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WHAT IS THE GENERALIZATION OF NATURAL RESIDUAL FUNCTION FOR NCP?

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Abstract: It is well known that the generalized Fischer-Burmeister is a natural extension of the popular Fischer-Burmeister function NCP-function, in which the 2-norm is replaced by general p-norm. As for another popular natural residual NCP-function, its generalization was unknown during the past three decades. In this short communication, we answer this long-standing open question. In particular, we propose the generalization of natural residual function for NCP, which possesses twice differentiability. This feature enables that many methods like Newton method can be employed directly for solving NCP. This is a new discovery to the literature and we believe that such generalized function can also be employed in many other contexts.

Key words: *NCP, natural residual, complementarity.*

Mathematics Subject Classification: *90C33, 90C56, 65K10.*

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1 Motivation

The nonlinear complementarity problem (NCP for short) is to find a point $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad F(x) \geq 0, \quad \langle x, F(x) \rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product and $F = (F_1, \ldots, F_n)^T$ is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$. The NCP has attracted much attention because of its wide applications in the fields of economics, engineering, and operations research, see [9, 10, 16] and references therein. For solving NCP, the so-called NCP-function $\phi : \mathbb{R}^2 \to \mathbb{R}$ defined as below

$$\phi(a, b) = 0 \quad \Longleftrightarrow \quad a, b \geq 0, \ ab = 0$$

plays a crucial role. With such NCP-functions, the NCP can be recast as nonsmooth equations [21, 22, 27] or unconstrained minimization [11, 12, 15, 18, 19, 23, 26]. During the past three decades, around thirty NCP-functions are proposed, see [14] for a survey. Among them, two popular NCP-functions, the Fischer-Burmeister function [12, 13] and the natural residual function [20, 24], are frequently employed and most of the existing NCP-functions are indeed variants of these two functions. The Fischer-Burmeister function $\phi_{\rm FB} : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\phi_{\rm FB}(a, b) = \sqrt{a^2 + b^2} - (a + b),$$

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whereas the natural residual function $\phi_{\rm NR} : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$\phi_{\rm NR}(a,b) = a - (a-b)_+ = \min\{a,b\}.$$

Recently, the generalized Fischer-Burmeister function $\phi^p_{\rm FB} : \mathbb{R}^2 \to \mathbb{R}$, which includes the Fischer-Burmeister as a special case, was considered in [1, 2, 3, 6, 17, 25]. Indeed, the function $\phi^p_{\rm FB}$ is a natural extension of the popular $\phi_{\rm FB}$ function, in which the 2-norm in $\phi_{\rm FB}(a,b)$ is replaced by general p-norm. In other words, $\phi^p_{\rm FB}$ is defined as

$$\phi^p_{\rm FB}(a,b) = \|(a,b)\|_p - (a+b), \quad p > 1 \tag{1.1}$$

and its geometric view is depicted in [25]. The effect of perturbing p for different kinds of algorithms are investigated in [4, 5, 6, 7, 8]. To the contrast, "Is there an extension of natural residual function?" and "If yes, how does the extension of $\phi_{\rm NR}$ look like?" remain open. As mentioned, there exist many NCP-functions which are variants of $\phi_{\rm NR}$, but there is no literature talking about the extension of natural residual function. The main hurdle lies on lacking continuous norm generalization like what we do for $\phi^p_{\rm FB}$. In this paper, we give an affirmative answer to the long-standing open question. In fact, the main ideas rely on "discrete generalization", not the "continuous generalization". More specifically, the generalized natural residual function, denoted by $\phi^p_{\rm NR}$, is defined by

$$\phi^p_{\rm NR}(a,b) = a^p - (a-b)^p_+ \quad \text{with } p > 1 \text{ being a positive odd integer}, \tag{1.2}$$

where $(a-b)^p_+ = [(a-b)_+]^p$ and $(a-b)_+ = \max\{a-b, 0\}$. Here p being a positive odd integer is necessary (that is, we require that $p = 2k+1$, where $k = 1,2,3,\cdots$). We will explain this in Section 2. Notice that when $p = 1$, $\phi^p_{\rm NR}$ reduces to the natural residual function $\phi_{\rm NR}$, i.e., when $k = 0$, it corresponds to

