + \|z_n - x_n\|$, we have that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.18}$$

Take $\lambda_0 \in [a, \frac{2\alpha}{\|A\|^2}]$. Putting $v_n = x_n - \lambda_n A^* U A x_n$, we have from Lemma 2.1 that

$$\begin{aligned} &\|\alpha_n u_n + (1-\alpha_n) J_{\lambda_0}(I - \lambda_0 A^* U A) x_n - y_n\| \\ &\quad = (1-\alpha_n)\|J_{\lambda_0}(I - \lambda_0 A^* U A) x_n - J_{\lambda_n}(I - \lambda_n A^* U A) x_n\| \\ &\quad = (1-\alpha_n)\|J_{\lambda_0}(I - \lambda_0 A^* U A) x_n - J_{\lambda_0}(I - \lambda_n A^* U A) x_n \\ &\qquad + J_{\lambda_0}(I - \lambda_n A^* U A) x_n - J_{\lambda_n}(I - \lambda_n A^* U A) x_n\| \\ &\quad \le (1-\alpha_n)\{\|(I - \lambda_0 A^* U A) x_n - v_n\| \\ &\qquad + \|J_{\lambda_0} v_n - J_{\lambda_n} v_n\|\} \\ &\quad \le (1-\alpha_n)\Big\{|\lambda_0 - \lambda_n|\|A^* U A x_n\| + \frac{|\lambda_0 - \lambda_n|}{\lambda_0}\|J_{\lambda_0} v_n - v_n\|\Big\}. \end{aligned} \tag{3.19}$$

We also have that

$$\begin{aligned} &\|x_n - J_{\lambda_0}(I - \lambda_0 A^* U A) x_n\| \\ &\quad \le \|x_n - y_n\| + \|y_n - \{\alpha_n u_n + (1-\alpha_n) J_{\lambda_0}(I - \lambda_0 A^* U A) x_n\}\| \\ &\qquad + \|\alpha_n u_n + (1-\alpha_n) J_{\lambda_0}(I - \lambda_0 A^* U A) x_n \\ &\qquad\quad - J_{\lambda_0}(I - \lambda_0 A^* U A) x_n\| \\ &\quad = \|x_n - y_n\| + \|y_n - (\alpha_n u_n + (1-\alpha_n) J_{\lambda_0}(I - \lambda_0 A^* U A) x_n)\| \\ &\qquad + \alpha_n \|u_n - J_{\lambda_0}(I - \lambda_0 A^* U A) x_n\|. \end{aligned} \tag{3.20}$$

We will use (3.19) and (3.20) later.

Let us show that $\limsup_{n\to\infty} \langle u - z_0, y_n - z_0\rangle \le 0$, where $z_0 = P_{B^{-1}0 \cap A^{-1}(U^{-1}0)} u$. Put $l = \limsup_{n\to\infty} \langle u - z_0, y_n - z_0\rangle$. Then without loss of generality, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $l = \lim_{i\to\infty} \langle u - z_0, y_{n_i} - z_0\rangle$ and $\{y_{n_i}\}$ converges weakly to some point $w \in H_1$. From $\|x_n - y_n\| \to 0$, we also have that $\{x_{n_i}\}$ converges weakly to $w \in H_1$. On the other hand, since $\{\lambda_n\} \subset (0, \infty)$ satisfies $0 < a \le \lambda_n \le \frac{2\alpha}{\|A\|^2}$, there exists a subsequence $\{\lambda_{n_{i_j}}\}$ of $\{\lambda_{n_i}\}$ such that $\{\lambda_{n_{i_j}}\}$ converges to a number $\lambda_0 \in [a, \frac{2\alpha}{\|A\|^2}]$. Using (3.19), we have that

$$\|\alpha_{n_{i_j}} u_{n_{i_j}} + (1-\alpha_{n_{i_j}}) J_{\lambda_0}(I - \lambda_0 A^* U A) x_{n_{i_j}} - y_{n_{i_j}}\| \to 0.$$

