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WEAKLY SHARP SOLUTIONS OF PRIMAL AND DUAL VARIATIONAL INEQUALITY PROBLEMS

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Abstract: We discuss relations between the Gâteaux differentiability of two gap functions of a variational inequality problem and its dual problem and present sufficient conditions for their locally Lipschitz property as well. Based on these results, we present sufficient conditions for the weakly sharp solutions of the variational inequality problems.

Key words: *variational inequality, gap functions, Gâteaux differentiability, locally Lipschitz property, weakly sharp solution, error bound.*

Mathematics Subject Classification: *90C33, 49J52.*

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1 Introduction

For a nonempty closed convex subset C in $\mathbb{R}^n$ and a mapping F from $\mathbb{R}^n$ into $\mathbb{R}^n$, the *variational inequality problem* (VIP(C, F)) aims to find $x^* \in C$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0 \quad \text{for all } x \in C,$$

and the *dual variational inequality problem* (DVIP(C, F)) is to find a vector $x_* \in C$ such that

$$\langle F(x), x - x_* \rangle \geq 0 \quad \text{for all } x \in C.$$

We use C^* to denote the solution set of the VIP(C, F) and C_* that of the DVIP(C, F). Throughout the paper, we assume that both C^* and C_* are nonempty.

The variational inequality problem was studied by Hartman and Stampacchia in [7] as a tool for solving partial differential equations with applications of mechanics in infinite-dimensional spaces. It was applied in finite-dimensional spaces in [4] by Dafermos in 1980. Variational inequalities provide us with a tool for a wide variety of problems in mathematical programming, including systems of nonlinear equations, optimization problems and fixed point theorems. For more details, the reader can refer to [6, 9, 14].

In order to solve the variational inequality, much work has been done to reformulate it as an optimization problem by utilizing different gap functions. Readers can refer to [10] and [12] for various gap functions for variational inequalities. We often use two gap functions to formulate the VIP. One is the *primal gap function* $g(x)$ defined as

$$g(x) := \max\{\langle F(x), x - c \rangle : c \in C\} = \langle F(x), x - y \rangle,$$

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where y is a point in $\Gamma(x) := \{c \in C : \langle F(x), x - c\rangle = g(x)\}$ for $x \in \mathbb{R}^n$. The other is the *dual gap function* $G(x)$ given by

$$G(x) := \max\{\langle F(c), x - c\rangle : c \in C\} = \langle F(y), x - y\rangle,$$

where y is a point in $\Lambda(x) := \{c \in C : \langle F(c), x - c\rangle = G(x)\}$ for $x \in \mathbb{R}^n$.

Next we review some concepts relevant to C and F which will be used in the paper.

The *normal cone* to C at $x \in C$ is defined and denoted by

$$N_C(x) := \{\xi \in \mathbb{R}^n : \langle \xi, c - x\rangle \leq 0 \quad \text{for all } c \in C\}.$$

The *tangent cone* to C at $x \in C$ is defined from $N_C(x)$ by *polarity*, which is denoted as:

$$T_C(x) := [N_C(x)]^\circ := \{v \in \mathbb{R}^n : \langle s, v\rangle \leq 0 \quad \text{for all } s \in N_C(x)\}.$$

The tangent cone can also be expressed by $d_C^\circ(x; \cdot)$,

$$T_C(x) = \{v \in \mathbb{R}^n : d_C^\circ(x; v) = 0\},$$

see [2], where d_C is the *distance function* associated with C:

$$d_C(x) := \min\{\|x - c\| : c \in C\} \text{ for } x \in \mathbb{R}^n,$$

and

$$d_C^\circ(x; v) = \limsup_{\substack{y \to x \\ \lambda \downarrow 0}} \frac{d_C(y + \lambda v) - d_C(y)}{\lambda},$$

see [2]. Since C is convex, $v \in T_C(x)$ if and only if $d_C^\circ(x; v) = d'_C(x; v) = 0$, where $d'_C(x; \cdot)$ is the directional derivative of d_C at x [2, Proposition 2.2.7].

A mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *pseudomonotone* on C if for all $x, y \in C$,

$$\langle F(x), y - x\rangle \geq 0 \Rightarrow \langle F(y), y - x\rangle \geq 0.$$

The mapping F is said to be *pseudomonotone*$^+$ on C if it is pseudomonotone on C and, for all $x, y \in C$ there holds

$$\langle F(x), y - x\rangle \geq 0, \langle F(y), y - x\rangle = 0 \Rightarrow F(y) = F(x).$$

Based on these basic definitions, we introduce the notion of weakly sharp solutions of the VIP and its dual problem. Recall that Burke and Ferris have extended the concept of sharp minimum solution to the notion of a nonunique solution set, see [1]. They have proposed that $\overline{S}$ is weakly sharp to minimize the function $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ on S if there exists $\alpha > 0$ such that

$$f(x) \geq f(\overline{x}) + \alpha d_{\overline{S}}(x) \quad \text{for all } x \in S \text{ and } \overline{x} \in \overline{S}.$$

If f is differentiable and convex, S and $\overline{S}$ are assumed to be nonempty closed and convex, then they have proved that $\overline{S}$ is weakly sharp if and only if the gradient ∇f of f satisfies

$$-\nabla f(\overline{x}) \in \operatorname{int} \bigcap_{x \in \overline{S}} [T_S(x) \cap N_{\overline{S}}(x)]^\circ \quad \text{for all } \overline{x} \in \overline{S}.$$

