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FIXED POINT THEOREMS AND MEAN CONVERGENCE THEOREMS FOR GENERALIZED HYBRID SELF MAPPINGS AND NON-SELF MAPPINGS IN HILBERT SPACES

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Abstract: In this paper we prove fixed point theorems and mean convergence theorems for widely more generalized hybrid self mappings and non-self mappings in Hilbert spaces.

Key words: *fixed point theorem, mean convergence theorem, widely more generalized hybrid mapping, Hilbert space*.

Mathematics Subject Classification: *47H10.*

1 Introduction

Let H be a real Hilbert space and let C be a non-empty subset of H. In 2010, Kocourek, Takahashi and Yao [16] defined a class of nonlinear mappings in a Hilbert space. A mapping T from C into H is said to be generalized hybrid if there exist real numbers α and β such that

$$\alpha\|Tx-Ty\|^2+(1-\alpha)\|x-Ty\|^2 \le \beta\|Tx-y\|^2+(1-\beta)\|x-y\|^2$$

for any $x, y \in C$. We call such a mapping an (α, β)-generalized hybrid mapping. We observe that the class of the mappings covers the classes of well-known mappings. For example, an (α, β)-generalized hybrid mapping is nonexpansive [21] for $\alpha = 1$ and $\beta = 0$, that is, $\|Tx - Ty\| \le \|x - y\|$ for any $x, y \in C$. It is nonspreading [18] for $\alpha = 2$ and $\beta = 1$, that is, $2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2$ for any $x, y \in C$. It is also hybrid [22] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, that is, $3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$ for any $x, y \in C$. They proved fixed point theorems for such mappings; see also Kohsaka and Takahashi [17] and Iemoto and Takahashi [9]. Moreover they proved a nonlinear ergodic theorem. Furthermore they defined a more broad class of nonlinear mappings than the class of generalized hybrid mappings. A mapping T from C into H is said to be super hybrid if there exist real numbers α, β and γ such that

$$\begin{aligned}
&\alpha\|Tx-Ty\|^2+(1-\alpha+\gamma)\|x-Ty\|^2 \\
&\quad \le (\beta+(\beta-\alpha)\gamma)\|Tx-y\|^2+(1-\beta-(\beta-\alpha-1)\gamma)\|x-y\|^2 \\
&\qquad +(\alpha-\beta)\gamma\|x-Tx\|^2+\gamma\|y-Ty\|^2
\end{aligned}$$

for any $x, y \in C$. We call such a mapping an (α, β, γ)-super hybrid mapping. A generalized hybrid mapping with a fixed point is quasinonexpansive. However a super hybrid mapping is not quasi-nonexpansive generally even if it has a fixed point. Very recently, the author [13] also defined a class of nonlinear mappings in a Hilbert space which covers the class of contractive mappings and the class of generalized hybrid mappings. A mapping T from C into H is said to be widely generalized hybrid if there exist real numbers $\alpha, \beta, \gamma, \delta, \varepsilon$ and ζ such that

$$\begin{aligned}\alpha\|Tx-Ty\|^2+\beta\|x-Ty\|^2+\gamma\|Tx-y\|^2+\delta\|x-y\|^2 \\ +\max\{\varepsilon\|x-Tx\|^2, \zeta\|y-Ty\|^2\} \leq 0\end{aligned}$$

for any $x, y \in C$. Furthermore the author [14] defined a class of nonlinear mappings in a Hilbert space which covers the class of super hybrid mappings and the class of widely generalized hybrid mappings. A mapping T from C into H is said to be widely more generalized hybrid if there exist real numbers $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ and η such that

$$\begin{aligned}\alpha\|Tx-Ty\|^2+\beta\|x-Ty\|^2+\gamma\|Tx-y\|^2+\delta\|x-y\|^2 \\ +\varepsilon\|x-Tx\|^2+\zeta\|y-Ty\|^2+\eta\|(x-Tx)-(y-Ty)\|^2 \leq 0\end{aligned}$$

for any $x, y \in C$. We call such a mapping an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping. Then we prove fixed point theorems for such new mappings in a Hilbert space. Furthermore we prove nonlinear ergodic theorems of Baillon's type in a Hilbert space. It seems that the results are new and useful. For example, using our fixed point theorems, we can directly prove Browder and Petryshyn's fixed point theorem [5] for strictly pseudocontractive mappings and Kocourek, Takahashi and Yao's fixed point theorem [16] for super hybrid mappings. On the other hand, Hojo, Takahashi and Yao [8] defined a more broad class of nonlinear mappings than the class of generalized hybrid mappings. A mapping T from C into H is said to be extended hybrid if there exist real numbers α, β and γ such that

$$\begin{aligned}\alpha(1+\gamma)\|Tx-Ty\|^2+(1-\alpha(1+\gamma))\|x-Ty\|^2 \\ \leq(\beta+\alpha\gamma)\|Tx-y\|^2+(1-(\beta+\alpha\gamma))\|x-y\|^2 \\ -(\alpha-\beta)\gamma\|x-Tx\|^2-\gamma\|y-Ty\|^2\end{aligned}$$

for any $x, y \in C$. We call such a mapping an (α, β, γ)-extended hybrid mapping. Furthermore they proved a fixed point theorem for generalized hybrid non-self mappings by using the extended hybrid mapping.

In this paper we prove fixed point theorems and mean convergence theorems for widely more generalized hybrid self mappings and non-self mappings in Hilbert spaces.

2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let C be a non-empty subset of H. We denote by $F(T)$ the set of fixed points of T. A mapping T from C into H with $F(T) \neq \emptyset$ is said to be quasi-nonexpansive if $\|x-Ty\| \leq \|x-y\|$ for any $x \in F(T)$ and for any $y \in C$. It is well-known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping T is closed and convex; see Ito and Takahashi [10]. It is not difficult to prove such a result in a Hilbert space; see, for instance, [26]. Let C be a non-empty closed convex subset of H and $x \in H$. Then, we know that there exists a unique nearest point $z \in C$ such that $\|x - z\| = \inf_{y \in C} \|x - y\|$. We denote such a correspondence by $z = P_C x$. The mapping P_C is said to be the metric projection from H onto C. It is known that P_C is nonexpansive and

$$\langle x - P_C x, P_C x - u \rangle \geq 0$$

for any $x \in H$ and for any $u \in C$; see [21] for more details.

3 Fixed Point Theorems for Self Mappings

In this section we consider fixed point theorems for widely more generalized hybrid self mappings.

