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WELL-POSEDNESS BY PERTURBATIONS FOR THE HEMIVARIATIONAL INEQUALITY GOVERNED BY A MULTI-VALUED MAP PERTURBED WITH A NONLINEAR TERM

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Abstract: In this paper, we introduce the notion of well-posedness by perturbations to the hemivariational inequality governed by a multi-valued map perturbed with a nonlinear term (HVIMN) in Banach spaces. Under very suitable conditions, we establish some metric characterizations for the well-posed (HVIMN). In the setting of finite-dimensional spaces, the strongly generalized well-posedness by perturbations for (HVIMN) are established. Our results are new and improve recent existing ones in the literature.

Key words: *well-posedness by perturbations, hemivariational inequality, multi-valued map, Banace space.*

Mathematics Subject Classification: *49J40, 49K40, 90C31.*

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1 Introduction

It is well known that the well-posedness is very important for both optimization theory and numerical methods of optimization problems, which guarantees that, for approximating solution sequences, there is a subsequence which converges to a solution. The study of well-posedness originates from Tikhonov [36], which means the existence and uniqueness of the solution and convergence of each minimizing sequence to the solution. Levitin-Polyak [21] introduced a new notion of well-posedness that strengthened Tykhonov's concept as it required the convergence to the optimal solution of each sequence belonging to a larger set of minimizing sequences.

Another important notion of well-posedness for a minimization problem is the well-posedness by perturbations or extended well-posedness due to Zolezzi [41, 42]. The notion of well-posedness by perturbations establishes a form of continuous dependence of the solutions upon a parameter. There are many other notions of well-posedness in optimization problems. For more details, see, e.g., [41, 42, 2, 6, 11, 15, 18, 27, 32, 37, 39]. Meanwhile, the concept of well-posedness has been generalized to other varia- tional problems such as variational inequalities [5, 10, 12, 13, 24, 25, 26, 27], saddle point problems [3], Nash equilibrium problems [26, 28, 29, 30, 31, 33], equilibrium problems [14], inclusion problems [22, 23] ,and fixed point problems [22, 23, 40]

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Lucchetti and Patrone [27] introduced the notion of well-posedness for variational inequalities and proved some related results by means of Ekeland ’ s variational principle. From then on, many papers have been devoted to the extensions of well-posedness of minimization problems to various variational inequalities. Lignola and Morgan [25] generalized the notion of well-posedness by perturbations to a variational inequality and established the equivalence between the well-posedness by perturbations of a variational inequality and the well-posedness by perturbations of the corresponding minimization problem. Lignola and Morgan [26] investigated the concepts of α-well-posedness for variational inequalities. Del Prete et al. [10] further proved that the α-well-posedness of variational inequalities is closely related to the well-posedness of minimization problems. Recently, Fang et al. [16] generalized the notions of well-posedness and α-well-posedness to a mixed variational inequality. In the setting of Hilbert spaces, Fang et al. [16] proved that under suitable conditions the well-posedness of a mixed variational inequality is equivalent to the existence and uniqueness of its solution. They also showed that the well-posedness of a mixed variational inequality has close links with the well-posedness of the corresponding inclusion problem and corresponding fixed point problem in the setting of Hilbert spaces. Very recently, Fang et al. [15] generalized the notion of well-posedness by perturbations to a mixed variational inequality in Banach spaces. In the setting of Banach spaces, they established some metric characterizations, and showed that the well-posedness by perturbations of a mixed variational inequality is closely related to the well-posedness by perturbations of the corresponding inclusion problem and corresponding fixed point problem. They also derived some conditions under which the well-posedness by perturbations of the mixed variational inequality is equivalent to the existence and uniqueness of its solution.

On the other hand, the notion of hemivariational inequality was introduced by Panagiotopoulos [34, 35] at the beginning of the 1980s as a variational formulation for several classes of mechanical problems with nonsmooth and nonconvex energy super-potentials. In the case of convex super-potentials, hemivariational inequalities reduce to variational inequalities which were studied earlier by many authors (see e.g. Fichera [17] or Hartman and Stampacchia [19]). Wangkeeree and Preechasilp [38] also introduced and studied some existence results for the hemivariational inequality governed by a multi-valued map perturbed with a nonlinear term in reflexive Banach spaces. Recently Ceng et al. [4] considered an extension of the notion of well-posedness by perturbations, introduced by Zolezzi for a minimization problem, to a class of variational-hemivariational inequalities with perturbations in Banach spaces. Under very mild conditions, they established some metric characterizations for the well-posed variational-hemivariational inequality, and proved that the well-posedness by perturbations of a variational hemivariational inequality is closely related to the well-posedness by perturbations of the corresponding inclusion problem. Furthermore, in the setting of finite-dimensional spaces they also derived some conditions under which the variational-hemivariational inequality is strongly generalized well-posed-like by perturbations.

The aim of this paper is to introduce the new notion of well-posedness by perturbations to the hemivariational inequality governed by a multi-valued map perturbed with a nonlinear term (HVIMN) in Banach spaces. Under very suitable conditions, we establish some metric characterizations for the well-posed (HVIMN). In the setting of finite-dimensional spaces, the strongly generalized well-posedness by perturbations for (HVIMN) are established. The example illustrating main results is established. Our results are new and improve recent existing ones in the literature.

2 Preliminaries

Let K be a nonempty, closed and convex subset of a real reflexive Banach space E with its dual E^*, $F : K \rightrightarrows 2^{E^*}$ a multivalued mapping. Let Ω be a bounded open set in $\mathbb{R}^N$, $T : E \to L^q(\Omega;\mathbb{R}^k)$ a linear continuous mapping, where $1 < q < \infty$, $k \geq 1$ and $j : \Omega\times\mathbb{R}^k \to \mathbb{R}$ a function. We shall denote $\hat{u} := Tu, j^\circ(x, y; h)$ denotes the Clarke's generalized directional derivative of a locally Lipschitz mapping $j(x, \cdot)$ at the point $y \in \mathbb{R}^k$ with respect to direction $h \in \mathbb{R}^k$, where $x \in \Omega$.