Since the VIP doesn't have an objective function, Patriksson [13, pp. 108] extended the definition of weakly sharp solutions to VIP by replacing ∇f with F, that is, C^* is weakly sharp if and only if

$$-F(x^*) \in \operatorname{int} \bigcap_{x\in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ \quad \text{for each } x^* \in C^*. \tag{1.1}$$

Our aim in this paper is to characterize the weak sharpness of C^* and C_* by considering the error bound of $g + G$ on C. Recently, Marcotte and Zhu [11] have stated the weak sharpness of C^* in terms of error bound of G on C under the condition that F is continuous and pseudomonotone$^+$ on a compact set C. Then Zhang et al. [17] have extended their result just under the condition that F is continuous and pseudomonotone on C. In addition, Wu and Wu [15] have studied this same result under some new conditions of G, i.e., G is Gâteaux differentiable and locally Lipschitz on C^*. Similarly, Hu and Song have also analyzed this result by considering the Gâteaux differentiability of G in [8]. It is noted that most of these results have been obtained based on properties of G while g is seldom used. Moreover, there have been many papers discussing properties of G, e.g., [16] and [17]. However, for a fixed point $x \in \mathbb{R}^n$, $g(x)$ is usually easier to be calculated since this is a linear program. We will characterize the weak sharpness of C^* and C_* by considering some properties of g and $g+G$ instead of those of G in this paper.

We begin this work by stating relations between the Gâteaux differentiability of g and that of G on C^* and C_* in section 2. We then discuss the sufficiency for the Lipschitz property of G and study relations between the locally Lipschitz properties of g and $g + G$ in section 3. As a result, we get an understanding of the weak sharpness of C^* and C_* as well as the error bound of $g + G$ in section 4.

2 Gâteaux Differentiability of Two Gap Functions

In this section, we characterize the Gâteaux differentiability of the two gap functions g and G on C^* and C_* and discuss their relations under certain conditions. Wu and Wu [16] have proposed several sufficient conditions for the Gâteaux differentiability of G at $x_* \in C_*$. By [5, pp. 23, Proposition 5.3], if G is Gâteaux differentiable at $x^* \in C^*$, then $\partial G(x^*) = \{\nabla G(x^*)\}$, where $\nabla G(x^*)$ is the gradient of G at x^*. Our first purpose in this section is to discuss relations between the Gâteaux differentiabilities of g and G. It is noted that the inequality $g(x) \geq G(x)$ for all $x \in \mathbb{R}^n$ ensures the nonemptiness of the subdifferential of g at $x^* \in C^*$. In this case, we show that for $x^* \in C^*$ the Gâteaux differentiability of g at x^* implies that of G at x^*. Similarly, under some condition, the Gâteaux differentiability of G at some $x_* \in C_*$ is sufficient for that of g at x_*. Moreover, the Gâteaux differentiability of $g + G$ is also presented.

We begin with the following result which is useful for presenting the weak sharpness of C^* and C_* after.

Proposition 2.1. *Let $g(x) \geq G(x)$ for all $x \in \mathbb{R}^n$. Suppose that g is Gâteaux differentiable at $x^* \in C^*$. Then G is Gâteaux differentiable at x^*,*

$$\{\nabla g(x^*)\} = \partial g(x^*) = \partial G(x^*) = \{\nabla G(x^*)\} = \{F(x^*)\}.$$

Proof. Let $x^* \in C^*$. Then by [15, Proposition 2.1] we have $0 = g(x^*) \geq G(x^*)$. Since G is nonnegative on C, we obtain $G(x^*) = 0$ and hence $x^* \in C_*$. Therefore for all $x \in \mathbb{R}^n$ we have

$$g(x) - g(x^*) \geq G(x) - G(x^*) \geq \langle F(x^*), x - x^*\rangle,$$

which implies that $F(x^*) \in \partial g(x^*)$.

Let $\xi \in \partial g(x^*)$. Then for all $v \in \mathbb{R}^n$ and $t > 0$,

$$g(x^* + tv) - g(x^*) \geq t\langle \xi, v\rangle.$$

Since g is Gâteaux differentiable at x^*, $\langle \nabla g(x^*), v\rangle \geq \langle \xi, v\rangle$. This implies that $\xi = \nabla g(x^*)$. So $\{F(x^*)\} = \partial g(x^*) = \{\nabla g(x^*)\}$.

By assumption, we have

$$\begin{aligned}\langle F(x^*), v\rangle = \langle \nabla g(x^*), v\rangle &= \lim_{t\to 0} \frac{g(x^* + tv) - g(x^*)}{t} \\ &\geq \lim_{t\to 0} \frac{G(x^* + tv) - G(x^*)}{t} \geq \langle F(x^*), v\rangle,\end{aligned}$$

so

$$\lim_{t\to 0} \frac{G(x^* + tv) - G(x^*)}{t} = \langle F(x^*), v\rangle.$$

This implies that G is Gâteaux differentiable at x^* with $\nabla G(x^*) = F(x^*)$. Hence the proof is complete. □

We note that Proposition 2.1 may fail if the inequality $g(x) \geq G(x)$ holds only for $x \in C$ not for all $x \in \mathbb{R}^n$.