Theorem 3.1. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself which satisfies the following condition* (1), (2) *or* (3)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$ *and* $\zeta + \eta \geq 0$*;*

(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$ *and* $\varepsilon + \eta \geq 0$*;*

(3) $\alpha + \beta + \gamma + \delta \geq 0$, $2\alpha + \beta + \gamma + \varepsilon + \zeta + 2\eta > 0$ *and* $\varepsilon + \zeta + 2\eta \geq 0$.

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the conditions (1), (2) *and* (3).

Proof. In [14] we obtained the results in the cases of (1) and (2). We show in the case of (3). Since T is an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping,

$$\begin{aligned} &\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ &\quad + \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0 \end{aligned}$$

for any $x, y \in C$. By replacing the variables x and y, we obtain

$$\begin{aligned} &\alpha\|Tx - Ty\|^2 + \gamma\|x - Ty\|^2 + \beta\|Tx - y\|^2 + \delta\|x - y\|^2 \\ &\quad + \zeta\|x - Tx\|^2 + \varepsilon\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0 \end{aligned}$$

for any $x, y \in C$. Therefore we obtain

$$\begin{aligned} &2\alpha\|Tx - Ty\|^2 + (\beta + \gamma)\|x - Ty\|^2 + (\beta + \gamma)\|Tx - y\|^2 + 2\delta\|x - y\|^2 \\ &\quad + (\varepsilon + \zeta)\|x - Tx\|^2 + (\varepsilon + \zeta)\|y - Ty\|^2 + 2\eta\|(x - Tx) - (y - Ty)\|^2 \leq 0 \end{aligned}$$

for any $x, y \in C$ and hence T is a $(2\alpha, \beta+\gamma, \beta+\gamma, 2\delta, \varepsilon+\zeta, \varepsilon+\zeta, 2\eta)$-widely more generalized hybrid mapping. Moreover, since

$$\begin{aligned} &2\alpha + (\beta + \gamma) + (\beta + \gamma) + 2\delta = 2(\alpha + \beta + \gamma + \delta) \geq 0, \\ &2\alpha + (\beta + \gamma) + (\varepsilon + \zeta) + 2\eta > 0, \\ &(\varepsilon + \zeta) + 2\eta \geq 0 \end{aligned}$$

hold, we obtain the desired result by the case of (1) or (2). □

Remark 3.2. The conditions (1), (2) and (3) in Theorem 3.1 are not contained each other. For instance, if $\alpha = 2, \beta = -2, \gamma = \delta = \zeta = \eta = 0, \varepsilon = -1$, then, since

$$\begin{aligned} &\alpha + \beta + \gamma + \delta = 0 \geq 0, \\ &\alpha + \gamma + \varepsilon + \eta = 1 > 0, \\ &\zeta + \eta = 0 \geq 0, \end{aligned}$$

(1) is satisfied and hence this mapping has a fixed point. However, since

$$\begin{aligned} &\varepsilon + \eta = -1 \ngeq 0, \\ &\varepsilon + \zeta + 2\eta = -1 \ngeq 0, \end{aligned}$$

(2) and (3) are not satisfied. Similarly, for instance, $\alpha = 2, \beta = \delta = \varepsilon = \eta = 0, \gamma = -2, \zeta = -1$, then, since

$$\begin{aligned} &\alpha + \beta + \gamma + \delta = 0 \geq 0, \\ &\alpha + \beta + \zeta + \eta = 1 > 0, \\ &\varepsilon + \eta = 0 \geq 0, \end{aligned}$$

(2) is satisfied and hence this mapping has a fixed point. However, since

$$\begin{aligned} &\zeta + \eta = -1 \ngeq 0, \\ &\varepsilon + \zeta + 2\eta = -1 \ngeq 0, \end{aligned}$$

(1) and (3) are not satisfied. Similarly, for instance, $\alpha = 2, \beta = \gamma = \zeta = -1, \delta = \eta = 0, \varepsilon = 1$, then, since

$$\begin{aligned} &\alpha + \beta + \gamma + \delta = 0 \geq 0, \\ &2\alpha + \beta + \gamma + \varepsilon + \zeta + 2\eta = 2 > 0, \\ &\varepsilon + \zeta + 2\eta = 0 \geq 0, \end{aligned}$$

(3) is satisfied and hence this mapping has a fixed point. However, since

$$\begin{aligned} &\zeta + \eta = -1 \ngeq 0, \\ &\alpha + \beta + \zeta + \eta = 0 \not> 0, \end{aligned}$$

(1) and (2) are not satisfied.

Such a phenomenon also occurs in Theorem 3.3, Theorem 3.4, Theorem 4.1, Theorem 4.2, Theorem 5.1, Theorem 5.2, Theorem 6.1, Theorem 6.2, Theorem 6.3 and Theorem 6.4.

As a direct consequence of Theorem 3.1, we obtain the following; see Kawasaki and Takahashi [14].

Theorem 3.3. *Let H be a real Hilbert space, let C be a bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself which satisfies the following condition* (1)*,* (2) *or* (3)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$ *and* $\zeta + \eta \geq 0$*;*

(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$ *and* $\varepsilon + \eta \geq 0$*;*

(3) $\alpha + \beta + \gamma + \delta \geq 0$, $2\alpha + \beta + \gamma + \varepsilon + \zeta + 2\eta > 0$ *and* $\varepsilon + \zeta + 2\eta \geq 0$.

Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha+\beta+\gamma+\delta>0$ on the conditions (1)*,* (2) *and* (3)*.*

Moreover, by Theorem 3.1, we obtain the following fixed point theorem; see Kawasaki and Takahashi [14, 15].

Theorem 3.4. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into itself which satisfies the following conditions* (1)*,* (2) *or* (3)*:*

(1) $\alpha+\beta+\gamma+\delta\geq 0$, $\alpha+\gamma+\varepsilon+\eta>0$, *and there exists* $\lambda\in[0,1)$ *such that* $(\alpha+\beta)\lambda+\zeta+\eta\geq 0$*;*

(2) $\alpha+\beta+\gamma+\delta\geq 0$, $\alpha+\beta+\zeta+\eta>0$, *and there exists* $\lambda\in[0,1)$ *such that* $(\alpha+\gamma)\lambda+\varepsilon+\eta\geq 0$*;*

(3) $\alpha+\beta+\gamma+\delta\geq 0$, $2\alpha+\beta+\gamma+\varepsilon+\zeta+2\eta>0$, *and there exists* $\lambda\in[0,1)$ *such that* $(2\alpha+\beta+\gamma)\lambda+\varepsilon+\zeta+2\eta\geq 0$.