For the given bifunction $f : K \times K \to [-\infty, +\infty]$ imposed the condition that the set $\mathcal{D}_1(f) = \{u \in K : \ f(u, v) \neq -\infty, \ \forall v \in K\}$ is nonempty, Wangkeeree and Preechasilp [38] introduced and studied the existence of a solution for the following hemivariational inequality governed by a multi-valued map perturbed with a nonlinear term

$$(\text{HVIMN}) \begin{cases} \text{Find } u \in \mathcal{D}_1(f) \text{ and } u^* \in F(u) \text{ such that} \\ \langle u^*, v-u\rangle + f(u, v) + \int_\Omega j^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq 0, \\ \forall v \in K. \end{cases} \tag{2.1}$$

Now, let us consider some special cases of the problem (2.1). If $f(u, v) = \phi(v) - \phi(u)$, where $\phi : X \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous function such that $K_\phi = K \cap \text{dom}\phi \neq \emptyset$, then $\mathcal{D}_1(f) = K_\phi$ and (2.1) is reduced to the following *variational-hemivariational inequality problem*: Find $u \in K_\phi$ such that

$$\langle u^*, v-u\rangle + \phi(v) - \phi(u) + \int_\Omega j(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq 0, \quad \forall v \in K. \tag{2.2}$$

The problem (2.2) was studied by Costea and Lupu [8] by assuming that F is monotone and lower hemicontinuous and several existence results were obtained. Furthermore, if $F \equiv 0$ and $f(u, v) = \Lambda(u, v) - \langle g^*, v-u\rangle$, where $\Lambda : K \times K \to \mathbb{R}$ and $g^* \in X^*$, then (2.1) reduces to the problem: Find $u \in K$ such that

$$\Lambda(u, v) + \int_\Omega j(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq \langle g^*, v-u\rangle, \quad \forall v \in K. \tag{2.3}$$

The problem (2.3) was studied by Costea and Radulescu [9] and it was called nonlinear hemivariational inequality (see also Andrei and Costea [1] for some applications of nonlinear hemivariational inequalities to Nonsmooth Mechanics).

Now, suppose that L is a parametric normed space, $P \subset L$ is a closed ball with positive radius $p^* \in P$ is a fixed point. Let $\tilde{F} : P \times K \to 2^{E^*}$ be multivalued mapping. Let $\tilde{T} : P \times E \to L^p(\Omega;\mathbb{R}^k)$ be a linear continuous mapping, where $1 < p < \infty, k \geq 1$ and $\tilde{j} : P \times \Omega \times \mathbb{R}^k \to \mathbb{R}$ a function. We denote $\tilde{j}^\circ_p(x, y; h)$ denotes the Clarke's generalized directional derivative of a locally Lipschitz mapping $\tilde{j}(p, x, \cdot)$ at the point $y \in \mathbb{R}^k$ with respect to direction $h \in \mathbb{R}^k$. For the given bifunction $\tilde{f} : P \times K \times K \to [-\infty, +\infty]$, we assume the condition

$$\tilde{\mathcal{D}}_1(\tilde{f}) = \{u \in K | \tilde{f}(p^*, u, v) \neq -\infty, \forall v \in K\} \neq \emptyset.$$

The perturbed problem of the HVIMN (2.1) is given by

$$(\text{HVIMN}_{p^*}) \begin{cases} \text{Find } u \in \tilde{\mathcal{D}}_1(\tilde{f}) \text{ and } u^* \in \tilde{F}(p^*, u) \text{ such that} \\ \langle u^*, v-u\rangle + \tilde{f}(p^*, u, v) + \int_\Omega \tilde{j}^\circ_{p*}(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq 0, \\ \forall v \in K. \end{cases} \tag{2.4}$$

Let $\bar{\partial} j : E \to 2^{E^*} \setminus \{0\}$ denote the Clarke's generalized gradient of locally Lipschitz functional j (see [7]). That is

$$\bar{\partial} j(x) = \{\xi \in E^* : \langle \xi, v\rangle \leq j^0(x,y), \forall y \in E\}.$$

The following useful results can be found in [7].

Proposition 2.1. *Let X be a Banach space, $x, y \in X$ and J be a locally Lipschitz functional defined on X. Then*

(i) *The function $y \mapsto j^\circ(x,y)$ is finite, positively homogeneous, subadditive and then convex on X;*

(ii) *$j^\circ(x,y)$ is upper semicontinuous as a function of (x,y), as a function of y alone, is Lipschitz continuous on X;*

(iii) *$j^\circ(x,-y) = (-j)^\circ(x,y)$;*

(iv) *$\bar{\partial} j(x)$ is a nonempty, convex, bounded, weak*-compact subset of X^*;*

(v) *For every $y \in X$, one has*

$$j^\circ(x,y) = \max\{\langle \xi, y\rangle : \xi \in \bar{\partial} j(x)\}.$$

Definition 2.2. The set-valued map F is said to be

(i) upper semicontinuous (usc) at $x \in \text{dom } F$ if for any open set U satisfying $F(x) \subset U$, there exists a $\delta > 0$ such that $F(y) \subset U$, for every $y \in B(x,\delta)$;

(ii) lower semicontinuous (lsc) at $x \in \text{dom } F$ if for any open set U satisfying $F(x) \cap U \neq \emptyset$, there exists a $\delta > 0$ such that $F(y) \cap U \neq \emptyset$, for every $y \in B(x,\delta)$;

(iii) closed at $x \in \text{dom } F$ if for each sequence $\{x_n\}$ in X converging to x and $\{y_n\}$ in Y converging to y such that $y_n \in F(x_n)$, we have $y \in F(x)$.

If $S \subseteq X$, then F is said to be usc (lsc, closed respectively) on the set S if F is usc (lsc, closed respectively) at every $x \in \text{dom } F \cap S$.