Example 2.2. Let $C = [0, 1]$ and

$$F(x) = \begin{cases} x & \text{for } x \in C; \\ -x & \text{for } x \notin C. \end{cases}$$

Then $C^* = \{0\}$,

$$g(x) = \begin{cases} -x^2 & \text{for } x < 0; \\ x^2 & \text{for } x \in C; \\ -x^2 + x & \text{for } x > 1, \end{cases}$$

and

$$G(x) = \begin{cases} 0 & \text{for } x < 0; \\ \frac{1}{4}x^2 & \text{for } 0 \leq x \leq 2; \\ x - 1 & \text{for } x > 2. \end{cases}$$

It is clear that $g(x) \geq G(x)$ holds for each $x \in C = [0, 1]$ but not for $x \in (-\infty, 0) \cup (1, +\infty)$. In this case, for $x^* \in C^*$, there exists no $\xi \in \mathbb{R}$ such that

$$\langle \xi, x - x^*\rangle \leq g(x) - g(x^*) \quad \text{for each } x \in \mathbb{R},$$

which implies that $\partial g(x^*)$ is empty. This shows that the condition $g(x) \geq G(x)$ for all $x \in C$ is not sufficient for $\partial g(x^*)$ to be nonempty.

Remark 2.3. For $x^* \in C^*$, $\Lambda(x^*)$ is the solution set to maximize $f(x) = \langle F(x), x^* - x\rangle$ subject to $x \in C$. Thus with the condition $g(x) \geq G(x)$ for all $x \in \mathbb{R}^n$ in Proposition 2.1, the solution set C^* can be obtained by $\Lambda(x^*)$ if g is Gâteaux differentiable at x^*, see [16, Theorem 2.3]. In this case, F is constant on $\Lambda(x^*)$ and $x^* \in C^* = \Gamma(x^*) \cap \Lambda(x^*)$. Furthermore, if each $x^* \in C^*$ and each $y^* \in \Gamma(x^*)$ satisfy

$$\{v \in \mathbb{R}^n : \langle F(x^*), v\rangle \geq 0\} = \{v \in \mathbb{R}^n : \langle F(y^*), v\rangle \geq 0\},$$

then $\Lambda(x^*) = C^* = C_* = \Gamma(x^*)$ from [15, Proposition 3.1] and hence F is constant on $\Gamma(x^*)$. So the solution sets to VIP and DVIP can be determined either by $\Lambda(x^*)$ or by $\Gamma(x^*)$.

Next we present an equivalent statement of the Gâteaux differentiability of g.

Proposition 2.4. *Let $x^* \in C^*$. Suppose that $g(x) \geq G(x)$ for all $x \in \mathbb{R}^n$. Then the following are equivalent:*

(i) *g is Gâteaux differentiable at x^*.*

(ii) *F is constant on $\Gamma(x^*) \cap \Lambda(x^*)$ and*
$g'(x^*; v) = \sup \{\langle F(x), v\rangle : x \in C^*\}$ for all $v \in \mathbb{R}^n$.

Proof. Since $(i) \Rightarrow (ii)$ is direct from Proposition 2.1 and Remark 2.3, it suffices to prove $(ii) \Rightarrow (i)$.

By assumption, we have $C^* \subseteq C_*$. Therefore, $C^* \subseteq \Lambda(x^*)$ and $C^* \subseteq \Gamma(x^*)$ by [15, Proposition 2.3]. This implies that $C^* \subseteq \Lambda(x^*) \cap \Gamma(x^*)$ and F is constant on C^* from (ii). By the expression of $g'(x^*; v)$ in (ii), g is Gâteaux differentiable at x^* with $\nabla g(x^*) = F(x^*)$. □

Similar to Proposition 2.1, the Gâteaux differentiability of G implies that of g under certain condition.

Proposition 2.5. *Let $g(x) \leq G(x)$ for all $x \in \mathbb{R}^n$. Suppose that G is Gâteaux differentiable at $x_* \in C_*$ and $\partial g(x_*) \neq \emptyset$. Then g is Gâteaux differentiable at x_*,*

$$\{\nabla g(x_*)\} = \partial g(x_*) = \partial G(x_*) = \{\nabla G(x_*)\} = \{F(x_*)\}$$

and F is constant on C^*.

Proof. Since $g(x) \leq G(x)$ for all $x \in \mathbb{R}^n$, by [15, Proposition 2.1], we have $C_* \subseteq C^*$. Applying [16, Theorem 2.3], the Gâteaux differentiability of G at x_* implies that $\partial G(x_*) = \{\nabla G(x_*)\} = \{F(x_*)\}$ and F is constant on C^*.

Let $\xi \in \partial g(x_*)$. Then for all $v \in \mathbb{R}^n$ and $t > 0$,

$$\langle \xi, tv\rangle \leq g(x_* + tv) - g(x_*) \leq G(x_* + tv) - G(x_*),$$

from which we obtain that

$$\langle \xi, v\rangle \leq \lim_{t\to 0} \frac{g(x_* + tv) - g(x_*)}{t} \leq G'(x_*; v) = \langle F(x_*), v\rangle.$$

This implies that $\xi = F(x_*)$. Thus g is Gâteaux differentiable at x_* and

$$\{\nabla g(x_*)\} = \partial g(x_*) = \partial G(x_*) = \{\nabla G(x_*)\} = \{F(x_*)\}.$$

□

Remark 2.6. Propositions 2.1 and 2.5 state the relationships between the Gâteaux differentiability of g and that of G on C^* and C_* and present sufficient conditions for F to be constant on C^*. Based on Proposition 2.1, the Gâteaux differentiability of g at $x^* \in C^*$ also implies that $F(c) = F(x^*)$ for all $c \in \Lambda(x^*)$.