Then T has a fixed point if and only if there exists $z\in C$ such that $\{((1-\lambda)T+\lambda I)^n z\mid n\in\mathbb{N}\cup\{0\}\}$ is bounded for $\lambda\in[0,1)\cap\{\lambda\mid(\alpha+\beta)\lambda+\zeta+\eta\geq 0\}$, $\lambda\in[0,1)\cap\{\lambda\mid(\alpha+\gamma)\lambda+\varepsilon+\eta\geq 0\}$ or $\lambda\in[0,1)\cap\{\lambda\mid(2\alpha+\beta+\gamma)\lambda+\varepsilon+\zeta+2\eta\geq 0\}$, respectively. In particular, a fixed point of T is unique in the case of $\alpha+\beta+\gamma+\delta>0$ on the condition (1)*,* (2) *and* (3)*.*

Proof. In [14] we obtained the results in the cases of (1) and (2). Moreover we can show in the case of (3) similarly to Theorem 3.1. □

4 Fixed Point Theorems for Non-Self Mappings

In this section we consider fixed point theorems for widely more generalized hybrid non-self mappings. By Theorem 3.3, we obtain a fixed point theorem for widely more generalized hybrid non-self mappings in a Hilbert space; see Kawasaki and Kobayashi [12].

Theorem 4.1. *Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into H which satisfies the following condition* (1)*,* (2) *or* (3)*:*

(1) $\alpha+\beta+\gamma+\delta\geq 0$, $\alpha+\gamma+\varepsilon+\eta>0$, *and there exists* $\lambda\in\mathbb{R}$ *such that* $\lambda\neq 1$ *and* $(\alpha+\beta)\lambda+\zeta+\eta\geq 0$*;*

(2) $\alpha+\beta+\gamma+\delta\geq 0$, $\alpha+\beta+\zeta+\eta>0$, *and there exists* $\lambda\in\mathbb{R}$ *such that* $\lambda\neq 1$ *and* $(\alpha+\gamma)\lambda+\varepsilon+\eta\geq 0$*;*

(3) $\alpha+\beta+\gamma+\delta\geq 0$, $2\alpha+\beta+\gamma+\varepsilon+\zeta+2\eta>0$, *and there exists* $\lambda\in\mathbb{R}$ *such that* $\lambda\neq 1$ *and* $(2\alpha+\beta+\gamma)\lambda+\varepsilon+\zeta+2\eta\geq 0$.

Suppose that for any $x\in C$ there exist $m\in\mathbb{R}$ and $y\in C$ such that $0\leq(1-\lambda)m\leq 1$ and $Tx=x+m(y-x)$. Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha+\beta+\gamma+\delta>0$ on the conditions (1)*,* (2) *and* (3)*.*

Proof. In [12] we obtained the results in the cases of (1) and (2). Moreover we can show in the case of (3) similarly to Theorem 3.1. □

By Theorem 4.1, we obtain a fixed point theorem for widely more generalized hybrid non-self mappings in a Hilbert space; see Kawasaki [11].

Theorem 4.2. *Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into H which satisfies the following condition* (1), (2) *or* (3)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma+\varepsilon+\eta > 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $\lambda \neq 1$ *and* $(\alpha+\beta)\lambda+\zeta+\eta \geq 0$*;*

(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta+\zeta+\eta > 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $\lambda \neq 1$ *and* $(\alpha+\gamma)\lambda+\varepsilon+\eta \geq 0$*;*

(3) $\alpha+\beta+\gamma+\delta \geq 0$, $2\alpha+\beta+\gamma+\varepsilon+\zeta+2\eta > 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $\lambda \neq 1$ *and* $(2\alpha+\beta+\gamma)\lambda+\varepsilon+\zeta+2\eta \geq 0$.

Suppose that there exists $N \in \mathbb{R}$ *with* $N > 0$ *such that for any* $x \in C$ *there exist* $m \in M(N)$ *and* $y \in C$ *such that* $Tx = x + m(y - x)$, *where*

$$M(N) = \begin{cases} \left[\frac{1}{1-\lambda}, 0\right] & \text{if } \alpha+\beta > 0 \text{ and } \lambda > 1, \\ [0, N] & \text{if } \alpha+\beta > 0 \text{ and } \lambda < 1, \\ [0, N] \text{ or } [-N, 0] & \text{if } \alpha+\beta = 0, \\ [-N, 0] & \text{if } \alpha+\beta < 0 \text{ and } \lambda > 1, \\ \left[0, \frac{1}{1-\lambda}\right] & \text{if } \alpha+\beta < 0 \text{ and } \lambda < 1 \end{cases}$$

in the case of (1),

$$M(N) = \begin{cases} \left[\frac{1}{1-\lambda}, 0\right] & \text{if } \alpha+\gamma > 0 \text{ and } \lambda > 1, \\ [0, N] & \text{if } \alpha+\gamma > 0 \text{ and } \lambda < 1, \\ [0, N] \text{ or } [-N, 0] & \text{if } \alpha+\gamma = 0, \\ [-N, 0] & \text{if } \alpha+\gamma < 0 \text{ and } \lambda > 1, \\ \left[0, \frac{1}{1-\lambda}\right] & \text{if } \alpha+\gamma < 0 \text{ and } \lambda < 1 \end{cases}$$

in the case of (2) *and*

$$M(N) = \begin{cases} \left[\frac{1}{1-\lambda}, 0\right] & \text{if } 2\alpha+\beta+\gamma > 0 \text{ and } \lambda > 1, \\ [0, N] & \text{if } 2\alpha+\beta+\gamma > 0 \text{ and } \lambda < 1, \\ [0, N] \text{ or } [-N, 0] & \text{if } 2\alpha+\beta+\gamma = 0, \\ [-N, 0] & \text{if } 2\alpha+\beta+\gamma < 0 \text{ and } \lambda > 1, \\ \left[0, \frac{1}{1-\lambda}\right] & \text{if } 2\alpha+\beta+\gamma < 0 \text{ and } \lambda < 1 \end{cases}$$

in the case of (3)*. Then T has a fixed point. In particular, a fixed point of T is unique in the case of* $\alpha+\beta+\gamma+\delta > 0$ *on* (1), (2) *and* (3).