Remark 2.3. An equivalent formulation of Definition 2.2(ii) is as follows: F is said to be lsc at $x \in \text{dom } F$ if for each sequence $\{x_n\}$ in dom F converging to x and for any $y \in F(x)$, there exists a sequence $\{y_n\}$ in $F(x_n)$ converging to y.

Definition 2.4 (see [20])**.** Let S be a nonempty subset of X. The measure, say μ, of noncompactness for the set S is defined by

$$\mu(S) := \inf\{\varepsilon > 0 : S \subset \cup_{i=1}^n S_i,\ \text{diam}|S_i| < \varepsilon, i = 1, 2, \ldots, n,\ \text{for some integer } n \geq 1\},$$

where diam$|S_i|$ means the diameter of set S_i.

Definition 2.5 (see[20])**.** Let A, B be nonempty subsets of X. The Hausdorff metric $H(\cdot,\cdot)$ between A and B is defined by

$$H(A,B) = \max\{e(A,B), e(B,A)\},$$

where $e(A,B) := \sup_{a\in A} d(a,B)$ with $d(a,B) = \inf_{b\in B} \|a-b\|$.

Let $\{A_n\}$ be a sequence of nonempty subsets of X. We say that A_n converges to A in the sense of Hausdorff metric if $H(A_n, A) \to 0$. It is easy to see that $e(A_n, A) \to 0$ if and only if $d(a_n, A) \to 0$ for all section $a_n \in A_n$. For more details on this topic, we refer the readers to [20].

3 Well-Posedness by Perturbations and Metric Characterizations

In this section, we generalize the concepts of well-posedness by perturbations to the variationalhemivariational inequality and establish their metric characterizations. In the sequel we always denote by $\to$ and $\rightharpoonup$ the strong convergence and weak convergence, respectively. Let $\alpha \geq 0$ be a fixed number.

Definition 3.1. Let $\{p_n\} \subset P$ be such that $p_n \to p^*$. A sequence $\{u_n\} \subset E$ is called an α-approximating sequence corresponding to $\{p_n\}$ for HVIMN (2.1) if there exist a sequence $\{\varepsilon_n\}$ of nonnegative numbers with $\varepsilon_n \to 0$, $u_n^* \in \tilde{F}(p_n, u_n)$ such that $u_n \in \tilde{D}_1(\tilde{f})$, and

$$\begin{aligned}&\langle u_n^*, v - u_n\rangle + \tilde{f}(p_n, u_n, v) + \int_\Omega \tilde{j}^\circ_{p_n}(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))dx \\ &\geq -\frac{\alpha}{2}\|v - u_n\|^2 - \varepsilon_n, \qquad \forall v \in K.\end{aligned}$$

for each $n \geq 1$. Whenever $\alpha = 0$, we say that $\{u_n\}$ is an approximating sequence corresponding to $\{p_n\}$ for HVIMN (2.1). Clearly, every α_2-approximating sequence corresponding to $\{p_n\}$ is α_1-approximating sequence corresponding to $\{p_n\}$ whenever $\alpha_1 > \alpha_2 \geq 0$.

Definition 3.2. We say that HVIMN (2.1) is strongly (resp., weakly) $\alpha-$well-posed by perturbations if

(i) HVIMN (2.1) has a unique solution

(ii) for any $\{p_n\} \subset P$ with $p_n \to p^*$, every $\alpha-$approximating sequence corresponding to $\{p_n\}$ converges strongly (resp., weakly) to the unique solution.

In the sequel, strong (resp., weak) $0-$well-posedness by perturbations is always called as strong (resp., weak) well-posedness by perturbations. If $\alpha_1 > \alpha_2 \geq 0$, then strong (resp., weak) α_1-well-posedness by perturbations implies strong (resp., weak) α_2-well-posedness by perturbations.

Definition 3.3. We say that HVIMN (2.1) is strongly (resp., weakly) generalized $\alpha-$well-posed by perturbations if

(i) HVIMN (2.1) has a nonempty solution set S

(ii) for any $\{p_n\} \subset P$ with $p_n \to p^*$, every $\alpha-$approximating sequence corresponding to $\{p_n\}$ has some subsequence which converges strongly (resp., weakly) to some point of S

In the sequel, strong (resp., weak) generalized $0-$well-posedness by perturbations is always called as strong (resp., weak) generalized well-posedness by perturbations.

If $\alpha_1 > \alpha_2 \geq 0$, then strong (resp., weak) generalized α_1-well-posedness by perturbations implies strong (resp., weak) generalized α_2-well-posedness by perturbations.

To derive the metric characterizations of α-well-posedness by perturbations, we consider the following approximating solution set of HVIMN (2.1):

$$\begin{aligned}\Omega_\alpha(\varepsilon) &= \bigcup_{p \in B(p^*, \varepsilon)} \{u \in \tilde{D}_1(\tilde{f}), u^* \in \tilde{F}(p, u) : \langle u^*, v - u\rangle + \tilde{f}(p, u, v) \\ &\quad + \int_\Omega \tilde{j}^\circ_p(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq -\frac{\alpha}{2}\|v - u\|^2 - \varepsilon, \forall v \in K.\}\end{aligned}$$

when $B(p^*, \varepsilon)$ denotes the closed ball centered at p^* with radius ε. In this section, we assume that $\bar{u}$ is a fixed solution of HVIMN (2.1). Define

$$\theta(\varepsilon) = \sup\{\|u - \bar{u}\| : u \in \Omega_\alpha(\varepsilon)\}, \ \ \forall \varepsilon \geq 0.$$

It is easy to see that $\theta(\varepsilon)$ is the radius of the smallest closed ball centered at $\bar{u}$ containing $\Omega_\alpha(\varepsilon)$. Now, we give a metric characterization of strong α-well-posedness by perturbations by considering the behavior of $\theta(\varepsilon)$ when $\varepsilon \to 0$.