Note that Proposition 2.1 implies that $g + G$ is Gâteaux differentiable at $x^* \in C^*$ and $\nabla(g+G)(x^*) = 2F(x^*)$. The following proposition presents weaker conditions for this result.

Proposition 2.7. *Let $g(x) \geq G(x)$ for all $x \in \mathbb{R}^n$. Suppose that $g + G$ is Gâteaux differentiable at $x^* \in C^*$. Then*

$$\partial(g + G)(x^*) = \{\nabla(g + G)(x^*)\} = \{2F(x^*)\}.$$

Proof. Let $x^* \in C^*$. Then, by assumption and definition, we have

$$g(x^*) = G(x^*) = 0,\ C^* \subseteq C_*,\ \text{and } g(x) \geq G(x) \geq \langle F(x^*), x - x^* \rangle \quad \text{for all } x \in \mathbb{R}^n,$$

from which we obtain that

$$(g+G)(x) - (g+G)(x^*) \geq \langle 2F(x^*), x - x^* \rangle \quad \text{for all } x \in \mathbb{R}^n.$$

This implies that $2F(x^*) \in \partial(g+G)(x^*)$.

Let $\xi \in \partial(g+G)(x^*)$. Then for any $v \in \mathbb{R}^n$ and $t > 0$ we have

$$(g+G)(x^* + tv) - (g+G)(x^*) \geq t\langle \xi, v \rangle.$$

If $g + G$ is Gâteaux differentiable at x^*, then

$$\langle \nabla(g+G)(x^*), v \rangle = \lim_{t \to 0} \frac{(g+G)(x^* + tv) - (g+G)(x^*)}{t} \geq \langle \xi, v \rangle.$$

This implies that $\xi = \nabla(g+G)(x^*)$ since v is arbitrary. Thus

$$\{2F(x^*)\} \subseteq \partial(g+G)(x^*) \subseteq \{\nabla(g+G)(x^*)\},$$

which implies $\partial(g+G)(x^*) = \{\nabla(g+G)(x^*)\} = \{2F(x^*)\}$. □

3 Locally Lipschitz Property of g and G

Like the Gâteaux differentiability of g and G, the locally Lipschitz property of these two gap functions are also very important for characterizing the weak sharpness of both C^* and C_*. In this section, we propose sufficient conditions for this property of g and G. Moreover, the relations between the locally Lipschitz properties of g and $g + G$ are also presented. The following proposition presents one sufficient condition for G to be Lipschitz.

Proposition 3.1. *Let F be bounded on C. Then G is Lipschitz.*

Proof. Let $y, z \in \mathbb{R}^n$. For $c \in \Lambda(y)$, we have

$$\begin{aligned} G(y) - G(z) &\leq \langle F(c), y - c \rangle - \langle F(c), z - c \rangle \\ &= \langle F(c), y - z \rangle \leq \|F(c)\| \|y - z\| \leq M\|y - z\|, \end{aligned}$$

where $M = \sup\{\|F(x)\| : x \in C\}$. This implies that G is Lipschitz. □

Since G is convex, its locally Lipschitz property can immediately be obtained by the following proposition.

Proposition 3.2 ([3, Corollary 2.35])**.** *If X is finite dimensional, then any convex function $f : X \to \mathbb{R}_\infty$ is locally Lipschitz in the set int dom$f$.*

So if G is bounded above on some set S and int$S$ is nonempty, then G is locally Lipschitz in int$S$. Based on this idea, we characterize the locally Lipschitz property of G and discuss relations between this property of g and that of $g + G$ since they are important for stating the weak sharpness of C^*.

Proposition 3.3. *Let $x^* \in C^*$. Suppose that there exists $\delta > 0$ such that $g(x) \geq G(x)$ for all $x \in B(x^*, \delta)$. Then the following hold:*

(i) If g is bounded in a neighbourhood of x^*, then G is Lipschitz near x^*.

(ii) $g+G$ is Lipschitz near x^* if and only if g is Lipschitz near x^*.

Proof. (i) Since g is bounded near x^*, there exist $0<\delta_1<\delta$ and $L\geq 0$ such that

$$\|g(x)\|\leq L \quad \text{for all } x\in B(x^*,\delta_1),$$

which implies that $\langle F(x^*),x-x^*\rangle\leq G(x)\leq L$ for all $x\in B(x^*,\delta_1)$. Then by Proposition 3.2, G is Lipschitz near x^*.

(ii) Since the sufficiency is direct from (i), it remains to show the necessity. If $g+G$ is Lipschitz near x^*, then there exist $0<\delta_1<\delta$ and $L\geq 0$ such that

$$2\langle F(x^*),x-x^*\rangle\leq 2G(x)\leq (g+G)(x)\leq L \quad \text{for all } x\in B(x^*,\delta_1).$$

So, by Proposition 3.2, G is Lipschitz near x^*. Hence $g=(g+G)-G$ is Lipschitz near x^*. □

4 Weak Sharpness of C^* and C_*

In [15, Theorem 5.1], Wu and Wu have proved that C^* is weakly sharp if and only if there exists $\mu>0$ such that $d_{C^*}(x)\leq \mu G(x)$ for each $x\in C$ under the conditions that G is Gâteaux differentiable and locally Lipschitz on C^* and for each $x^*\in C^*\cup C_*$ and each $y^*\in\Lambda(x^*)$ there hold

$$\{v\in\mathbb{R}^n:\langle F(x^*),v\rangle\geq 0\}=\{v\in\mathbb{R}^n:\langle F(y^*),v\rangle\geq 0\}$$

and

$$\langle F(x^*),x^*-y^*\rangle=0 \text{ and } \langle F(y^*),x^*-y^*\rangle=0\Rightarrow F(x^*)=F(y^*).$$

In this section, we use similar proofs to show the weak sharpness of C^* and C_* in terms of the error bound of $g+G$ on C.