Proof. In [11] we obtained the results in the cases of (1) and (2). Moreover we can show in the case of (3) similarly to Theorem 3.1. □

5 Mean Convergence Theorems for Self Mappings

In this section we consider mean convergence theorems for widely more generalized hybrid self mappings.

Theorem 5.1. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into itself which has a fixed point and satisfies the conditions* (1), (2) *or* (3)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta+\zeta+\eta > 0$, $\alpha+\gamma > 0$ *and* $\varepsilon+\eta \geq 0$*;*

(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma+\varepsilon+\eta > 0$, $\alpha+\beta > 0$ *and* $\zeta+\eta \geq 0$*;*

(3) $\alpha+\beta+\gamma+\delta \geq 0$, $2\alpha+\beta+\gamma > 0$ *and* $\varepsilon+\zeta+2\eta \geq 0$.

Then for any $x \in C$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

is weakly convergent to $p \in F(T)$, *where* P *is the metric projection of* H *onto* $F(T)$ *and* $p = \lim_{n\to\infty} PT^n x$.

Proof. In [14] we obtained the results in the cases of (1) and (2). Moreover we can show in the case of (3) similarly to Theorem 3.1. □

By Theorem 5.1, we obtain the following mean convergence theorem; see Kawasaki and Takahashi [14, 15].

Theorem 5.2. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into itself which has a fixed point and satisfies the conditions* (1), (2) *or* (3)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta+\zeta+\eta > 0$, *and there exists* $\lambda \in [0,1)$ *such that* $0 \leq (\alpha+\gamma)\lambda+\varepsilon+\eta < \alpha+\gamma+\varepsilon+\eta$*;*

(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma+\varepsilon+\eta > 0$, *and there exists* $\lambda \in [0,1)$ *such that* $0 \leq (\alpha+\beta)\lambda+\zeta+\eta < \alpha+\beta+\zeta+\eta$*;*

(3) $\alpha+\beta+\gamma+\delta \geq 0$, *and there exists* $\lambda \in [0,1)$ *such that* $0 \leq (2\alpha+\beta+\gamma)\lambda+\varepsilon+\zeta+2\eta < 2\alpha+\beta+\gamma+\varepsilon+\zeta+2\eta$.

Then for any real number $\lambda \in [0,1) \cap \{\lambda \mid 0 \leq (\alpha+\gamma)\lambda+\varepsilon+\eta < \alpha+\gamma+\varepsilon+\eta\}$, $\lambda \in [0,1) \cap \{\lambda \mid 0 \leq (\alpha+\beta)\lambda+\zeta+\eta < \alpha+\beta+\zeta+\eta\}$ *or* $\lambda \in [0,1) \cap \{\lambda \mid 0 \leq (2\alpha+\beta+\gamma)\lambda+\varepsilon+\zeta+2\eta < 2\alpha+\beta+\gamma+\varepsilon+\zeta+2\eta\}$, *respectively, and for any* $x \in C$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} ((1-\lambda)T+\lambda I)^k x$$

is weakly convergent to $p \in F(T)$, *where* P *is the metric projection of* H *onto* $F(T)$ *and* $p = \lim_{n\to\infty} P((1-\lambda)T+\lambda I)^n x$.

Proof. In [14] we obtained the results in the cases of (1) and (2). Moreover we can show in the case of (3) similarly to Theorem 3.1. □

6 Mean Convergence Theorems for Non-Self Mappings

In this section we consider mean convergence theorems for widely more generalized hybrid non-self mappings.

Theorem 6.1. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into H which has a fixed point and satisfies the condition* (1), (2) *or* (3)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta+\zeta+\eta > 0$, $\alpha+\gamma > 0$ *and* $\varepsilon+\eta \geq 0$;

(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma+\varepsilon+\eta > 0$, $\alpha+\beta > 0$ *and* $\zeta+\eta \geq 0$;

(3) $\alpha+\beta+\gamma+\delta \geq 0$, $2\alpha+\beta+\gamma > 0$ *and* $\varepsilon+\zeta+2\eta \geq 0$.

Then for any $x \in C(T;0) = \{z \mid T^n z \in C \text{ for any } n \in \mathbb{N}\cup\{0\}\}$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

is weakly convergent to $p \in F(T)$*, where* P *is the metric projection of* H *onto* $F(T)$ *and* $p = \lim_{n\to\infty} PT^n x$.

Proof. In [12] we obtained the results in the cases of (1) and (2). Moreover we can show in the case of (3) similarly to Theorem 3.1. □

Theorem 6.2. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into H which has a fixed point and satisfies the condition* (1), (2) *or* (3)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta+\zeta+\eta > 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (\alpha+\gamma)\lambda+\varepsilon+\eta < \alpha+\gamma+\varepsilon+\eta$;

(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma+\varepsilon+\eta > 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (\alpha+\beta)\lambda+\zeta+\eta < \alpha+\beta+\zeta+\eta$;

(3) $\alpha+\beta+\gamma+\delta \geq 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (2\alpha+\beta+\gamma)\lambda+\varepsilon+\zeta+2\eta < 2\alpha+\beta+\gamma+\varepsilon+\zeta+2\eta$.

Then for any $x \in C(T;\lambda) = \{z \mid ((1-\lambda)T+\lambda I)^n z \in C \text{ for any } n \in \mathbb{N}\cup\{0\}\}$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} ((1-\lambda)T+\lambda I)^k x$$

is weakly convergent to $p \in F(T)$*, where* P *is the metric projection of* H *onto* $F(T)$ *and* $p = \lim_{n\to\infty} P((1-\lambda)T+\lambda I)^n x$.

Proof. In [12] we obtained the results in the cases of (1) and (2). Moreover we can show in the case of (3) similarly to Theorem 3.1. □

Theorem 6.3. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into H which satisfies the following condition* (1), (2) *or* (3)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta+\zeta+\eta > 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (\alpha+\gamma)\lambda+\varepsilon+\eta < \alpha+\gamma+\varepsilon+\eta$*;*

(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma+\varepsilon+\eta > 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (\alpha+\beta)\lambda+\zeta+\eta < \alpha+\beta+\zeta+\eta$*;*

(3) $\alpha+\beta+\gamma+\delta \geq 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (2\alpha+\beta+\gamma)\lambda+\varepsilon+\zeta+2\eta < 2\alpha+\beta+\gamma+\varepsilon+\zeta+2\eta$.

