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SUMS OF SQUARES CHARACTERIZATIONS OF CONTAINMENT OF CONVEX SEMIALGEBRAIC SETS

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Abstract: In this paper, we establish numerically checkable sums of squares characterizations of containment of a convex semialgebraic set in another reverse convex semialgebraic set, described by SOS-convex polynomials. The significance of these characterizations is that they hold without any qualifications. In particular, when the semialgebraic sets are described by convex quadratic functions, we obtain a simple linear matrix inequality characterization for the containment. We also present robust set containment characterizations for convex semialgebraic sets in the face of data uncertainty of the SOS-convex polynomials that define the convex semialgebraic sets.

Key words: *set containment, SOS-convex polynomials, sums of squares polynomials, convex semialgebraic sets.*

Mathematics Subject Classification: *14P10, 90C25, 90C46.*

1 Introduction

The problem of characterizing containment of a polyhedral set or a convex set, defined by linear systems, in a closed half-space is fundamental to solving linear programming or convex optimization problems and has been extensively studied in the literature [9, 10, 28]. Solutions to this problem have generated various forms of Farkas' lemma [11] that have led to the development of duality theory and methods of convex programming [6, 12, 13, 18]. Containment problems of convex sets are also the classical problems in convex geometry and they arise in many different applications, e.g. shape fitting and packing problems, clustering and pattern recognition (see [8] for a survey and [20, 21] for current studies).

Recently, motivated by knowledge-based classification problems of data mining [3, 24], computationally tractable characterizations of the containment of a polyhedral set in another or in a reverse-convex set, defined by convex quadratic functions, were given in [23]. Since then various set containment characterizations have been given (see [6, 7, 12, 26, 27]). More recently, the computational problem of whether a given polytope or spectrahedron (which defines the feasible region of a semidefinite program) is contained in another one was examined in [20, 21]. In this paper we study characterizations of containment of a convex semialgebraic set, described by SOS-convex polynomials, in a reverse convex semialgebraic

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set. We also examine robust containment [13] of a convex semialgebraic set in a reverse convex semialgebraic set in the face of data uncertainty of the SOS-convex polynomials that define the convex semialgebraic set.

Our study was motivated by the desire to obtain numerically checkable sums of squares set containment characterizations involving semialgebraic sets, described by SOS-convex polynomial inequalities [1, 16, 22]. We make use of a hyperplane separation theorem and exploit algebraic properties of SOS-convexity to derive such sums of squares characterizations. For recent work on SOS-convex polynomials and convex optimization, see [16, 17].

Our contributions to convex analysis and geometry [9] are outlined below.

(i) In Section 2, using conjugate convex analysis, we present numerically checkable sum of squares characterizations of containment of a convex semialgebraic set in a reverse convex semialgebraic set. The significance of these characterizations is that they hold without any qualifications. In particular, when the semialgebraic sets are described by convex quadratic functions, we obtain a simple semidefinite characterization for the containment. Importantly, the sum of squares conditions characterizing the set containment can be numerically checked via semidefinite programming because whether a polynomial is a sum of squares of polynomials or not can be verified by solving related semidefinite programs [22].

(ii) In Section 3, we also extend some of our set containment characterizations to robust set containment characterizations where a given convex semialgebraic set is affected by data uncertainty of the SOS-convex polynomials. In this case, we present characterizations for containment of the robust counterpart [4] of the uncertain semialgebraic set in a reverse convex semialgebraic set for various classes of commonly used data uncertainty. Such robust set containment characterizations for uncertain polyhedral sets were given in [13] and their applications to data classification problems were described in [19].

2 Semialgebraic Set Containment Characterizations

In this section, we derive set containment characterizations of semialgebraic sets as a consequence of results of the preceding section. It should be noted that set containment characterizations for general (not necessarily semialgebraic) convex sets and reverse-convex sets are known (see [6, 7, 12, 26, 27]). However, those characterizations are often hard to check numerically. The sum of squares characterizations of the set containment, presented in this Section, can be checked by solving semidefinite linear programs.