Furthermore, using (3.20), we have that

$$\begin{aligned} \|x_{n_{i_j}} - J_{\lambda_0}(I - \lambda_0 A^* U A) x_{n_{i_j}}\| &\le \|x_{n_{i_j}} - y_{n_{i_j}}\| \\ &\quad + \|y_{n_{i_j}} - \{\alpha_{n_{i_j}} u_{n_{i_j}} + (1-\alpha_{n_{i_j}}) J_{\lambda_0}(I - \lambda_0 A^* U A) x_{n_{i_j}}\}\| \\ &\quad + \alpha_{n_{i_j}} \|u_{n_{i_j}} - J_{\lambda_0}(I - \lambda_0 A^* U A) x_{n_{i_j}}\| \to 0. \end{aligned}$$

Since $J_{\lambda_0}(I-\lambda_0 A^*UA)$ is nonexpansive, we have $w = J_{\lambda_0}(I-\lambda_0 A^*UA)w$ from [28, p.114]. From Lemma 2.7 we have that $w \in B^{-1}0 \cap A^{-1}(U^{-1}0)$. Then we have that

$$l = \lim_{j\to\infty}\langle u - z_0, y_{n_{i_j}} - z_0\rangle = \langle u - z_0, w - z_0\rangle \leq 0.$$

Since $y_n - z_0 = \alpha_n(u_n - z_0) + (1-\alpha_n)\{J_{\lambda_n}(x_n - \lambda_n A^*UAx_n) - z_0\}$, we have from (2.1) that

$$\|y_n - z_0\|^2 \leq (1-\alpha_n)^2\left\|J_{\lambda_n}(x_n - \lambda_n A^*UAx_n) - z_0\right\|^2 + 2\alpha_n\left\langle u_n - z_0, y_n - z_0\right\rangle.$$

From (3.7), we have

$$\|y_n - z_0\|^2 \leq (1-\alpha_n)^2\|x_n - z_0\|^2 + 2\alpha_n\left\langle u_n - z_0, y_n - z_0\right\rangle.$$

This implies that

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &\leq \beta_n\|x_n - z_0\|^2 + (1-\beta_n)\|y_n - z_0\|^2 \\
&\leq \beta_n\|x_n - z_0\|^2 + (1-\beta_n)\Big((1-\alpha_n)^2\|x_n - z_0\|^2 + 2\alpha_n\left\langle u_n - z_0, y_n - z_0\right\rangle\Big) \\
&= \big(\beta_n + (1-\beta_n)(1-\alpha_n)^2\big)\|x_n - z_0\|^2 + 2(1-\beta_n)\alpha_n\left\langle u_n - z_0, y_n - z_0\right\rangle \\
&\leq (\beta_n + (1-\beta_n)(1-\alpha_n))\|x_n - z_0\|^2 + 2(1-\beta_n)\alpha_n\left\langle u_n - z_0, y_n - z_0\right\rangle \\
&= (1-(1-\beta_n)\alpha_n)\|x_n - z_0\|^2 + 2(1-\beta_n)\alpha_n\left\langle u_n - u, y_n - z_0\right\rangle \\
&\quad + 2(1-\beta_n)\alpha_n\left\langle u - z_0, y_n - z_0\right\rangle.
\end{aligned}$$

Since $\sum_{n=1}^{\infty}(1-\beta_n)\alpha_n = \infty$, by Lemma 2.2 we obtain that $x_n \to z_0$.

Case 2: Suppose that there exists a subsequence $\{\Gamma_{n_i}\} \subset \{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then we have from Lemma 2.3 that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$. Thus we have from (3.14) that for all $n \in \mathbb{N}$,

$$\begin{aligned}
\beta_{\tau(n)}(1-\beta_{\tau(n)})\|z_{\tau(n)} - x_{\tau(n)}\|^2 \leq\ & (1-\beta_{\tau(n)})^2 2\alpha_{\tau(n)}\|z_{\tau(n)} - x_{\tau(n)}\|\|u_{\tau(n)} - z_{\tau(n)}\| \\
& + (1-\beta_{\tau(n)})^2\alpha_{\tau(n)}^2\|u_{\tau(n)} - z_{\tau(n)}\|^2 \\
& + 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}\langle u_{\tau(n)}, x_{\tau(n)} - z_0\rangle \\
& - 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}\langle z_{\tau(n)}, x_{\tau(n)} - z_0\rangle.
\end{aligned} \tag{3.21}$$