Following the definition of the weak sharpness of C^* in (1.1), since

$$\operatorname{int}\bigcap_{x\in C^*}[T_C(x)\cap N_{C^*}(x)]^\circ\subseteq \operatorname{int}\bigcap_{x\in C^*\cap C_*}[T_C(x)\cap N_{C^*\cup C_*}(x)]^\circ,$$

we extend this definition as follows.

Definition 4.1. C^* is said to be *weakly sharp* provided that

$$-F(x^*)\in \operatorname{int}\bigcap_{x\in C^*\cap C_*}[T_C(x)\cap N_{C^*\cup C_*}(x)]^\circ \quad \text{for each } x^*\in C^*.$$

This is equivalent to saying that for each $x^*\in C^*$ there exists $\alpha>0$ such that

$$\alpha B\subseteq F(x^*)+\bigcap_{x\in C^*\cap C_*}[T_C(x)\cap N_{C^*\cup C_*}(x)]^\circ,$$

where B denotes the closed unit ball in $\mathbb{R}^n$. Similarly, C_* is said to be *weakly sharp* provided that

$$-F(x_*)\in \operatorname{int}\bigcap_{x\in C^*\cap C_*}[T_C(x)\cap N_{C^*\cup C_*}(x)]^\circ \quad \text{for each } x_*\in C_*.$$

The advantage of this extended definition is that the relationship between the weak sharpness of C^* and C_* can immediately be obtained as the following proposition states.

Proposition 4.2. *(i) Let $C^* \subseteq C_*$. If C_* is weakly sharp, then C^* is weakly sharp as well.*

(ii) Let $C_* \subseteq C^*$. If C^* is weakly sharp, then so is C_*.

Proof. The proof is straightforward, so it is omitted. □

Next we present our main results which characterize the weak sharpness of C^* and C_* in terms of the error bound of $g + G$ on C based on Definition 4.1.

Theorem 4.3. *Let F be constant on C^*. Suppose that $g(x) \geq G(x)$ for all $x \in \mathbb{R}^n$ and that $g + G$ is Gâteaux differentiable and locally Lipschitz on C^*. If there exists $\alpha > 0$ such that*

$$\alpha d_{C^* \cap C_*}(x) \leq (g + G)(x) \quad \textit{for each } x \in C,$$

then C^* is weakly sharp. In particular, if $C^* = C_*$, then the above sufficient condition is also necessary.

Proof. By assumption, we have $C^* \subseteq C_*$, so $C^* \cap C_* = C^*$ and $C^* \cup C_* = C_*$.

Suppose that there exists $\alpha > 0$ such that

$$\alpha d_{C^* \cap C_*}(x) \leq (g + G)(x) \quad \text{for each } x \in C.$$

Since F is constant on C^*, it suffices to show that there holds

$$\delta B \subseteq F(\overline{x}) + [T_C(\overline{x}) \cap N_{C^* \cup C_*}(\overline{x})]^\circ \quad \text{for each } \overline{x} \in C^* \text{ with } \delta = \frac{\alpha}{2}. \tag{4.1}$$

It is obvious that (4.1) holds if $T_C(\overline{x}) \cap N_{C^* \cup C_*}(\overline{x}) = \{0\}$ for $\overline{x} \in C^*$.

If $0 \neq v \in T_C(\overline{x}) \cap N_{C^* \cup C_*}(\overline{x})$ for $\overline{x} \in C^*$, then

$$\langle v, v \rangle > 0 \quad \text{and} \quad \langle v, \overline{y} - \overline{x} \rangle \leq 0 \quad \text{for each } \overline{y} \in C^* \cup C_*,$$

which implies that C^* is separated from $\overline{x} + v$ by the hyperplane

$$H_v = \{x \in \mathbb{R}^n : \langle v, x - \overline{x} \rangle = 0\}.$$

Since $v \in T_C(\overline{x})$, according to [2, Theorem 2.4.5], there exist a sequence $\{v_i\}$ converging to v and a positive sequence $\{t_i\}$ decreasing to 0 such that for each index i we have $\overline{x} + t_i v_i \in C$. Therefore,

$$d_{C^*}(\overline{x} + t_i v_i) \geq d_{H_v}(\overline{x} + t_i v_i) = t_i \frac{\langle v, v_i \rangle}{\|v\|}.$$

By assumption, we have

$$(g + G)(\overline{x} + t_i v_i) - (g + G)(\overline{x}) \geq \alpha d_{C^* \cap C_*}(\overline{x} + t_i v_i) = \alpha d_{C^*}(\overline{x} + t_i v_i).$$

Since $g + G$ is Gâteaux differentiable and locally Lipschitz on C^*, by Proposition 2.7, we have

$$\langle 2F(\overline{x}), v \rangle = \langle \nabla (g + G)(\overline{x}), v \rangle = \lim_{i \to \infty} \frac{(g + G)(\overline{x} + t_i v_i) - (g + G)(\overline{x})}{t_i} \geq \alpha \|v\|.$$

Let $w \in B$. Then

$$\left\langle \frac{\alpha}{2} w - F(\overline{x}), v \right\rangle = \frac{\alpha}{2} \langle w, v \rangle - \langle F(\overline{x}), v \rangle \leq \frac{\alpha}{2} \|v\| - \frac{\alpha}{2} \|v\| = 0.$$

Hence $\frac{\alpha}{2}B - F(\overline{x}) \subseteq [T_C(\overline{x}) \cap N_{C^*\cup C_*}(\overline{x})]^\circ$.