Theorem 3.4. *HVIMN (2.1) is strongly $\alpha-$well-posed by perturbations if and only if $\theta(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

Proof. Assume that HVIMN (2.1) is strongly $\alpha-$well-posed by perturbations. Then $\bar{u} \in E$ is the unique solution of HVIMN (2.1). Suppose to the contrary that $\theta(\varepsilon) \not\to 0$ as $\varepsilon \to 0$. There exist $\delta > 0$ and $0 < \varepsilon_n \to 0$ such that

$$\theta(\varepsilon_n) > \delta > 0.$$

By the definition of θ, there exists $u_n \in \Omega_\alpha(\varepsilon_n)$ such that

$$\|u_n - \bar{u}\| > \delta. \tag{3.1}$$

Since $u_n \in \Omega_\alpha(\varepsilon_n)$, there exist $p_n \in B(p^*, \varepsilon_n), u_n^* \in \tilde{F}(p_n, u_n)$ such that

$$\langle u_n^*, v - u_n \rangle + \tilde{f}(p_n, u_n, v) + \int_\Omega \tilde{j}^\circ_{p_n}(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))dx \geq -\frac{\alpha}{2}\|v - u_n\|^2 - \varepsilon,$$

for all $v \in K$ and $n \geq 1$. Since $p_n \in B(p^*, \varepsilon_n)$, we have $p_n \to p^*$. Then $\{u_n\}$ is an α approximating sequence corresponding to $\{p_n\}$ for HVIMN (2.1). Since HVIMN (2.1) is strongly $\alpha-$well-posed by perturbations, we can get that $\|u_n - \bar{u}\| \to 0$, which leads to a contradiction with (3.1).

Conversely, suppose that $\theta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then $\bar{u} \in E$ is the unique solution of HVIMN (2.1). Indeed, if $\hat{u}$ is another solution of HVIMN (2.1) with $\hat{u} \neq \bar{u}$, then by definition,

$$\theta(\varepsilon) \geq \|\bar{u} - \hat{u}\| > 0, \ \ \forall \varepsilon \geq 0,$$

a contradiction. Let $p_n \in P$ be such that $p_n \to p^*$ and let $\{u_n\}$ be an $\alpha-$approximating sequence corresponding to $\{p_n\}$ for HVIMN (2.1). Then there exist $0 < \varepsilon_n \to 0, u_n^* \in \tilde{F}(p_n, u_n)$ such that $u_n \in \tilde{D}_1(\tilde{f})$ and

$$\langle u_n^*, v - u_n \rangle + \tilde{f}(p_n, u_n, v) + \int_\Omega \tilde{j}^\circ_{p_n}(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))dx \geq -\frac{\alpha}{2}\|v - u_n\|^2 - \varepsilon_n,$$

for all $v \in K$ and $n \geq 1$. Take $\delta_n = \|p_n - p^*\|$ and $\varepsilon'_n = \max\{\delta_n, \varepsilon_n\}$. It is easy to verify that $u_n \in \Omega_\alpha(\varepsilon'_n)$ with $\varepsilon'_n \to 0$. Put

$$t_n = \|u_n - \bar{u}\|,$$

by definition of θ, we can get that

$$\theta(\varepsilon'_n) \geq t_n = \|u_n - \bar{u}\|.$$

Since $\theta(\varepsilon'_n) \to 0$, we have $\|u_n - \bar{u}\| \to 0$ as $n \to \infty$. So, HVIMN (2.1) is strongly $\alpha-$well-posed by perturbations. □

Now, we give an example to illustrate Theorem 3.4.

Example 3.5. Let $E = \mathbb{R}, P = [-1,1], K = \mathbb{R}, p^* = 0, \alpha = 2, \tilde{F}(p,u) = \{2u\}, \tilde{j} = 0, \tilde{f}(p,u,v) = (1 - \frac{(p^2+1)^2}{4})u^2$ for all $p \in P, u, v \in K$. Clearly $u = 0$ is a solution of HVIMN (2.1). For any $\varepsilon > 0$, it follows that

$$\begin{aligned}
\Omega_\alpha^p(\varepsilon) &= \Big\{u \in \tilde{D}_1(\tilde{f}), u^* \in \tilde{F}(p) : \langle u^*, v-u\rangle + u^2 - \frac{(p^2+1)^2}{4}u^2 \geq -(v-u)^2 - \varepsilon,\ \forall v \in K\Big\}\\
&= \Big\{u \in \mathbb{R} : 2u(v-u) + u^2 - \frac{(p^2+1)^2}{4}u^2 \geq -(v-u)^2 - \varepsilon,\ \ \forall v \in \mathbb{R}\Big\}\\
&= \Big\{u \in \mathbb{R} : -u^2 + 2uv - \frac{(p^2+1)^2}{3}u^2 \geq -(v-u)^2 - \varepsilon,\ \ \forall v \in \mathbb{R}\Big\}\\
&= \Big\{u \in \mathbb{R} : v^2 - (v-u)^2 - \frac{(p^2+1)^2}{4}u^2 \geq -(v-u)^2 - \varepsilon,\ \ \forall v \in \mathbb{R}\Big\}\\
&= \Big\{u \in \mathbb{R} : -v^2 + \frac{(p^2+1)^2}{4}u^2 \leq +\varepsilon,\ \ \forall v \in \mathbb{R}\Big\}\\
&= \Big[-\frac{2\sqrt{\varepsilon}}{p^2+1}, \frac{2\sqrt{\varepsilon}}{p^2+1}\Big].
\end{aligned}$$

Therefore,

$$\Omega_\alpha(\varepsilon) = \bigcup_{p \in B(0,\varepsilon)} \Omega_\alpha^p(\varepsilon) = \Big[-2\sqrt{\varepsilon}, 2\sqrt{\varepsilon}\Big],$$

for sufficiently small $\varepsilon > 0$. By trivial computation, we have

$$\theta(\varepsilon) = \sup\{u - u^* : u \in \Omega_\alpha(\varepsilon)\} = 2\sqrt{\varepsilon} \to 0 \text{ as } \varepsilon \to 0.$$

By Theorem 3.4, HVIMN (2.1) is 2-well-posed by perturbations

To derive a characterization of strong generalized $\alpha-$well-posedness by perturbations, we need another function q which is defined by

$$q(\varepsilon) = e(\Omega_\alpha(\varepsilon), S),\ \ \forall \varepsilon \geq 0,$$

where S is the solution set of HVIMN (2.1) and e is defined as in definition 2.5.