Suppose that for any $x \in C$ *there exist* $m \in \mathbb{R}$ *and* $y \in C$ *such that* $0 \leq (1-\lambda)m \leq 1$ *and* $Tx = x + m(y-x)$. *Then for any* $x \in C$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1}((1-\lambda)T+\lambda I)^k x$$

is weakly convergent to $p \in F(T)$, *where* P *is the metric projection of* H *onto* $F(T)$ *and* $p = \lim_{n\to\infty} P((1-\lambda)T+\lambda I)^n x$.

Proof. In [12] we obtained the results in the cases of (1) and (2). Moreover we can show in the case of (3) similarly to Theorem 3.1. □

By Theorem 6.3, we obtain the following mean convergence theorem; see Kawasaki [11].

Theorem 6.4. *Let* H *be a real Hilbert space, let* C *be a non-empty closed convex subset of* H *and let* T *be an* $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$*-widely more generalized hybrid mapping from* C *into* H *which satisfies the following condition* (1), (2) *or* (3)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta+\zeta+\eta > 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (\alpha+\gamma)\lambda+\varepsilon+\eta < \alpha+\gamma+\varepsilon+\eta$*;*

(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma+\varepsilon+\eta > 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (\alpha+\beta)\lambda+\zeta+\eta < \alpha+\beta+\zeta+\eta$*;*

(3) $\alpha+\beta+\gamma+\delta \geq 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (2\alpha+\beta+\gamma)\lambda+\varepsilon+\zeta+2\eta < 2\alpha+\beta+\gamma+\varepsilon+\zeta+2\eta$.

Suppose that there exists $N \in \mathbb{R}$ *with* $N > 0$ *such that for any* $x \in C$ *there exist* $m \in M(N)$ *and* $y \in C$ *such that* $Tx = x + m(y-x)$, *where*

$$M(N) = \begin{cases} [0,N] & \text{if } \alpha+\gamma > 0, \\ [-N,0] & \text{if } \alpha+\gamma < 0 \end{cases}$$

in the case of (1) *and let*

$$M(N) = \begin{cases} [0,N] & \text{if } \alpha+\beta > 0, \\ [-N,0] & \text{if } \alpha+\beta < 0 \end{cases}$$

in the case of (2) *and let*

$$M(N) = \begin{cases} [0,N] & \text{if } 2\alpha+\beta+\gamma > 0, \\ [-N,0] & \text{if } 2\alpha+\beta+\gamma < 0 \end{cases}$$

in the case of (3). *Then for any* $x \in C$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1}((1-\lambda)T+\lambda I)^k x$$

is weakly convergent to $p \in F(T)$, *where* P *is the metric projection from* H *onto* $F(T)$ *and* $p = \lim_{n\to\infty} P((1-\lambda)T + \lambda I)^n x$.

Proof. In [11] we obtained the results in the cases of (1) and (2). Moreover we can show in the case of (3) similarly to Theorem 3.1. □

7 Applications

In this section we discuss a strong convergence theorem with implicit iteration for mappings in a Hilbert space. Let H be a real Hilbert space and let C be a non-empty subset of H. A mapping T from C into H is said to be strictly pseudocontractive [5] if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(x - Tx) - (y - Ty)\|^2$$

for any $x, y \in C$. If $k = 0$, T is nonexpansive. In 1967 Browder [4] proved the strong convergence theorem with implicit iteration in a Hilbert space.

Theorem 7.1. *Let* H *be a Hilbert space, let* C *be a non-empty bounded closed convex subset of* H, *let* T *be a nonexpansive mapping from* C *into itself, let* $u \in C$ *and let* $\{\alpha_n\}$ *be a sequence in* $(0, 1)$. *Define a sequence* $\{z_n\}$ *in* C *by*

$$z_n = \alpha_n u + (1 - \alpha_n)Tz_n.$$

If $\{\alpha_n\}$ *is convergent to* 0, *then* $\{z_n\}$ *is convergent strongly to* $P_{F(T)}u$, *where* $P_{F(T)}$ *is the metric projection of* H *onto* $F(T)$.

In 2014, using widely more generalized hybrid mappings, Takahashi [23] proved the strong convergence theorem for strictly pseudocontractive mappings in a Hilbert space. This theorem is an extension of Theorem 7.1.

Theorem 7.2. *Let* H *be a Hilbert space, let* C *be a non-empty bounded closed convex subset of* H, *let* T *be a strictly pseudocontractive mapping from* C *into itself, that is, there exists* $k \in [0, 1)$ *such that*

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(x - Tx) - (y - Ty)\|^2$$

for any $x, y \in C$, *let* $u \in C$ *and let* $\{\alpha_n\}$ *be a sequence in* $(0, 1)$. *Define a mapping* U_n *as follows:*

$$U_n x = \alpha_n u + (1 - \alpha_n)Tx$$

for any $x \in C$ *and for any* $n \in \mathbb{N}$. *Then the following hold:*

(i) U_n *has a unique fixed point* z_n *in* C;

(ii) *if* $\{\alpha_n\}$ *is convergent to* 0, *then* $\{z_n\}$ *is convergent strongly to* $P_{F(T)}u$, *where* $P_{F(T)}$ *is the metric projection of* H *onto* $F(T)$.

Let H be a real Hilbert space and let C be a non-empty subset of H. A mapping T from C into H is said to be pseudocontractive-type if there exist real numbers α, δ and η such that

$$\alpha\|Tx - Ty\|^2 + \delta\|x - y\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \le 0$$

for any $x, y \in C$. We call such a mapping an (α, δ, η)-pseudocontractive-type mapping. Then we prove a following extension of Theorem 7.2.

Theorem 7.3. *Let H be a Hilbert space, let C be a non-empty bounded closed convex subset of H, let T be an (α,δ,η)-pseudocontractive-type mapping from C into itself, let $u\in C$ and let $\{\alpha_n\}$ be a sequence in $(0,1)$. Define a mapping U_n as follows:*

$$U_n x = \alpha_n u + (1-\alpha_n)Tx$$

for any $x\in C$ and for any $n\in\mathbb{N}$. Suppose that $\alpha+\delta\geq 0$, $\alpha+\eta>0$ and $\alpha>0$. Then the following hold:

(i) *U_n has a unique fixed point z_n in C;*

(ii) *if $\{\alpha_n\}$ is convergent to 0, then $\{z_n\}$ is convergent strongly to $P_{F(T)}u$, where $P_{F(T)}$ is the metric projection of H onto $F(T)$.*

The proof of Theorem 7.3 is almost the same as that of Theorem 7.2. However for the proof we provide the following lemmas.