Using $\lim_{n\to\infty}\alpha_n = 0$ and $0 < c \leq \beta_n \leq d < 1$, we have from (3.21) that

$$\lim_{n\to\infty}\|z_{\tau(n)} - x_{\tau(n)}\| = 0. \tag{3.22}$$

As in the proof of Case 1 we have that

$$\lim_{n\to\infty}\|x_{\tau(n)+1} - x_{\tau(n)}\| = 0. \tag{3.23}$$

and

$$\lim_{n\to\infty}\|y_{\tau(n)} - z_{\tau(n)}\| = 0. \tag{3.24}$$

Since $\|y_{\tau(n)} - x_{\tau(n)}\| \leq \|y_{\tau(n)} - z_{\tau(n)}\| + \|z_{\tau(n)} - x_{\tau(n)}\|$, we have that

$$\lim_{n\to\infty}\|y_{\tau(n)} - x_{\tau(n)}\| = 0. \tag{3.25}$$

For $z_0 = P_{B^{-1}0\cap A^{-1}(U^{-1}0)}u$, let us show that $\limsup_{n\to\infty} \langle z_0 - u, y_{\tau(n)} - z_0\rangle \geq 0$. Put

$$l = \limsup_{n\to\infty} \langle z_0 - u, y_{\tau(n)} - z_0\rangle .$$

Without loss of generality, there exists a subsequence $\{y_{\tau(n_i)}\}$ of $\{y_{\tau(n)}\}$ such that $l = \lim_{i\to\infty} \langle u - z_0, y_{\tau(n_i)} - z_0\rangle$ and $\{y_{\tau(n_i)}\}$ converges weakly to some point $w \in H_1$. From $\|y_{\tau(n)} - x_{\tau(n)}\| \to 0$, we also have that $\{x_{\tau(n_i)}\}$ converges weakly to $w \in H_1$. As in the proof of Case 1 we have that $w \in B^{-1}0 \cap A^{-1}(U^{-1}0)$. Then we have

$$l = \lim_{i\to\infty} \langle z_0 - u, y_{\tau(n_i)} - z_0\rangle = \langle z_0 - u, w - z_0\rangle \geq 0.$$

As in the proof of Case 1, we also have that

$$\left\|y_{\tau(n)} - z_0\right\|^2 \leq (1-\alpha_{\tau(n)})^2 \left\|x_{\tau(n)} - z_0\right\|^2 + 2\alpha_{\tau(n)} \langle u_{\tau(n)} - z_0, y_{\tau(n)} - z_0\rangle$$

and then

$$\begin{aligned}\|x_{\tau(n)+1} - z_0\|^2 &\leq \beta_{\tau(n)} \left\|x_{\tau(n)} - z_0\right\|^2 + (1-\beta_{\tau(n)}) \left\|y_{\tau(n)} - z_0\right\|^2 \\ &\leq \left(1 - (1-\beta_{\tau(n)})\alpha_{\tau(n)}\right) \left\|x_{\tau(n)} - z_0\right\|^2 \\ &\quad + 2(1-\beta_{\tau(n)})\alpha_{\tau(n)} \langle u_{\tau(n)} - z_0, y_{\tau(n)} - z_0\rangle .\end{aligned}$$

From $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, we have that

$$(1-\beta_{\tau(n)})\alpha_{\tau(n)} \left\|x_{\tau(n)} - z_0\right\|^2 \leq 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}\langle u_{\tau(n)} - z_0, y_{\tau(n)} - z_0\rangle.$$

Since $(1-\beta_{\tau(n)})\alpha_{\tau(n)} > 0$, we have that

$$\begin{aligned}\left\|x_{\tau(n)} - z_0\right\|^2 &\leq 2\langle u_{\tau(n)} - z_0, y_{\tau(n)} - z_0\rangle \\ &= 2\langle u_{\tau(n)} - u, y_{\tau(n)} - z_0\rangle + 2\langle u - z_0, y_{\tau(n)} - z_0\rangle\end{aligned}$$

Thus we have that

$$\limsup_{n\to\infty} \left\|x_{\tau(n)} - z_0\right\|^2 \leq 0$$

and hence $\|x_{\tau(n)} - z_0\| \to 0$. From (3.23), we have also that $x_{\tau(n)} - x_{\tau(n)+1} \to 0$. Thus $\|x_{\tau(n)+1} - z_0\| \to 0$ as $n \to \infty$. Using Lemma 2.3 again, we obtain that

$$\|x_n - z_0\| \leq \|x_{\tau(n)+1} - z_0\| \to 0$$

as $n \to \infty$. This completes the proof. □

4 Applications

Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $T : C \to H$ be a strict pseudo-contraction, that is, there exists $k \in \mathbb{R}$ with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in C.$$