We begin by fixing notation and preliminaries of convex sets, functions and polynomials. Throughout this paper, $\mathbb{R}^n$ denotes the Euclidean space with dimension n. The inner product in $\mathbb{R}^n$ is defined by $\langle x, y\rangle := x^T y$ for all $x, y \in \mathbb{R}^n$. The nonnegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$ and is defined by $\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_i \geqq 0\}$. We say A is convex whenever $\mu a_1 + (1-\mu)a_2 \in A$ for all $\mu \in [0, 1]$, $a_1, a_2 \in A$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be convex if for all $\mu \in [0, 1]$,

$$f((1-\mu)x + \mu y) \leqq (1-\mu)f(x) + \mu f(y)$$

for all $x, y \in \mathbb{R}^n$. The positive semi-definiteness of an $n \times n$ matrix B, denoted by $B \succeq 0$, is defined by $\langle x, Bx\rangle \geqq 0$, for each $x \in \mathbb{R}^n$.

A semialgebraic subset of $\mathbb{R}^n$ is a set, satisfying a Boolean combination of polynomial equations and inequalities with real coefficients [5]. We say that a real polynomial f is sum of squares if there exist real polynomials f_j, $j = 1, \ldots, r$, such that $f = \sum_{j=1}^{r} f_j^2$. The set consisting of all sum of squares real polynomials is denoted by Σ^2. Moreover, the set consisting of all sum of squares real polynomials with degree at most d is denoted by Σ_d^2.

The following useful result of convex polynomial systems was given in [2] and will play an important role later in the paper.

Lemma 2.1 ([2])**.** *Let $f_0, f_1, \ldots, f_m$ be convex polynomials on $\mathbb{R}^n$. Let $C := \{x \in \mathbb{R}^n : f_i(x) \leqq 0,\ i = 1, \ldots, m\}$. Suppose that $\inf_{x \in C} f_0(x) > -\infty$. Then, $\arg\min_{x \in C} f_0(x) \neq \emptyset$.*

We now recall the notion of SOS-convexity for polynomials.

Definition 2.2 (SOS-convexity)**.** A real polynomial f on $\mathbb{R}^n$ is SOS-convex whenever, for all $x, y \in \mathbb{R}^n$, and for all $\lambda \in [0, 1]$,

$$\lambda f(x) + (1 - \lambda) f(y) - f(\lambda x + (1 - \lambda) y)$$

is a *sum of squares polynomial in* $\mathbb{R}[x; y]$. Equivalently, f is SOS-convex whenever, for all $x, y \in \mathbb{R}^n$,

$$f(y) - f(x) - \langle \nabla f(x), y - x \rangle$$

is a *sum of squares polynomial in* $\mathbb{R}[x; y]$.

The notion was introduced in [1] and further studied recently in [14, 16, 17]. Note that a SOS-convex polynomial is convex and that convex quadratic and separable convex polynomials are SOS-convex polynomials. However, the converse is not true. Thus, there exists a convex polynomial which is not SOS-convex [1]. The degree of a polynomial g is denoted by $\deg g$.

Theorem 2.3 (Containment of a convex set in a reverse convex set)**.** *Let f_j, $j = 1, \ldots, p$, be SOS-convex polynomials and let g_i, $i = 1, \ldots, m$, be SOS-convex polynomials. Assume that $K := \{x \in \mathbb{R}^n : g_i(x) \leqq 0, i = 1, \ldots, m\} \neq \emptyset$. Let $I = \{1, 2, \ldots, m\}$ and let $J = \{1, 2, \ldots, p\}$. Let $d = \max\{\deg f_1, \ldots, \deg f_p, \deg g_1, \ldots, \deg g_m\}$. Then the following statements are equivalent:*

(i) $\{x \in \mathbb{R}^n : g_i(x) \leqq 0, i \in I\} \subset \{x \in \mathbb{R}^n : f_j(x) > 0, j \in J\}$*;*

(ii) $(\forall j \in J)(\exists (\lambda^j, \delta_j) \in \mathbb{R}_+^m \times \mathbb{R}_+ \backslash \{0\})\quad f_j + \sum_{i=1}^{m} \lambda_i^j g_i - \delta_j \in \Sigma_d^2$.