Theorem 3.6. *HVIMN (2.1) is strongly generalized $\alpha-$well-posed by perturbations if and only if S is nonempty compact and $q(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

Proof. Assume that HVIMN (2.1) is strongly generalized $\alpha-$well-posed by perturbations. Clearly, S is nonempty. Let $\{u_n\}$ be any sequence in S and $\{p_n\} \subset P$ be such that $p_n = p^*$. Then $\{u_n\}$ is an α-approximating sequence corresponding to $\{p_n\}$ for HVIMN (2.1). Since HVIMN (2.1) is strongly generalized $\alpha-$well-posed by perturbations, we have $\{u_n\}$ has a subsequence which converges strongly to some point of S. Thus S is compact. Next, we suppose that $q(\varepsilon) \not\to 0$ as $\varepsilon \to 0$, then there exist $l > 0, 0 < \varepsilon_n \to 0$ and $u_n \in \Omega_\alpha(\varepsilon_n)$ such that

$$u_n \notin S + B(0,l),\ \ \forall n \geq 1. \tag{3.2}$$

Since $u_n \in \Omega_\alpha(\varepsilon_n)$, there exist $p_n \in B(p^*, \varepsilon), u_n^* \in \tilde{F}(p_n, u_n)$ such that $u_n \in \tilde{D}_1(\tilde{f})$ and

$$\langle u_n^*, v - u_n\rangle + \tilde{f}(p_n, u_n, v) + \int_\Omega \tilde{j}_{p_n}^\circ(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))dx \geq -\frac{\alpha}{2}\|v - u_n\|^2 - \varepsilon,$$

for all $v \in K$ and $n \geq 1$. Since $p_n \in B(p^*, \varepsilon_n)$, we have $p_n \to p^*$. Then $\{u_n\}$ is an α approximating sequence corresponding to $\{p_n\}$ for HVIMN (2.1). Since HVIMN (2.1) is strongly generalized $\alpha-$well-posed by perturbations, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converging strongly to some point of S, which leads to a contradiction with (3.2) and so $q(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Conversely, we assume that S is nonempty compact and $q(\varepsilon) \to 0$ as $\varepsilon \to 0$. Let $\{p_n\} \subset P$ be such that $p_n \to p^*$ and let $\{u_n\}$ be an $\alpha-$approximating sequence corresponding to $\{p_n\}$. Take $\varepsilon_n' = \max\{\varepsilon_n, \|p_n - p^*\|\}$. Thus $\varepsilon_n' \to 0$ and $x_n \in \Omega_\alpha(\varepsilon_n')$. It follows that

$$d(u_n, S) \geq e(\Omega_\alpha(\varepsilon_n'), S) = q(\varepsilon_n') \to 0.$$

Since S is compact, there exists $\bar{u}_n \in S$ such that

$$\|u_n - \bar{u}_n\| = d(x_n, S) \to 0.$$

Again from the compactness of S, $\{\bar{u}_n\}$ has a subsequence $\{\bar{u}_{n_k}\}$ which converges to $\bar{u}$. Thus HVIMN (2.1) is strongly generalized $\alpha-$well-posed by perturbations. □

The following example is shown for illustrating the metric characterizations in Theorem 3.6.

Example 3.7. Let $E = \mathbb{R}, P = [-1, 1], K = \mathbb{R}, p^* = 0, \alpha = 2, \tilde{F}(p, u) = \{2u\}, \tilde{j} = 0, \tilde{f}(p, u, v) = (1 - \frac{(p^2+1)^2}{4})u^2$ for all $p \in P, u, v \in K$. It is easy to see that $u = 0$ is a solution of HVIMN (2.1). Repeating the same argument as in Example 3.5, we obtain that

$$\Omega_\alpha(\varepsilon) = \bigcup_{p \in B(0,\varepsilon)} \Omega_\alpha^p(\varepsilon) = \Big[-2\sqrt{\varepsilon}, 2\sqrt{\varepsilon} \Big],$$

for sufficiently small $\epsilon > 0$. By trivial computation, we have

$$q(\epsilon) = e(\Omega_\alpha(\epsilon), S) = \sup_{u(\epsilon) \in \Omega_\alpha(\epsilon)} d(u(\epsilon), S) \to 0 \text{ as } \epsilon \to 0.$$

By Theorem 3.6, HVIMN (2.1) is strongly generalized $\alpha-$well-posed by perturbations.

The strong generalized α-well-posedness by perturbations can be also characterized by the behavior of the noncompactness measure $\mu(\Omega_\alpha(\epsilon))$.