Lemma 7.4. *Let H be real Hilbert space, let C be a non-empty closed convex subset of H and let T be an (α,δ,η)-pseudocontractive-type mapping from C into H. Suppse that $\alpha+\delta\geq 0$, $\alpha+\eta>0$ and $\alpha>0$. Then $I-T$ is $\frac{\alpha+\eta}{2\alpha}$-inverse strongly monotone.*

Proof. Suppose that T is an (α,δ,η)-pseudocontractive-type mapping, that is,

$$\alpha\|Tx-Ty\|^2+\delta\|x-y\|^2+\eta\|(I-T)x-(I-T)y\|^2\leq 0$$

for any $x,y\in C$. Since $Tx-Ty=(x-y)-((I-T)x-(I-T)y)$, we obtain

$$\begin{aligned}
&\alpha(\|x-y\|^2+\|(I-T)x-(I-T)y\|^2-2\langle x-y,(I-T)x-(I-T)y\rangle)\\
&\quad+\delta\|x-y\|^2+\eta\|(I-T)x-(I-T)y\|^2\\
&=(\alpha+\eta)\|(I-T)x-(I-T)y\|^2+(\alpha+\delta)\|x-y\|^2\\
&\quad-2\alpha\langle x-y,(I-T)x-(I-T)y\rangle\\
&\leq 0.
\end{aligned}$$

Since $\alpha+\delta\geq 0$ and $\alpha>0$, we obtain

$$\frac{\alpha+\eta}{2\alpha}\|(I-T)x-(I-T)y\|^2\leq\langle x-y,(I-T)x-(I-T)y\rangle,$$

that is, $I-T$ is $\frac{\alpha+\eta}{2\alpha}$-inverse strongly monotone. □

Lemma 7.5. *Let H be real Hilbert space, let C be a non-empty closed convex subset of H and let T be an (α,δ,η)-pseudocontractive-type mapping from C into H. Suppose that $\alpha+\delta\geq 0$ and $\alpha+\eta>0$. If $\{x_n\}$ is convergent weakly to $z\in H$ and $\{x_n-Tx_n\}$ is convergent strongly to 0, then z is a fixed point of T.*

Proof. We know, if $\{x_n\}$ is convergent weakly to $z\in H$, then

$$\limsup_{n\to\infty}\|x_n-x\|^2=\limsup_{n\to\infty}\|x_n-z\|^2+\|x-z\|^2$$

for any $x\in H$. In particular,

$$\limsup_{n\to\infty}\|x_n-Tz\|^2=\limsup_{n\to\infty}\|x_n-z\|^2+\|Tz-z\|^2.$$

Since $\{x_n - Tx_n\}$ is convergent strongly to 0, we obtain

$$\limsup_{n\to\infty} \|Tx_n - Tz\|^2 = \limsup_{n\to\infty} \|x_n - Tz\|^2.$$

Therefore for any positive number ρ there exists a natural number N_1 such that, for any $n \geq N_1$

$$\|Tx_n - Tz\|^2 < \limsup_{n\to\infty} \|x_n - z\|^2 + \|Tz - z\|^2 + \rho$$

and there exists a natural number m with $m \geq n$ such that

$$\limsup_{n\to\infty} \|x_n - z\|^2 + \|Tz - z\|^2 - \rho < \|Tx_m - Tz\|^2.$$

Moreover there exists a natural number N_2 such that, for any $n \geq N_2$

$$\|x_n - z\|^2 < \limsup_{n\to\infty} \|x_n - z\|^2 + \rho$$

and there exists a natural number m with $m \geq n$ such that

$$\limsup_{n\to\infty} \|x_n - z\|^2 - \rho < \|x_m - z\|^2.$$

Since $\{x_n - Tx_n\}$ is convergent strongly to 0, for any positive number ρ there exists a natural number N_3 such that

$$\|Tz - z\|^2 - \rho < \|(I - T)x_n - (I - T)z\|^2 < \|Tz - z\|^2 + \rho$$

for any $n \geq N_3$. Put $N = \max(N_1, N_2, N_3)$ and take a natural number n with $n \geq N$. If $\alpha \geq 0$, then there exists a natural numbers m with $m \geq n$ such that

$$\|Tx_m - Tz\|^2 > \limsup_{n\to\infty} \|x_n - z\|^2 + \|Tz - z\|^2 - \rho.$$

Moreover, since

$$\limsup_{n\to\infty} \|x_n - z\|^2 > \|x_m - z\|^2 - \rho,$$

we obtain

$$\begin{aligned}\alpha\|Tx_m - Tz\|^2 &\geq \alpha\left((\|x_m - z\|^2 - \rho) + \|Tz - z\|^2 - \rho\right)\\ &= \alpha\|x_m - z\|^2 + \alpha\|Tz - z\|^2 - 2|\alpha|\rho.\end{aligned}$$

If $\alpha < 0$, then there exists a natural numbers m with $m \geq n$ such that

$$\limsup_{n\to\infty} \|x_n - z\|^2 < \|x_m - z\|^2 + \rho.$$

Moreover, since

$$\|Tx_m - Tz\|^2 < \limsup_{n\to\infty} \|x_n - z\|^2 + \|Tz - z\|^2 + \rho,$$

we obtain

$$\begin{aligned}\alpha\|Tx_m - Tz\|^2 &\geq \alpha\left((\|x_m - z\|^2 + \rho) + \|Tz - z\|^2 + \rho\right)\\ &= \alpha\|x_m - z\|^2 + \alpha\|Tz - z\|^2 - 2|\alpha|\rho.\end{aligned}$$

Therefore we obtain

$$\begin{aligned} 0 &\geq \alpha\|Tx_m - Tz\|^2 + \delta\|x_m - z\|^2 + \eta\|(I-T)x_m - (I-T)z\|^2 \\ &\geq \alpha\|x_m - z\|^2 + \alpha\|Tz - z\|^2 - 2|\alpha|\rho + \delta\|x_m - z\|^2 + \eta\|Tz - z\|^2 - |\eta|\rho \\ &= (\alpha+\delta)\|x_m - z\|^2 + (\alpha+\eta)\|Tz - z\|^2 - (2|\alpha| + |\eta|)\rho. \end{aligned}$$