$$\phi^1_{\rm NR}(a,b) = a - (a-b)_+ = \min\{a,b\} = \phi_{\rm NR}(a,b).$$

This is why we call it the "generalized natural residual function". We point it out again that the considered extension is based on "discrete generalization". For different values of p, it is no longer an NCP-function. A special feature of $\phi^p_{\rm NR}$ is that it is twice differentiable which will be proved in Section 2. It is well known that the generalized Fischer-Burmeister $\phi^p_{\rm FB}$ given as in (1.1) is not differentiable, while $\|\phi^p_{\rm FB}(a,b)\|^2$ is differentiable everywhere. This yields that $\|\phi^p_{\rm FB}(a,b)\|^2$ is usually adapted when using merit function approach and $\phi^p_{\rm FB}(a,b)$ is employed when applying nonsmooth function approach. Compared to the non-differentiability of $\phi^p_{\rm FB}$, the function $\phi^p_{\rm NR}$ with $p = 2k+1$ is twice continuously differentiable. This feature enables that many methods like Newton method can be employed directly for solving NCP. This is a new discovery to the literature and is the main contribution of this paper.

$\boxed{2}$ Generalized Natural Residual Function

In this section, we show that the function $\phi^p_{\rm NR}$ defined as in (1.2) is an NCP-function and present its twice differentiability.

Proposition 2.1. *Let $\phi^p_{\rm NR}$ be defined as in (1.2). Then, $\phi^p_{\rm NR}$ is an NCP-function.*

Proof. First, we note that for any fixed real number $\xi \geq 0$ and odd integer p, the equation $t^p - \xi^p = 0$ has exactly one real solution $t = \xi$ because the function t^p is strictly monotone.

Thus, we observe that

$$\begin{aligned}
& \phi^p_{\rm NR}(a,b) = 0 \\
\iff & a^p - (a-b)^p_+ = 0 \\
\iff & a - (a-b)_+ = 0 \\
\iff & \min\{a,b\} = 0 \\
\iff & a,\, b \geq 0,\ ab = 0.
\end{aligned}$$

This shows that $\phi^p_{\rm NR}$ is an NCP-function. □

Remarks: We elaborate more about the function $\phi^p_{\rm NR}$.

(a) For p being an even integer, $\phi^p_{\rm NR}$ is not a NCP-function. A counterexample is given as below.

$$\phi^2_{\rm NR}(-2,-4) = (-2)^2 - (-2+4)^2_+ = 0.$$

(b) The function $\phi^p_{\rm NR}$ is neither convex nor concave function. To see this, taking $p=3$ and using the following argument verify the assertion.

$$-1 = \phi^3_{\rm NR}(-1,-1) > \frac{1}{2}\phi^3_{\rm NR}(-2,-1) + \frac{1}{2}\phi^3_{\rm NR}(0,-1) = \frac{-8}{2} + \frac{-1}{2} = -\frac{9}{2}$$

Proposition 2.2. *Let $p>1$ be a positive odd integer. Then, we have*

$$[(a-b)_+]^p = [(a-b)^p]_+, \tag{2.1}$$

and hence

$$\phi^p_{\rm NR}(a,b) = a^p - [(a-b)_+]^p = a^p - [(a-b)^p]_+.$$

Proof. For any $\alpha \in \mathbb{R}$, we know that $[\alpha]_+ = \frac{1}{2}(\alpha + |\alpha|)$. In addition, looking the coefficients of the binomial $(1+x)^p$, we have

$$\sum_{j=0,\,\text{even}}^{p} C(p,j) = \sum_{j=0,\,\text{odd}}^{p} C(p,j) = \frac{1}{2}\sum_{j=0}^{p} C(p,j) = \frac{2^p}{2} = 2^{p-1}.$$

These two facts lead to

$$\begin{aligned}
& [(a-b)_+]^p \\
= & \frac{1}{2^p}\left(a-b+|a-b|\right)^p \\
= & \frac{1}{2^p}\left(\sum_{j=0}^{p} C(p,j)|a-b|^j(a-b)^{p-j}\right) \\
= & \frac{1}{2^p}\left(\sum_{j=0,\,\text{even}}^{p} C(p,j)|a-b|^j(a-b)^{p-j} + \sum_{j=0,\,\text{odd}}^{p} C(p,j)|a-b|^j(a-b)^{p-j}\right) \\
= & \frac{1}{2^p}\left(\sum_{j=0,\,\text{even}}^{p} C(p,j)(a-b)^p + \sum_{j=0,\,\text{odd}}^{p} C(p,j)|a-b|(a-b)^{p-1}\right) \\
= & \frac{1}{2^p}\left(2^{p-1}(a-b)^p + 2^{p-1}|a-b|(a-b)^{p-1}\right) \\
= & \frac{1}{2}\left((a-b)^p + |a-b|(a-b)^{p-1}\right) \\
= & [(a-b)^p]_+
\end{aligned}$$

where the last equality holds because p is an positive odd integer. Thus, the proof is complete. □