Theorem 3.8. *Let L be finite-dimensional, $\tilde{j}_p^\circ(x, y)$ be upper semicontinuous as a functional of $(p, x, y) \in P \times E \times E$ and f is convex. Let $\tilde{F}$ is closed on $P \times K$ and $\tilde{f}$ be continuous on $P \times K \times K$. Then HVIMN (2.1) is strongly generalized $\alpha-$well-posed by perturbations if and only if $\Omega_\alpha(\varepsilon) \neq \emptyset, \forall \varepsilon > 0$ and $\mu(\Omega_\alpha(\varepsilon)) \to 0$ as $\varepsilon \to 0$.*

Proof. First, we will prove that $\Omega_\alpha(\varepsilon)$ is closed for all $\varepsilon \geq 0$. Let $\{u_n\} \subset \Omega_\alpha(\varepsilon)$ with $u_n \to \bar{u}$. Then there exist $p_n \in B(p^*, \varepsilon), u_n^* \in \tilde{F}(p_n, u_n)$ such that $u_n \in \tilde{D}_1(\tilde{f})$ and

$$\langle u_n^*, v - u_n \rangle + \tilde{f}(p_n, u_n, v) + \int_\Omega \tilde{j}_{p_n}^\circ(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))dx \geq -\frac{\alpha}{2}\|v - u_n\|^2 - \varepsilon, \quad (3.3)$$

for all $v \in K$ and $n \geq 1$. Without loss of generality, we may assume that $p_n \to \bar{p} \in B(p^*, \varepsilon)$ because L is finite dimensional. Since $\tilde{j}_p(x, y)$ is upper semicontinuous as a functional of $(p, x, y) \in P \times E \times E$. Hence it follows from (3.3) and the continuity of $\tilde{f}$ that

$$\langle u^*, v - \bar{u} \rangle + \tilde{f}(\bar{p}, \bar{u}, v) + \int_\Omega \tilde{j}_{\bar{p}}^\circ(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))dx$$

$$
\begin{aligned}
&\geq \limsup_{n\to\infty}\langle u_n^*, v-u_n\rangle + \tilde{f}(p_n,u_n,v) + \int_\Omega \tilde{j}^\circ_{p_n}(x,\hat{u}_n(x);\hat{v}(x)-\hat{u}_n(x))dx \\
&\geq \limsup_{n\to\infty} -\frac{\alpha}{2}\|v-u_n\|^2 - \varepsilon, \\
&= -\frac{\alpha}{2}\|v-\bar{u}\|^2 - \varepsilon \ \ \forall v\in K.
\end{aligned}
$$

Thus $\bar{u}\in\Omega_\alpha(\varepsilon)$. Hence $\Omega_\alpha(\varepsilon)$ is closed.

Next, we show that

$$S = \bigcap_{\varepsilon>0}\Omega_\alpha(\varepsilon). \tag{3.4}$$

It is easy to see that $S\subseteq \cap_{\varepsilon>0}\Omega_\alpha(\varepsilon)$. Thus, we show that $\cap_{\varepsilon>0}\Omega_\alpha(\varepsilon)\subseteq S$. Let $\bar{u}\in\cap_{\varepsilon>0}\Omega_\alpha(\varepsilon)$. Let $\{\varepsilon_n\}$ be a sequence of positive real numbers such that $\varepsilon_n\to 0$. Thus

$$\bar{u}\in\Omega_\alpha(\varepsilon_n)$$

and so there exist $p_n\in B(p^*,\varepsilon_n)$ and $u^*\in\tilde{F}(p_n,\bar{u})$ such that $\bar{u}\in\tilde{D}_1(\tilde{f})$ and

$$\langle u^*, v-\bar{u}\rangle + \tilde{f}(p_n,\bar{u},v) + \int_\Omega \tilde{j}^\circ_{p_n}(x,\hat{\bar{u}}(x);\hat{v}(x)-\hat{\bar{u}}(x))dx \geq -\frac{\alpha}{2}\|v-\bar{u}\|^2 - \varepsilon_n, \tag{3.5}$$

for all $v\in K$ and $n\geq 1$. It is easy to verify that $p_n\to p^*$. Taking limit as $n\to\infty$, we can get that

$$
\begin{aligned}
&\langle u^*, v-\bar{u}\rangle + f(\bar{u},v) + \int_\Omega j^\circ(x,\hat{\bar{u}}(x);\hat{v}(x)-\hat{\bar{u}}(x))dx \\
&= \langle u^*, v-\bar{u}\rangle + \tilde{f}(p^*,\bar{u},v) + \int_\Omega \tilde{j}^\circ_{p^*}(x,\hat{\bar{u}}(x);\hat{v}(x)-\hat{\bar{u}}(x))dx \\
&\geq -\frac{\alpha}{2}\|v-\bar{u}\|^2, \quad \forall v\in K
\end{aligned} \tag{3.6}
$$

Since $\tilde{F}$ is closed on $P\times K$, we have $u^*\in F(\bar{u})$ and for any $z\in K$ and $t\in(0,1)$, letting $v=\bar{u}+t(z-\bar{u})$ in (3.6), we can get from T is linear, f is convex and definition of j° that

$$
\begin{aligned}
&t\langle u^*, z-\bar{u}\rangle + tf(\bar{u},z) + \int_\Omega j^\circ(x,\hat{\bar{u}}(x);\hat{v}(x)-\hat{\bar{u}}(x))dx \\
&\geq t\langle u^*, z-\bar{u}\rangle + f(\bar{u},\bar{u}+t(z-\bar{u})) + \int_\Omega j^\circ(x,\hat{\bar{u}}(x);\hat{z}(x)-\hat{\bar{u}}(x))dx \\
&\geq -\frac{\alpha t^2}{2}\|z-\bar{u}\|^2.
\end{aligned}
$$

This implies that

$$\langle u^*, z-\bar{u}\rangle + tf(\bar{u},z) + \int_\Omega j^\circ(x,\hat{\bar{u}}(x);\hat{v}(x)-\hat{\bar{u}}(x))dx \geq -\frac{\alpha t}{2}\|z-\bar{u}\|^2 \ \ \forall z\in K.$$

As $t\to 0$ in the last inequality, we get

$$\langle u^*, z-\bar{u}\rangle + tf(\bar{u},z) + \int_\Omega j^\circ(x,\hat{\bar{u}}(x);\hat{v}(x)-\hat{\bar{u}}(x))dx \geq 0 \ \ \forall z\in K.$$