Since $\alpha + \delta \geq 0$ and $\alpha + \eta > 0$, we obtain

$$\|Tz - z\|^2 \leq \frac{2|\alpha| + |\eta|}{\alpha + \eta}\rho.$$

Since ρ is arbitrary, we obtain $\|Tz - z\|^2 = 0$ and hence z is a fixed point of T. □

Proof of Theorem 7.3. Since $\alpha+\eta > 0$, there exists $\lambda \in [0,1)$ such that $\alpha\lambda+\eta \geq 0$. Moreover, since $\alpha + \delta \geq 0$ and $\alpha + \eta > 0$, by Theorem 3.4 $F(T)$ is not empty. Since $\alpha + \eta > 0$, by [12, Lemma 4.1] $F(T)$ is closed. Since $\alpha + \delta \geq 0$ and $\alpha + \eta > 0$, by [12, Lemma 4.2] $F(T)$ is convex. Therefore $P_{F(T)}$ is well-defined. Since

$$U_n x = \alpha_n u + (1 - \alpha_n)Tx$$

we obtain

$$\begin{aligned} 0 &\geq \alpha\left\|\left(\frac{U_n x - \alpha_n u}{1-\alpha_n}\right) - \left(\frac{U_n y - \alpha_n u}{1-\alpha_n}\right)\right\|^2 + \delta\|x - y\|^2 \\ &\quad +\eta\left\|\left(x - \frac{U_n x - \alpha_n u}{1-\alpha_n}\right) - \left(y - \frac{U_n y - \alpha_n u}{1-\alpha_n}\right)\right\|^2 \\ &= \frac{\alpha}{(1-\alpha_n)^2}\|U_n x - U_n y\|^2 + \delta\|x - y\|^2 \\ &\quad +\eta\left(\frac{\alpha_n}{1-\alpha_n}\right)^2\left\|-\frac{1-\alpha_n}{\alpha_n}(x-y) + \frac{1}{\alpha_n}(U_n x - U_n y)\right\|^2. \end{aligned}$$

Since $\|(1-\lambda)u + \lambda v\|^2 = (1-\lambda)\|u\|^2 + \lambda\|v\|^2 - (1-\lambda)\lambda\|u - v\|^2$ for any $\lambda \in \mathbb{R}$, we obtain

$$\begin{aligned} &\frac{\alpha + \eta\alpha_n}{(1-\alpha_n)^2}\|U_n x - U_n y\|^2 + \left(\delta - \frac{\eta\alpha_n}{1-\alpha_n}\right)\|x - y\|^2 \\ &\quad + \frac{\eta}{1-\alpha_n}\|(I - U_n)x - (I - U_n)y\|^2 \leq 0. \end{aligned}$$

We obtain

$$\frac{\alpha + \eta\alpha_n}{(1-\alpha_n)^2} + \left(\delta - \frac{\eta\alpha_n}{1-\alpha_n}\right) = \frac{(\delta+\eta)\alpha_n^2 - 2\delta\alpha_n + \alpha + \delta}{(1-\alpha_n)^2}.$$

Put $f(x) = (\delta+\eta)x^2 - 2\delta x + \alpha + \delta$. If $\delta + \eta > 0$, then f is convex. Moreover, if $\frac{\delta}{\delta+\eta} \leq 0$, then f is increasing on $(0,1)$ and $f(0) = \alpha + \delta \geq 0$, and hence $f(x) > 0$ for any $x \in (0,1)$; if $0 < \frac{\delta}{\delta+\eta} < 1$, then $\delta > 0$ and $\eta > 0$, and hence $f(x) \geq f\left(\frac{\delta}{\delta+\eta}\right) = -\frac{\delta^2}{\delta+\eta} + \alpha + \delta > \alpha > 0$ for any $x \in (0,1)$; if $1 \leq \frac{\delta}{\delta+\eta}$, then f is decreasing and $f(1) = \alpha + \eta > 0$, and hence $f(x) > 0$ for any $x \in (0,1)$. If $\delta + \eta \leq 0$, then f is concave. Moreover, since $f(0) = \alpha + \delta \geq 0$ and $f(1) = \alpha + \eta > 0$, $f(x) > 0$ for any $x \in (0,1)$. Therefore we obtain

$$\frac{\alpha + \eta\alpha_n}{(1-\alpha_n)^2} + \left(\delta - \frac{\eta\alpha_n}{1-\alpha_n}\right) > 0.$$

Since $\alpha+\eta>0$, we obtain

$$\frac{\alpha+\eta\alpha_n}{(1-\alpha_n)^2}+\frac{\eta}{1-\alpha_n}=\frac{\alpha+\eta}{(1-\alpha_n)^2}>0.$$

Moreover we obtain

$$\begin{aligned}\frac{\alpha+\eta\alpha_n}{(1-\alpha_n)^2}\lambda+\frac{\eta}{1-\alpha_n} &= \frac{\alpha\lambda+\eta-\eta(1-\lambda)\alpha_n}{(1-\alpha_n)^2}\\ &\geq \frac{\alpha\lambda+\eta-|\eta|(1-\lambda)}{(1-\alpha_n)^2}\\ &= \frac{(\alpha+|\eta|)\lambda-(|\eta|-\eta)}{(1-\alpha_n)^2}.\end{aligned}$$

Since $\alpha+\eta>0$, we obtain $\frac{|\eta|-\eta}{\alpha+|\eta|}<1$. Therefore, if $\frac{|\eta|-\eta}{\alpha+|\eta|}\leq\lambda<1$, then

$$\frac{\alpha+\eta\alpha_n}{(1-\alpha_n)^2}\lambda+\frac{\eta}{1-\alpha_n}\geq 0.$$