Putting $U = I - T$, we have that

$$\|(I-U)x - (I-U)y\|^2 \leq \|x-y\|^2 + k\|Ux - Uy\|^2, \quad \forall x, y \in C.$$

Thus we have that

$$\|x-y\|^2 + \|Ux-Uy\|^2 - 2\langle x-y, Ux-Uy\rangle \leq \|x-y\|^2 + k\|Ux-Uy\|^2.$$

Then

$$\frac{1-k}{2}\|Ux-Uy\|^2 \leq \langle x-y, Ux-Uy\rangle.$$

Therefore, $U = I - T$ is $\frac{1-k}{2}$-inverse strongly monotone.

Let H be a Hilbert space and let f be a proper, lower semicontinuous and convex function of H into $(-\infty, \infty]$. Then the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{z \in H : f(x) + \langle z, y - x\rangle \leq f(y), \quad \forall y \in H\}$$

for all $x \in H$. By Rockafellar [22], it is shown that ∂f is maximal monotone. Let C be a nonempty, closed and convex subset of H and let i_C be the indicator function of C, i.e.,

$$i_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ \infty, & \text{if } x \notin C. \end{cases}$$

Then $i_C : H \to (-\infty, \infty]$ is a proper, lower semicontinuous and convex function on H and hence ∂i_C is a maximal monotone operator. Thus we can define the resolvent J_λ of ∂i_C for $\lambda > 0$ as follows:

$$J_\lambda x = (I + \lambda \partial i_C)^{-1} x, \quad \forall x \in H,\ \lambda > 0.$$

We know that $J_\lambda x = P_C x$ for all $x \in H$ and $\lambda > 0$; see [28]. From Theorem 3.1 we obtain the following strong convergence theorem which is a generalization of [1].

Theorem 4.1. Let H_1 and H_2 be Hilbert spaces. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $T : H_2 \to H_2$ be a k-strict pseudo-contraction with $0 \leq k < 1$. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H_1 such that $u_n \to u$. Let $x_1 = x \in H_1$ and let $\{x_n\} \subset H_1$ be a sequence generated by

$$x_{n+1} = \alpha_n u_n + (1-\alpha_n) J_{\lambda_n}(I - \lambda_n A^*(I-T)A)x_n$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0,1)$ satisfy

$$0 < a \leq \lambda_n \leq \frac{1-k}{\|A\|^2}, \quad \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty,$$

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \textit{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in B^{-1}0 \cap A^{-1}F(T)$, where $z_0 = P_{B^{-1}0 \cap A^{-1}F(T)} u$.

Proof. Suppose that T is a k-strict pseudo-contraction with $0 \leq k < 1$. Then $U = I - T$ is $\frac{1-k}{2}$-inverse strongly monotone. Thus we obtain the desired result by Theorem 3.1. □

Similarly, from Theorem 3.2 we get the following theorem which is another generalization of [1].

Theorem 4.2. Let H_1 and H_2 be Hilbert spaces. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $T : H_2 \to H_2$ be a k-strict pseudo-contraction with $0 \le k < 1$. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H_1 such that $u_n \to u$. Let $x_1 = x \in H_1$ and let $\{x_n\} \subset H_1$ be a sequence generated by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u_n + (1 - \alpha_n) J_{\lambda_n}(I - \lambda_n A^*(I - T)A)x_n)$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, \infty)$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy

$$0 < a \le \lambda_n \le \frac{1-k}{\|A\|^2}, \quad 0 < c \le \beta_n \le d < 1,$$

$$\lim_{n\to\infty} \alpha_n = 0 \quad \textit{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in B^{-1}0 \cap A^{-1}F(T)$, where $z_0 = P_{B^{-1}0 \cap A^{-1}F(T)} u$.

Let C be a nonempty, closed and convex subset of a Hilbert space H and let $f : C \times C \to \mathbb{R}$ be a bifunction. Then we consider the following equilibrium problem: Find $z \in C$ such that

$$f(z, y) \ge 0, \quad \forall y \in C. \tag{4.1}$$

The set of such $z \in C$ is denoted by $EP(f)$, i.e.,

$$EP(f) = \{z \in C : f(z, y) \ge 0, \ \forall y \in C\}.$$

For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e., $f(x, y) + f(y, x) \le 0$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$,
$$\limsup_{t\downarrow 0} f(tz + (1 - t)x, y) \le f(x, y);$$

(A4) $f(x, \cdot)$ is convex and lower semicontinuous for all $x \in C$.