Hence $\bar{u}\in S$ and thus (3.4) is proved. Next, we suppose that HVIMN (2.1) is strongly generalized $\alpha-$well-posed by perturbations. By Theorem 3.6, we can get that S is nonempty compact and $q(\varepsilon)\to 0$. Since $S\subset\Omega_\alpha(\varepsilon)$ for all $\varepsilon>0$, we have

$$\Omega_\alpha(\varepsilon)\neq\emptyset, \quad \forall\varepsilon>0.$$

We observe that for each $\varepsilon > 0$,

$$H(\Omega_\alpha(\varepsilon), S) = \max\{e(\Omega_\alpha(\varepsilon), S), e(S, \Omega_\alpha(\varepsilon))\} = e(\Omega_\alpha(\varepsilon), S).$$

By the compactness of S, we have

$$\mu(\Omega_\alpha(\varepsilon)) \leq 2H(\Omega_\alpha(\varepsilon), S) = 2q(\varepsilon) \to 0.$$

Conversely, we suppose that $\Omega_\alpha(\varepsilon) \neq \emptyset, \ \forall \varepsilon > 0$ and $\mu(\Omega_\alpha(\varepsilon)) \to 0$ as $\varepsilon \to 0$. Since $\Omega_\alpha(\cdot)$, by the Kuratowski theorem, we can get from (3.4) that

$$q(\varepsilon) = H(\Omega_\alpha(\varepsilon), S) \to 0 \quad \text{as} \quad \varepsilon \to 0$$

and S is nonempty compact. Hence HVIMN (2.1) is strongly generalized $\alpha-$well-posed by perturbations by Theorem 3.6. □

The following example is given for illustrating the measure in Theorem 3.8.

Example 3.9. Let $E = \mathbb{R}, P = [-1, 1], K = \mathbb{R}, p^* = 0, \alpha = 2, \tilde{F}(p, u) = \{2u\}, \tilde{j} = 0, \tilde{f}(p, u, v) = (1 - \frac{(p^2+1)^2}{4})u^2$ for all $p \in P, u, v \in K$. It is easy to see that $u = 0$ is a solution of HVIMN (2.1). Repeating the same argument as in Example 3.5, we obtain that

$$\Omega_\alpha(\varepsilon) = \bigcup_{p \in B(0,\varepsilon)} \Omega^p_\alpha(\varepsilon) = \Big[-2\sqrt{\varepsilon}, 2\sqrt{\varepsilon} \Big].$$

We will show that $\mu(\Omega_\alpha(\epsilon)) = 0$ for each $\epsilon > 0$. Let $\epsilon > 0$. Consider

$$\mu(\Omega_\alpha(\epsilon)) = \inf\{\lambda > 0 : [-2\sqrt{\epsilon}, 2\sqrt{\epsilon}] \subseteq \bigcup_{k=1}^{n} [a_k, b_k], \text{ with } \operatorname{diam}[a_k, b_k] < \lambda, \forall i = 1, \dots, n, \exists n \in \mathbb{N}\}.$$

For every $\lambda > 0$, we can find $n \in \mathbb{N}$ with $a_1 = -2\sqrt{\epsilon}, b_n = 2\sqrt{\epsilon}$ such that

$$[-2\sqrt{\epsilon}, 2\sqrt{\epsilon}] \subseteq \bigcup_{k=1}^{n} [a_k, b_k] \quad \text{and} \quad \operatorname{diam}[a_k, b_k] < \lambda.$$

This implies that $\mu(\Omega_\alpha(\epsilon)) = 0$ for each $\epsilon > 0$. Then HVIMN (2.1) is strongly generalized $\alpha-$well-posed by perturbations.

Remark 3.10. Any solution of HVIMN (2.1) is a solution of the α problem: find $u \in D_1(f)$ and $u^* \in F(u)$ such that

$$\langle u^*, v - u \rangle + f(u, v) + \int_\Omega j^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq -\frac{\alpha}{2}\|y - x\|^2, \ \ \forall v \in K,$$

but the converse is not true in general. To show this, let $K = \mathbb{R}$,

$$F(u) = \{u\}, f(u, v) = 2u^2 - v \text{ and } j = 0,$$

for all $u, v \in K$. It is easy to see that the solution set of HVIMN (2.1) is empty and $u^* = u = 0$ is the unique solution of the corresponding α problem with $\alpha = 2$.

References

[1] I. Andre and, N. Costea, Nonlinear hemivariational inequalities and applications to Nonsmooth Mechanics, *Adv. Nonlinear Var. Inequal.* 13 (2010) 1–17.

[2] E. Bednarczuk and J.P. Penot, Metrically well-set minimization problems, Appl. Math. Optim. 26 (1992) 273–285.

[3] E. Cavazzuti and J. Morgan, Well-posed saddle point problems, in *Optimization, Theory and Algorithms*, J.B. Hirriart-Urruty, W. Oettli, J. Stoer (eds.), Marcel Dekker, New York, NY, 1983, pp. 61–76.

[4] L.C. Ceng, H. Gupta and C.F. Wen, Well-posedness by perturbations of variational-hemivariational inequalities with perturbations, *Filomat* 26 (2012) 881–895.

[5] L.C. Ceng, N. Hadjisavvas, S. Schaible and J.C. Yao, Well-posedness for mixed quasivariational-like inequalities, *J. Optim. Theory Appl.* 139 (2008) 109–125.

[6] L.C. Ceng and J.C. Yao, Well-posedness of generalized mixed variational inequalities, inclusion problems and fixed point problems, *Nonlinear Anal. TMA* 69 (2008) 4585–4603.

[7] F.H. Clarke, *Optimization and Nonsmooth Analysis*, SIAM, Philadelphia, 1990.

[8] N. Costea and C. Lupu, On a class of variational-hemivariational inequalities involving set valued mappings, *Adv. Pure Appl. Math.* 1 (2010) 233–246.

[9] N. Costea and V. Radulescu, Hartman–Stampacchia results for stably pseudomonotone operators and nonlinear hemivariational inequalities, *Appl. Anal.* 89 (2010) 175–188.