Therefore by Theorem 3.4 U_n has a unique fixed point $z_n\in C$. To show that $\{z_n\}$ is convergent strongly to $P_{F(T)}u$, we may show that any subsequence $\{z_{n(i)}\}$ of $\{z_n\}$ has a subsequence $\{z_{n(i,j)}\}$ of $\{z_{n(i)}\}$ such that $\{z_{n(i,j)}\}$ is convergent to $P_{F(T)}u$. Without loss of generality, we may assume that $\{z_{n(i)}\}$ is convergent weakly to $v\in C$. Let us show $v\in F(T)$. Since $\{\alpha_n\}$ is convergent to 0, we obtain $\{z_n-Tz_n\}$ is convergent strongly to 0. In fact, since

$$z_n=U_nz_n=\alpha_nu+(1-\alpha_n)Tz_n,$$

we obtain

$$z_n-Tz_n=\alpha_n(u-Tz_n).$$

Since $\{Tz_n\}$ is bounded and $\{\alpha_n\}$ is convergent to 0, we obtain $\{z_n-Tz_n\}$ is convergent strongly to 0. Since $\{z_{n(i)}\}$ is convergent weakly to $v\in C$, by Lemma 7.5 we obtain $v\in F(T)$. Since $z_{n(i)}\in F(U_{n(i)})$, we obtain

$$z_{n(i)}=\alpha_{n(i)}u+(1-\alpha_{n(i)})Tz_{n(i)}$$

and hence

$$\alpha_{n(i)}z_{n(i)}+(1-\alpha_{n(i)})(z_{n(i)}-Tz_{n(i)})=\alpha_{n(i)}u.$$

Moreover, since $P_{F(T)}u\in F(T)$, we obtain

$$\alpha_{n(i)}P_{F(T)}u+(1-\alpha_{n(i)})(P_{F(T)}u-TP_{F(T)}u)=\alpha_{n(i)}P_{F(T)}u.$$

Therefore we obtain

$$\begin{aligned}&\alpha_{n(i)}\langle z_{n(i)}-P_{F(T)}u,z_{n(i)}-P_{F(T)}u\rangle\\ &\quad+(1-\alpha_{n(i)})\langle(I-T)z_{n(i)}-(I-T)P_{F(T)}u,z_{n(i)}-P_{F(T)}u\rangle\\ &=\alpha_{n(i)}\langle u-P_{F(T)}u,z_{n(i)}-P_{F(T)}u\rangle.\end{aligned}$$

By Lemma 7.4, $I-T$ is $\frac{\alpha+\eta}{2\alpha}$-inverse strongly monotone and hence

$$\frac{\alpha+\eta}{2\alpha}\|z_{n(i)}-P_{F(T)}u\|^2 \leq \langle (I-T)z_{n(i)}-(I-T)P_{F(T)}u, z_{n(i)}-P_{F(T)}u\rangle.$$

Therefore we obtain

$$\begin{aligned}\|z_{n(i)}-P_{F(T)}u\|^2 &\leq \langle u-P_{F(T)}u, z_{n(i)}-P_{F(T)}u\rangle \\ &= \langle u-P_{F(T)}u, z_{n(i)}-v\rangle + \langle u-P_{F(T)}u, v-P_{F(T)}u\rangle.\end{aligned}$$

Since $v\in F(T)$, we obtain

$$\langle u-P_{F(T)}u, v-P_{F(T)}u\rangle \leq 0.$$

Therefore we obtain

$$\|z_{n(i)}-P_{F(T)}u\|^2 \leq \langle u-P_{F(T)}u, z_{n(i)}-v\rangle.$$

Since $\{z_{n(i)}\}$ is convergent weakly to $v\in C$, we obtain $\{z_{n(i)}\}$ is convergent strongly to $P_{F(T)}u$. □

Moreover, we obtain the following.

Theorem 7.6. *Let H be a Hilbert space, let C be a non-empty bounded closed convex subset of H, let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into itself, let $u\in C$ and let $\{\alpha_n\}$ be a sequence in $(0,1)$. Define a mapping U_n as follows:*

$$U_n x = \alpha_n u + (1-\alpha_n)Tx$$

for any $x\in C$ and for any $n\in\mathbb{N}$. Suppose that $\beta+\gamma+\varepsilon+\zeta\geq 0$, $\alpha+\beta+\gamma+\delta\geq 0$, $\alpha+\eta>0$ and $2\alpha+\beta+\gamma>0$. Then the following hold:

(i) *U_n has a unique fixed point z_n in C;*

(ii) *if $\{\alpha_n\}$ is convergent to 0, then $\{z_n\}$ is convergent strongly to $P_{F(T)}u$, where $P_{F(T)}$ is the metric projection of H onto $F(T)$.*

Proof. Since T is an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping,

$$\begin{aligned}&\alpha\|Tx-Ty\|^2+\beta\|x-Ty\|^2+\gamma\|Tx-y\|^2+\delta\|x-y\|^2 \\ &\quad +\varepsilon\|x-Tx\|^2+\zeta\|y-Ty\|^2+\eta\|(x-Tx)-(y-Ty)\|^2\leq 0\end{aligned}$$

for any $x,y\in C$. By replacing the variables x and y, we obtain

$$\begin{aligned}&\alpha\|Tx-Ty\|^2+\gamma\|x-Ty\|^2+\beta\|Tx-y\|^2+\delta\|x-y\|^2 \\ &\quad +\zeta\|x-Tx\|^2+\varepsilon\|y-Ty\|^2+\eta\|(x-Tx)-(y-Ty)\|^2\leq 0\end{aligned}$$

for any $x,y\in C$. Therefore we obtain

$$\begin{aligned}&2\alpha\|Tx-Ty\|^2+(\beta+\gamma)\|x-Ty\|^2+(\beta+\gamma)\|Tx-y\|^2+2\delta\|x-y\|^2 \\ &\quad +(\varepsilon+\zeta)\|x-Tx\|^2+(\varepsilon+\zeta)\|y-Ty\|^2+2\eta\|(x-Tx)-(y-Ty)\|^2\leq 0\end{aligned}$$

for any $x, y \in C$ and hence T is a $(2\alpha, \beta+\gamma, \beta+\gamma, 2\delta, \varepsilon+\zeta, \varepsilon+\zeta, 2\eta)$-widely more generalized hybrid mapping. Since $\beta + \gamma + \varepsilon + \zeta \geq 0$, we obtain $-(\beta + \gamma) \leq \varepsilon + \zeta$ and hence

$$\begin{aligned} &2\alpha\|Tx - Ty\|^2 + (\beta + \gamma)\|x - Ty\|^2 + (\beta + \gamma)\|Tx - y\|^2 + 2\delta\|x - y\|^2 \\ &\qquad -(\beta + \gamma)\|x - Tx\|^2 - (\beta + \gamma)\|y - Ty\|^2 + 2\eta\|(x - Tx) - (y - Ty)\|^2 \\ &= (2\alpha + \beta + \gamma)\|Tx - Ty\|^2 + (2\delta + \beta + \gamma)\|x - y\|^2 \\ &\qquad +(2\eta - \beta - \gamma)\|(x - Tx) - (y - Ty)\|^2 \\ &\leq 0. \end{aligned}$$

Since $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\eta > 0$ and $2\alpha+\beta+\gamma > 0$, by Theorem 7.3 we obtain the desired results. □

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