Next if C^* is weakly sharp and $C^* = C_*$, then by Definition 4.1 there exists $\delta > 0$ such that

$$\delta B \subseteq F(x^*) + \bigcap_{x \in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ \quad \text{for each } x^* \in C^*$$

since F is constant on C^*. From the proof of [11, Theorem 4.1], this is equivalent to saying that

$$\langle F(x^*), z\rangle \geq \delta\|z\| \quad \text{for each } z \in T_C(x^*) \cap N_{C^*}(x^*) \text{ and each } x^* \in C^*.$$

Since C_* is closed and convex and $C^* = C_*$, for each $x \in C$ there exists unique $c^* \in C^*$ such that $d_{C^*}(x) = \|x - c^*\|$. It follows that

$$x - c^* \in T_C(c^*) \cap N_{C^*}(c^*).$$

Hence the point c^* satisfies

$$(g + G)(x) \geq 2G(x) \geq 2\langle F(c^*), x - c^*\rangle \geq 2\delta\|x - c^*\| = 2\delta d_{C^*}(x).$$

Taking $\alpha = 2\delta$, we have

$$\alpha d_{C^*\cap C_*}(x) = \alpha d_{C^*}(x) \leq (g + G)(x) \quad \text{for each } x \in C.$$

The proof is complete. □

Remark 4.4. As mentioned above, Wu and Wu have characterized the weak sharpness of C^* in [15, Theorem 5.1] under the condition that G is Gâteaux differentiable and locally Lipschitz on C^*. By presenting relations between g and G in Theorem 4.3, the same result was proposed in terms of the error bound of $g + G$ on C. Under the conditions of Theorem 4.3, the existence of positive μ satisfying $d_{C^*\cap C_*}(x) \leq \mu G(x)$ for all $x \in C$ is also sufficient for the weak sharpness of C^* since $G(x) \leq g(x)$ for $x \in \mathbb{R}^n$. In this case, $(g + G)(x) \leq 2g(x)$ for all $x \in \mathbb{R}^n$. So if the condition that $g + G$ is Gâteaux differentiable and locally Lipschitz on C^* is replaced by a stronger one that g is Gâteaux differentiable and locally Lipschitz on C^*, by Propositions 2.1 and 3.3, we find that $d_{C^*\cap C_*}(x) \leq \mu g(x)$ for each $x \in C$ with some $\mu > 0$ is still a sufficient condition for the weak sharpness of C^*.

Similar proofs can be applied to the theorem below for discussing sufficient conditions for the weak sharpness of C_*.

Theorem 4.5. *Let $\partial g(x_*) \neq \emptyset$ for each $x_* \in C_*$. Suppose that $g(x) \leq G(x)$ for all $x \in \mathbb{R}^n$ and that G is Gâteaux differentiable and $g + G$ is locally Lipschitz on C_*. If there exists $\alpha > 0$ such that*

$$\alpha d_{C^*\cap C_*}(x) \leq (g + G)(x) \quad \textit{for each } x \in C,$$

then C_* is weakly sharp.

Proof. By assumption, we have $C_* \subseteq C^*$, that is, $C^* \cap C_* = C_*$.

Suppose that there exists $\alpha > 0$ such that

$$\alpha d_{C^*\cap C_*}(x) \leq (g + G)(x) \quad \text{for each } x \in C.$$

We claim that

$$\delta B \subseteq F(\overline{x}) + [T_C(\overline{x}) \cap N_{C^*\cup C_*}(\overline{x})]^\circ \quad \text{for each } \overline{x} \in C_* \text{ with } \delta = \frac{\alpha}{2}. \tag{4.2}$$

Obviously, (4.2) holds if $T_C(\overline{x}) \cap N_{C^*\cup C_*}(\overline{x}) = \{0\}$ for $\overline{x} \in C_*$.

If $0 \neq v \in T_C(\overline{x}) \cap N_{C^*\cup C_*}(\overline{x})$ for $\overline{x} \in C_*$, then

$$\langle v, v\rangle > 0 \text{ and } \langle v, \overline{y} - \overline{x}\rangle \leq 0 \quad \text{for all } \overline{y} \in C^* \cup C_* = C^*.$$

Therefore, C_* is separated from $\overline{x} + v$ by the hyperplane

$$H_v = \{x \in \mathbb{R}^n : \langle v, x - \overline{x}\rangle = 0\}.$$

Since $v \in T_C(\overline{x})$, there exist a sequence $\{v_i\}$ converging to v and a positive sequence $\{t_i\}$ decreasing to 0 such that for each index i there holds $\overline{x} + t_i v_i \in C$. Hence we have

$$(g+G)(\overline{x} + t_i v_i) - (g+G)(\overline{x}) \geq \alpha d_{C_*}(\overline{x} + t_i v_i) \geq \alpha d_{H_v}(\overline{x} + t_i v_i) = \alpha t_i \frac{\langle v, v_i\rangle}{\|v\|}.$$