Remark: In Proposition 2.2, note that the equality in (2.1) holds only when p is positive odd integer. When p is an even integer, $[(a-b)_+]^p \neq [(a-b)^p]_+$. This also explains that requiring p being an positive odd integer is necessary in the definition of $\phi^p_{\rm NR}$. Next, we provide an alternative expression for $\phi^p_{\rm NR}$ and show its twice differentiability. To this end, we need a technical lemma.

Lemma 2.3. *Let $p > 1$. Then,*

(a) *the function $f(t) = |t|^p$ is differentiable and $f'(t) = p\,\mathrm{sgn}(t)|t|^{p-1}$;*

(b) *the function $f(t) = t^p|t|$ is differentiable and $f'(t) = (p+1)t^{p-1}|t|$.*

Proof. The proofs are straightforward which are omitted here. □

Proposition 2.4. *Let $p = 2k+1$ where $k = 1, 2, 3\cdots$. Then, we have*

(a) $\phi^p_{\rm NR}(a,b) = a^{2k+1} - \frac{1}{2}\left((a-b)^{2k+1} + (a-b)^{2k}|a-b|\right)$;

(b) $\phi^p_{\rm NR}$ *is continuously differentiable with*

$$\begin{aligned} & \nabla\phi^p_{\rm NR}(a,b) \\ = \;& p\begin{bmatrix} a^{p-1} - (a-b)^{p-2}(a-b)_+ \\ (a-b)^{p-2}(a-b)_+ \end{bmatrix}; \end{aligned}$$

(c) $\phi^p_{\rm NR}$ *is twice continuously differentiable with*

$$\begin{aligned} & \nabla^2\phi^p_{\rm NR}(a,b) \\ = \;& p(p-1)\begin{bmatrix} a^{p-2} - (a-b)^{p-3}(a-b)_+ & (a-b)^{p-3}(a-b)_+ \\ (a-b)^{p-3}(a-b)_+ & -(a-b)^{p-3}(a-b)_+ \end{bmatrix}. \end{aligned}$$

Proof. (a) This alternative expression follows from Proposition 2.2.
(b) From Lemma 2.3, we compute that

$$\begin{aligned} & \frac{\partial\phi^p_{\rm NR}}{\partial a}(a,b) \\ = \;& \frac{\partial}{\partial a}\left(a^{2k+1} - \frac{1}{2}((a-b)^{2k+1} + (a-b)^{2k}|a-b|\right) \\ = \;& (2k+1)a^{2k} - \frac{(2k+1)}{2}(a-b)^{2k} - \frac{(2k+1)}{2}(a-b)^{2k-1}|a-b| \end{aligned}$$

and

$$\begin{aligned} & \frac{\partial\phi^p_{\rm NR}}{\partial b}(a,b) \\ = \;& \frac{\partial}{\partial b}\left(a^{2k+1} - \frac{1}{2}((a-b)^{2k+1} + (a-b)^{2k}|a-b|\right) \\ = \;& \frac{(2k+1)}{2}(a-b)^{2k} + \frac{(2k+1)}{2}(a-b)^{2k-1}|a-b|. \end{aligned}$$

Hence, we obtain

$$
\begin{aligned}
& \nabla\phi^p_{\rm NR}(a,b) \\
= & \frac{2k+1}{2}\left[\begin{array}{c} 2a^{2k}-(a-b)^{2k}-(a-b)^{2k-1}|a-b| \\ (a-b)^{2k}+(a-b)^{2k-1}|a-b| \end{array}\right] \\
= & \frac{2k+1}{2}\left[\begin{array}{c} 2a^{2k}-2(a-b)^{2k-1}(a-b)_+ \\ 2(a-b)^{2k-1}(a-b)_+ \end{array}\right] \\
= & \ p\left[\begin{array}{c} a^{p-1}-(a-b)^{p-2}(a-b)_+ \\ (a-b)^{p-2}(a-b)_+ \end{array}\right]
\end{aligned}
$$

which proves part (b).
(c) Similarly, with Lemma 2.3 again, the Hessian matrix can be calculated as below.