[10] I. Del Prete, M.B. Lignola and J. Morgan, New concepts of well-posedness for optimization Problems with variational inequality constraints, *J. Inequal. Pure Appl. Math.* 4 (2003).

[11] A.L. Dontchev and T. Zolezzi, in *Well-Posed Optimization Problems*, Lecture Notes in Math. vol. 1543, Springer-Verlag, Berlin, 1993.

[12] Y.P. Fang and R. Hu, Parametric well-posedness for variational inequalities defined by bifunctions, *Comput. Math. Appl.* 53 (2007) 1306–1316.

[13] Y.P. Fang and R. Hu, Estimates of approximate solutions and well-posedness for variational inequalities, *Math. Methods Oper. Res.* 65 (2007) 281–291.

[14] Y.P. Fang, R. Hu and N.J. Huang, Well-posedness for equilibrium problems and for optimization problems with equilibrium constraints, *Comput. Math. Appl.* 55 (2008) 89–100.

[15] Y.P. Fang, N.J. Huang and J.C. Yao, Well-posedness by perturbations of mixed variational inequalities in Banach spaces, *Euro. J. Oper. Res.* 201 (2010) 682–692.

[16] Y.P. Fang, N.J. Huang and J.C. Yao, Well-posedness of mixed variational inequalities, inclusion problems and fixed point problems, *J. Global Optim.* 41 (2008) 117–133.

[17] G. Fichera, Problemi elettrostatici con vincoli unilaterali: il problema di Signorini con ambigue condizioni al contorno, *Mem. Acad. Naz. Lincei.* 7 (1964) 91–140.

[18] X.X. Huang, Extended and strongly extended well-posedness of set-valued optimization problems, *Math. Methods Oper. Res.* 53 (2001) 101–116.

[19] G.J. Hartman and G. Stampacchia, On some nonlinear elliptic differential equations, *Acta Math.* 112 (1966) 271–310.

[20] K. Kuratowski, Topology, vols. 1 and 2, Academic Press, New York, NY, 1968.

[21] E.S. Levitin and B.T. Polyak, Convergence of minimizing sequences in conditional extremum problems, *Soviet Math. Dokl.* 7 (1966) 764–767.

[22] B. Lemaire, Well-posedness, conditioning, and regularization of minimization, inclusion, and fixed point problems, *Pliska Studia Math. Bulgaria* 12 (1998) 71–84.

[23] B. Lemaire, C.O.A. Salem and J.P. Revalski, Well-posedness by perturbations of variational problems, *J. Optim. Theory Appl.* 115 (2002) 345–368.

[24] M.B. Lignola, Well-posedness and L-well-posedness for quasivariational inequalities, *J. Optim. Theory Appl.* 128 (2006) 119–138.

[25] M.B. Lignola and J. Morgan, Well-posedness for optimization problems with constraints defined by variational inequalities having a unique solution, *J. Global Optim.* 16 (2000) 57–67.

[26] M.B. Lignola and J. Morgan, Approximating solutions and α-well-posedness for variational inequalities and Nash equilibria, in: *Decision and Control in Management Science*, Kluwer Academic Publishers, 2002, pp. 367–378.

[27] R. Lucchetti and F. Patrone, A characterization of Tikhonov well-posedness for minimum problems, with applications to variational inequalities, *Numer. Funct. Anal. Optim.* 3 (1981) 461–476.

[28] R. Lucchetti and J. Revalski (Eds.), Recent Developments in Well-Posed Variational Problems, Kluwer Academic Publishers, Dordrecht, Holland, 1995.

[29] M. Margiocco, F. Patrone and L. Pusillo, A new approach to Tikhonov well-posedness for Nash equilibria, *Optimization* 40 (1997) 385–400.

[30] M. Margiocco, F. Patrone and L. Pusillo, Metric characterizations of Tikhonov well-posedness in value, *J. Optim. Theory Appl.* 100 (1999) 377–387.

[31] M. Margiocco, F. Patrone and L. Pusillo, On the Tikhonov well-posedness of concave games and Cournot oligopoly games, *J. Optim. Theory Appl.* 112 (2002) 361–379.

[32] E. Miglierina and E. Molho, Well-posedness and convexity in vector optimization, Math. Methods Oper. Res. 58 (2003) 375–385.

[33] J. Morgan, Approximations and well-posedness in multicriteria games, *Ann. Oper. Res.* 137 (2005) 257–268.

[34] P.D. Panagiotopoulos, *Nonconvex energy functions, hemivariational inequalities and substationarity principles*, Acta Mechanica 42 (1983) 160–183

[35] P.D. Panagiotopoulos, *Hemivariational inequalities, Applications to Mechanics and Engineering*, Springer Verlag, Berlin, 1993.

[36] A.N. Tykhonov, On the stability of functional optimization problems, *USSR Comput. Math. Math. Phys.* 6 (1966) 28–33.

[37] A.N. Tykhonov, On the stability of the functional optimization problem, *USSR J. Comput. Math. Math. Phys.* 6 (1966) 631–634.

[38] R. Wangkeeree and P. Pakkapon, Existence theorems of the hemivariational inequality governed by a multi-valued map perturbed with a nonlinear term in Banach spaces, *J Glob Optim.*, DOI 10.1007/s10898-012-0018-x (2012).

[39] Y.B. Xiao and N.J. Huang, Well-posedness for a class of variational-hemivariational inequalities with perturbations, *J. Optim. Theory Appl.* (2011) to appear.

[40] H. Yang and J. Yu, Unified approaches to well-posedness with some applications, *J. Global Optim.* 31 (2005) 317–381.

[41] T. Zolezzi, Well-posedness criteria in optimization with application to the calculus of variations, *Nonlinear Anal.* TMA 25 (1995) 437–453.

[42] T. Zolezzi, Extended well-posedness of optimization problems, *J. Optim. Theory Appl.* 91 (1996) 257–266.

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