We know the following lemmas; see, for instance, [6] and [12].

Lemma 4.3 ([6])**.** Let C be a nonempty, closed and convex subset of H, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0$$

for all $y \in C$.

Lemma 4.4 ([12])**.** For $r > 0$ and $x \in H$, define the resolvent $T_r : H \to C$ of f for $r > 0$ as follows:

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C \right\}$$

for all $x \in H$. Then, the following hold:

(i) T_r is single-valued;

(ii) T_r is firmly nonexpansive, i.e., for all $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(iii) $F(T_r) = EP(f)$;

(iv) $EP(f)$ is closed and convex.

Takahashi, Takahashi and Toyoda [25] showed the following.

Lemma 4.5 ([25])**.** Let C be a nonempty, closed and convex subset of a Hibert space H and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (A1)-(A4). Define A_f as follows:

$$A_f(x) = \begin{cases} \{z \in H : f(x, y) \geq \langle y - x, z\rangle,\ \forall y \in C\}, & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

Then $EP(f) = A_f^{-1}(0)$ and A_f is maximal monotone with the domain of A_f in C. Furthermore,

$$T_r(x) = (I + rA_f)^{-1}(x), \quad \forall x \in H,\ r > 0.$$

We obtain the following theorem from Theorem 3.1.

Theorem 4.6. Let H_1 and H_2 be Hilbert spaces. Let C be a nonempty, closed and convex subset of H_1. Let $f : C \times C \to \mathbb{R}$ satisfy the conditions (A1)-(A4) and let T_{λ_n} be the resolvent of A_f for $\lambda_n > 0$ in Lemma 4.5. Let $U : H_2 \to H_2$ be an α-inverse strongly monotone mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $EP(f) \cap A^{-1}(U^{-1}0) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H_1 such that $u_n \to u$. Let $x_1 = x \in H_1$ and let $\{x_n\} \subset H_1$ be a sequence generated by

$$x_{n+1} = \alpha_n u_n + (1 - \alpha_n) T_{\lambda_n}(I - \lambda_n A^* U A)x_n$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy

$$0 < a \leq \lambda_n \leq \frac{2\alpha}{\|A\|^2}, \quad \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty,$$

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \textit{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Then the sequence $\{x_n\}$ converges strongly to a point z_0 of $EP(f) \cap A^{-1}(U^{-1}0)$, where $z_0 = P_{EP(f)\cap A^{-1}(U^{-1}0)}u$.

Proof. Define A_f for the bifunction f and set $B = A_f$ in Theorem 3.1. Thus we have the desired result from Theorem 3.1. □

As in the proof of Theorem 4.6, we obtain the following result from Theorem 3.2.

Theorem 4.7. Let H_1 and H_2 be Hilbert spaces. Let C be a nonempty, closed and convex subset of a real Hilbert space H_1. Let $f : C \times C \to \mathbb{R}$ satisfy the conditions (A1)-(A4) and let T_{λ_n} be the resolvent of A_f for $\lambda_n > 0$ in Lemma 4.5. Let $U : H_2 \to H_2$ be an α-inverse strongly monotone mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $EP(f) \cap A^{-1}(U^{-1}0) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H_1 such that $u_n \to u$. Let $x_1 = x \in H_1$ and let $\{x_n\} \subset H_1$ be a sequence generated by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u_n + (1 - \alpha_n)T_{\lambda_n}(I - \lambda_n A^* UA)x_n)$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, \infty)$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy

$$0 < a \leq \lambda_n \leq \frac{2\alpha}{\|A\|^2}, \quad 0 < c \leq \beta_n \leq d < 1,$$

$$\lim_{n\to\infty} \alpha_n = 0 \quad \textit{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then the sequence $\{x_n\}$ converges strongly to a point z_0 of $EP(f) \cap A^{-1}(U^{-1}0)$, where $z_0 = P_{EP(f)\cap A^{-1}(U^{-1}0)}u$.

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Manuscript received 7 February 2015
revised 7 April 2015
accepted for publication 10 April 2015

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