$$
\begin{aligned}
& \nabla^2\phi^p_{\rm NR}(a,b) \\
= & \ k(2k+1) \\
& \left[\begin{array}{cc} 2a^{2k-1}-(a-b)^{2k-1}-(a-b)^{2k-2}|a-b| & (a-b)^{2k-1}+(a-b)^{2k-2}|a-b| \\ (a-b)^{2k-1}+(a-b)^{2k-2}|a-b| & -(a-b)^{2k-1}-(a-b)^{2k-2}|a-b| \end{array}\right] \\
= & \ p(p-1)\left[\begin{array}{cc} a^{p-2}-(a-b)^{p-3}(a-b)_+ & (a-b)^{p-3}(a-b)_+ \\ (a-b)^{p-3}(a-b)_+ & -(a-b)^{p-3}(a-b)_+ \end{array}\right]
\end{aligned}
$$

□

Finally, we present some other variants of $\phi^p_{\rm NR}$. Indeed, analogous to those functions in [24], the variants of $\phi^p_{\rm NR}$ as below can be verified being NCP-functions.

$$
\begin{aligned}
\varphi_1(a,b) &= \phi^p_{\rm NR}(a,b)+\alpha(a)_+(b)_+, \ \alpha>0. \\
\varphi_2(a,b) &= \phi^p_{\rm NR}(a,b)+\alpha\left((a)_+(b)_+\right)^2, \ \alpha>0. \\
\varphi_3(a,b) &= \left(\phi^p_{\rm NR}(a,b)\right)^2+\alpha\left((ab)_+\right)^4, \ \alpha>0. \\
\varphi_4(a,b) &= \left(\phi^p_{\rm NR}(a,b)\right)^2+\alpha\left((ab)_+\right)^2, \ \alpha>0.
\end{aligned}
$$

Lemma 2.5. *The value of $\phi^p_{\rm NR}(a,b)$ is positive only in the first quadrant, i.e., $\phi^p_{\rm NR}(a,b)>0$ if and only if $a>0$, $b>0$.*

Proof. We know that $f(t)=t^p$ is a strictly increasing function since p is odd. Using this fact yields

$$
\begin{aligned}
& a>0, \ b>0 \\
\Longleftrightarrow \ & a+b>|a-b| \\
\Longleftrightarrow \ & a>\frac{a-b+|a-b|}{2} \\
\Longleftrightarrow \ & a>(a-b)_+ \\
\Longleftrightarrow \ & a^p>(a-b)^p_+ \\
\Longleftrightarrow \ & \phi^p_{\rm NR}(a,b)>0,
\end{aligned}
$$

which is the desired result. □

Proposition 2.6. *All the above functions φ_i, $i\in\{1,2,3,4\}$ are NCP-functions.*

Proof. We will only show that φ_1 is an NCP-function and the same argument can be applied to the other cases. Let $\Omega := \{(a,b)\,|\, a > 0,\ b > 0\}$ and suppose $\varphi_1(a,b) = 0$. If $(a,b) \in \Omega$, then $\phi^p_{\rm NR}(a,b) > 0$ by Lemma 2.5; and hence, $\varphi_1(a,b) > 0$. This is a contradiction. Therefore, there must have $(a,b) \in \Omega^c$ which says $(a)_+(b)_+ = 0$. This further implies $\phi^p_{\rm NR}(a,b) = 0$ which is equivalent to $a, b \geq 0,\ ab = 0$. Then, one direction is proved. The converse direction is straightforward. □

3 Geometric View of $\phi^p_{\rm NR}$

In this section, we depict the surfaces of $\phi^p_{\rm NR}$ with various values of p so that we may have more insight for this new family of NCP-functions. Figure 1 is the surface if $\phi_{\rm NR}(a,b)$ from which we see that it is concave and increasing along the direction (t,t) in the first quadrant. Figure 2 presents the surface of $\phi^p_{\rm NR}(a,b)$ in which we see that it is neither convex nor concave. In addition, the value of $\phi^p_{\rm NR}(a,b)$ is positive only when $a > 0$ and $b > 0$ as mentioned in Lemma 2.5. The surfaces of $\phi^p_{\rm NR}$ with various values of p is shown in Figure 3.