By Proposition 2.5, the Gâteaux differentiability of G on C_* implies that g is Gâteaux differentiable on C_* with $\nabla g(x_*) = \nabla G(x_*) = F(x_*)$ for each $x_* \in C_*$ and F is constant on C_*, that is, we have

$$\nabla(g+G)(x_*) = 2F(x_*) \quad \text{for each } x_* \in C_*.$$

If $g + G$ is locally Lipschitz on C_*, then

$$\langle 2F(\overline{x}), v\rangle = \langle \nabla(g+G)(\overline{x}), v\rangle = \lim_{i\to\infty} \frac{(g+G)(\overline{x} + t_i v_i) - (g+G)(\overline{x})}{t_i} \geq \alpha\|v\|.$$

Let $u \in B$. Then

$$\left\langle \frac{\alpha}{2}u - F(\overline{x}), v \right\rangle = \frac{\alpha}{2}\langle u, v\rangle - \langle F(\overline{x}), v\rangle \leq \frac{\alpha}{2}\|v\| - \frac{\alpha}{2}\|v\| = 0.$$

Thus $\frac{\alpha}{2}B - F(\overline{x}) \subseteq [T_C(\overline{x}) \cap N_{C^*\cap C_*}(\overline{x})]^\circ$. This implies that (4.2) holds. And hence C_* is weakly sharp since F is constant on C_*. □

Theorem 4.5 characterizes the weak sharpness of C_* under the assumption that $g(x) \leq G(x)$ for all $x \in \mathbb{R}^n$. In [15, Theorem 5.4], Wu and Wu have stated two equivalent statements for the weak sharpness of C^*. Motivated by their results, we note that

$$-F(x^*) \in \operatorname{int} \bigcap_{x \in C^*\cap C_*} T_{C^*}(x)$$

is also sufficient for the weak sharpness of C^* since

$$\operatorname{int} \bigcap_{x\in C^*\cap C_*} T_{C^*}(x) = \operatorname{int} \bigcap_{x\in C^*\cap C_*} [N_{C^*}(x)]^\circ \subseteq \operatorname{int} \bigcap_{x\in C^*\cap C_*} [T_{C^*}(x) \cap N_{C^*\cup C_*}(x)]^\circ.$$

Then we use similar proof of [15, Theorem 5.4] to present this equivalence with the error bounds of G and $g + G$ on $\mathbb{R}^n$.

Theorem 4.6. *Let C^* be closed and convex and F constant on C^*. Suppose that $g(x) \geq G(x)$ for all $x \in \mathbb{R}^n$ and that $(g+G)(x)$ is Gâteaux differentiable on C^*. Then the following are equivalent:*

(i) $-F(x^*) \in$ *int* $\bigcap_{x\in C^*\cap C_*} T_{C^*}(x)$ *for each* $x^* \in C^*$.

(*ii*) *There exists* $\alpha > 0$ *such that* $\alpha d_{C^* \cap C_*}(x) \leq G(x)$ *for each* $x \in \mathbb{R}^n$.

(*iii*) *There exists* $\alpha > 0$ *such that*

$$\alpha d_{C^* \cap C_*}(x) \leq (g+G)(x) \quad \textit{for each } x \in \mathbb{R}^n. \tag{4.3}$$

Proof. By assumption, we have $0 \leq G(x^*) \leq g(x^*) = 0$ for all $x^* \in C^*$, so $C^* \subseteq C_*$ and $C^* \cap C_* = C^*$.

$(i) \Rightarrow (ii)$: Let (i) hold. Then since F is assumed to be constant on C^*, there exists $\alpha > 0$ such that

$$\alpha B \subseteq F(x^*) + T_{C^*}(x^*) = F(x^*) + [N_{C^*}(x^*)]^\circ \quad \text{for each } x^* \in C^*.$$

This implies that for each $x^* \in C^*$ and each $u \in B$ we have

$$\langle \alpha u - F(x^*), v \rangle \leq 0 \quad \text{for each } v \in N_{C^*}(x^*).$$

Let $u = \frac{v}{\|v\|}$ for $v \neq 0$. Then

$$\langle F(x^*), v \rangle \geq \alpha \|v\| \quad \text{for each } v \in N_{C^*}(x^*).$$

Since C^* is closed and convex, for each $x \in \mathbb{R}^n$ there exists a unique $\overline{x} \in C^*$ such that

$$d_{C^* \cap C_*}(x) = d_{C^*}(x) = \|x - \overline{x}\|,$$

which yields that $x - \overline{x} \in N_{C^*}(\overline{x})$. Therefore,

$$G(x) \geq \langle F(\overline{x}), x - \overline{x} \rangle \geq \alpha \|x - \overline{x}\| = \alpha d_{C^* \cap C_*}(x).$$

$(ii) \Rightarrow (iii)$ is immediate from the inequality $G(x) \leq g(x)$ for all $x \in \mathbb{R}^n$. It remains to prove $(iii) \Rightarrow (i)$.

Suppose that (4.3) holds for some $\alpha > 0$. We claim that

$$\delta B \subseteq F(x^*) + T_{C^*}(x^*) = F(x^*) + [N_{C^*}(x^*)]^\circ \tag{4.4}$$

for each $x^* \in C^*$ with $\delta = \frac{\alpha}{2}$.

It is clear that (4.4) holds for $x^* \in C^*$ if $N_{C^*}(x^*) = \{0\}$. It remains to prove that (4.4) holds for $x^* \in C^*$ with $N_{C^*}(x^*) \neq \{0\}$.

Let $0 \neq v \in N_{C^*}(x^*)$. Then

$$\langle v, v \rangle > 0 \text{ and } \langle v, y^* - x^* \rangle \leq 0 \quad \text{for each } y^* \in C^*.$$

Thus C^* is separated from $x^* + v$ by the hyperplane

$$H_v = \{x \in \mathbb{R}^n : \langle v, x - x^* \rangle = 0\}.$$

Therefore for each positive sequence $\{t_i\}$ decreasing to 0, $x^* + t_i v$ lies in the open set $\{x \in \mathbb{R}^n : \langle v, x - x^* \rangle > 0\}$. Hence

$$d_{C^* \cap C_*}(x^* + t_i v) = d_{C^*}(x^* + t_i v) \geq d_{H_v}(x^* + t_i v) = t_i \|v\|.$$

From (4.3) we have

$$(g+G)(x^* + t_i v) - (g+G)(x^*) \geq \alpha d_{C^* \cap C_*}(x^* + t_i v) \geq \alpha t_i \|v\|.$$

Since $(g+G)(x)$ is Gâteaux differentiable on C^*, by Proposition 2.7,

$$\langle 2F(x^*), v\rangle = \lim_{i\to\infty} \frac{(g+G)(x^*+t_i v)-(g+G)(x^*)}{t_i} \geq \alpha\|v\|.$$

Therefore for each $u \in B$ we have

$$\left\langle \frac{1}{2}\alpha u - F(x^*), v \right\rangle = \frac{1}{2}\alpha\langle u, v\rangle - \langle F(x^*), v\rangle \leq \frac{\alpha}{2}\|v\| - \frac{\alpha}{2}\|v\| = 0,$$

from which we obtain that $\delta B - F(x^*) \subseteq T_{C^*}(x^*)$. This implies that (4.4) holds since F is constant on C^*. The proof is complete. □

In Theorem 4.6, we present the equivalence of three sufficient conditions for the weak sharpness of C^*. We note that the equivalence $(i) \Leftrightarrow (ii)$ in this theorem has been proved by Wu and Wu in [15] in terms of some restrictions of the relevant mapping F and the Gâteaux differentiability of G. By considering the Gâteaux differentiability of $g+G$ instead, we state that $(i)-(iii)$ are equivalent under the assumption that $g(x) \geq G(x)$ for all $x \in \mathbb{R}^n$. Then we conclude this paper by a finite convergence theorem for solving VIP under the condition that either C^* is weakly sharp or $g+G$ has an error bound on C.

Theorem 4.7. *Let $\{x_k\}$ be a sequence in C such that $d_{C^*}(x_k)$ converges to 0 and let F be constant on C^* and uniformly continuous in an open set containing $\{x_k\}$ and C_*. Suppose that $g(x) \geq G(x)$ for all $x \in \mathbb{R}^n$ and that $g+G$ is Gâteaux differentiable and locally Lipschitz on C^*. If*

- (i) *C^* is weakly sharp, or*
- (ii) *there exists $\alpha > 0$ such that $\alpha d_{C^*\cap C_*}(x) \leq (g+G)(x)$ for each $x \in C$,*

then $\arg\min\{\langle F(x_k), x\rangle : x \in C\} \subseteq C^*$ *for sufficiently large k.*

Proof. Let (i) hold. Then there exists $\alpha > 0$ such that

$$-F(x^*) + \alpha B \subseteq \bigcap_{x\in C^*\cap C_*} [T_C(x) \cap N_{C^*\cup C_*}(x)]^\circ \quad \text{for each } x^* \in C^*$$

since F is constant on C^*. Under the given conditions, we have $C^* = C_*$ and since C_* is closed and convex, for each x_k there exists a unique $x_k^* \in C^*$ such that $d_{C^*}(x_k) = \|x_k - x_k^*\|$. Therefore the uniformly continuity of F in an open set containing $\{x_k\}$ and C^* implies that

$$\|F(x_k) - F(x^*)\| = \|F(x_k) - F(x_k^*)\| < \alpha \text{ for sufficiently large } k.$$

Thus $-F(x^*) + F(x^*) - F(x_k) \subseteq \operatorname{int} \bigcap_{x\in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ$, that is,

$$-F(x_k) \in \operatorname{int} \bigcap_{x\in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ.$$

By [15, Theorem 3.2], $\arg\min\{\langle F(x_k), x\rangle : x \in C\} \subseteq C^*$ for sufficiently large k.

Now suppose that (ii) holds. Then, by Theorem 4.3, the weak sharpness of C^* can be proved. Hence we get the desired result. □

Remark 4.8. Under the conditions of Theorem 4.7, (i), (ii) or (iii) in Theorem 4.6 implies that both (i) and (ii) in Theorem 4.7 hold. Hence under the same conditions of Theorem 4.7, (i), (ii) and (iii) in Theorem 4.6 are all sufficient for the finite convergence algorithm presented in Theorem 4.7.

5 Conclusion

In this paper, weakly sharp solutions of variational inequalities in terms of primal and dual gap functions are studied.

We discuss relations between the Gâteaux differentiabilities of g and G on C^* and C_*, respectively. We also present the sufficiency for locally Lipschitz property of $g + G$ on C^* under the assumption that $g(x) \geq G(x)$, where x is near $x^* \in C^*$. Under conditions of the constancy of F on C^* and Gâteaux differentiability and locally Lipschitz property of $g + G$ on C^*, if $g(x) \geq G(x)$ for all $x \in \mathbb{R}^n$, then the existence of the error bound of $g + G$ implies the weak sharpness of C^*. In this case, the error bound of G on C is also sufficient for this weakly sharp result, as discussed in [15].

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