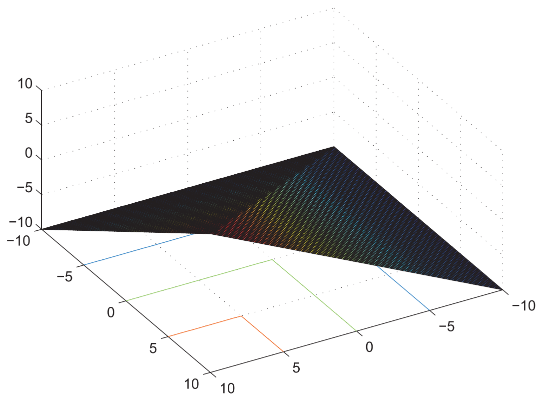


Figure 1: The surface of $z = \phi^p_{\rm NR}(a,b)$ with $p = 1$ and $(a,b) \in [-10,10] \times [-10,10]$

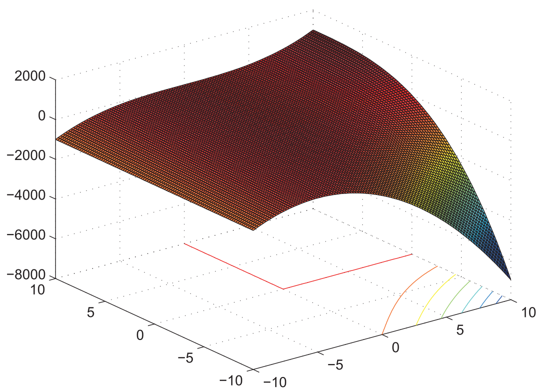
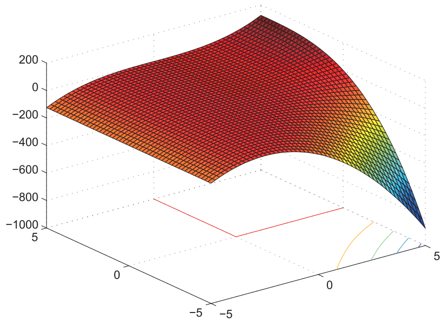
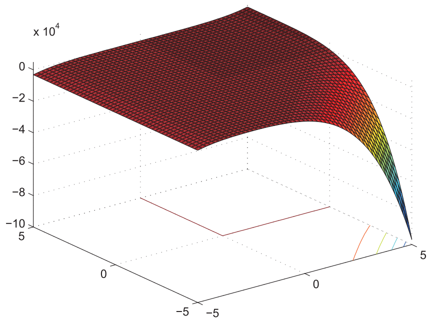


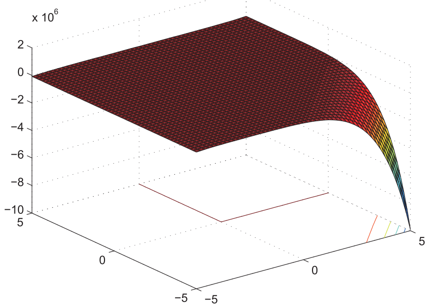
Figure 2: The surface of $z = \phi^p_{\rm NR}(a,b)$ with $p = 3$ and $(a,b) \in [-10,10] \times [-10,10]$



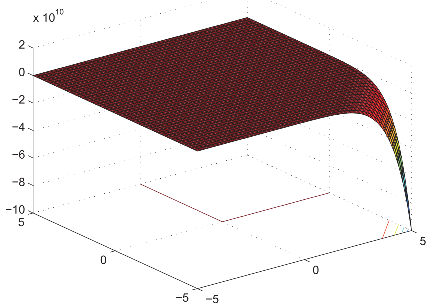
(a) $z = \phi^3_{\mathrm{NR}}(a,b)$



(b) $z = \phi^5_{\mathrm{NR}}(a,b)$



(c) $z = \phi^7_{\mathrm{NR}}(a,b)$



(d) $z = \phi^{11}_{\mathrm{NR}}(a,b)$

Figure 3: The surface of $z = \phi^p_{\mathrm{NR}}(a,b)$ with different values of p

To sum up, we propose a new family of new NCP-functions in this short paper. This answers a long-standing open question: what is the generalization of natural residual NCP-function? With this new discovery, many directions can be explored in the future, including numerical comparisons between ϕ^p_{FB} and ϕ^p_{NR} involved in various algorithms, studying the effect when perturbing the parameter p, applying this new family of NCP-functions to suitable optimization problems, and extending it as complementarity function associated with second-order cone and symmetric cone.

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