Proof. [(ii) $\Rightarrow$ (i)] Suppose that for each $j \in J$, there exist $\delta_j > 0$, $\lambda_i^j \geqq 0$, $i \in I$, and $\sigma_j \in \Sigma_d^2$ such that

$$f_j + \sum_{i=1}^{m} \lambda_i^j g_i - \delta_j = \sigma_j.$$

So, if $g_i(x) \leqq 0$, $i \in I$, then for any $x \in \mathbb{R}^n$,

$$f_j(x) = -\sum_{i=1}^{m} \lambda_i^j g_i(x) + \delta_j + \sigma_j(x) > 0.$$

Thus (i) holds.

[(i) $\Rightarrow$ (ii)] Assume that (i) holds. Let $j \in J$ be fixed. Then $\inf_{x \in K} f_j(x) \geqq 0$. Thus by Lemma 2.1, there exists $x^* \in K$ such that $f_j(x^*) = \inf_{x\in K} f_j(x)$. Let $\beta^j = f_j(x^*) = \inf_{x\in K} f_j(x)$. Then it follows from (i) that $\beta^j > 0$. Consider the following set

$$C_j = \{(y_j, z) \in \mathbb{R} \times \mathbb{R}^m \mid \exists x \in \mathbb{R}^m \ s.t.\ f_j(x) \leqq y_j,\ g_i(x) \leqq z_i,\ i = 1, \ldots, m\}.$$

Then we can check that C_j is a closed and convex subset of $\mathbb{R}^{m+1}$. Since $\beta^j = \inf_{x\in K} f_j(x) > 0$, $0 \neq (\frac{\beta^j}{2}, 0, \ldots, 0) \notin C_j$. By the strong separation theorem [28, Theorem 1.1.3], there exist $(\mu_j, v_1, \ldots, v_m) \neq 0$, $\alpha \in \mathbb{R}$, $\delta_0 > 0$ such that for all $(y_j, z) \in C_j$, $j = 1, \ldots, p$,

$$\Big\langle \Big(\frac{\beta^j}{2}, 0, \ldots, 0\Big), (\mu_j, v_1, \ldots, v_m) \Big\rangle \leqq \alpha < \alpha + \delta_0 \leqq \langle (y_j, z_1, \ldots, z_m), (\mu_j, v_1, \ldots, v_m) \rangle.$$

Since $C_j + \mathbb{R}^{m+1} \subset C_j$, then $\mu_j \geqq 0$, $j = 1, \ldots, p$, $v_i \geqq 0$, $i = 1, \ldots, m$ and

$$\frac{\beta^j}{2}\mu_j \leqq \alpha < \alpha + \delta_0 \leqq \mu_j y_j + \sum_{i=1}^{m} v_i z_i. \tag{2.1}$$

Since $K \neq \emptyset$, there exists $\hat{x} \in \mathbb{R}^n$ such that $g_i(\hat{x}) \leqq 0$, $i \in I$, and so, if $\mu_j = 0$, it follows from (2.1) that $0 \leqq \alpha < 0$. This is a contradiction. So, $\mu_j > 0$, $j \in J$. Let $\lambda_i^j = \frac{v_i}{\mu_j}$, $i \in I$. Since $(f_j(x), g_1(x), \ldots, g_m(x)) \in C_j$, for any $x \in \mathbb{R}^n$, from (2.1), $f(x) + \sum_{i=1}^m \lambda_i g_i(x) - \frac{\beta^j}{2} > 0$. Since $f_j + \sum_{i=1}^m \lambda_i^j g_i - \frac{\beta^j}{2}$ is bounded bellow and SOS-convex, it follows from Lemma 2.1 that there exists $\bar{x} \in \mathbb{R}^n$ such that

$$f_j(\bar{x}) + \sum_{i=1}^{m} \lambda_i^j g_i(\bar{x}) - \frac{\beta^j}{2} = \inf_{x\in\mathbb{R}^n} \Big\{ f_j(x) + \sum_{i=1}^{m} \lambda_i^j g_i(x) - \frac{\beta^j}{2} \Big\}.$$

Let $h^j(x) = f_j(x) + \sum_{i=1}^m \lambda_i^j g_i(x) - (f(\bar{x}) + \sum_{i=1}^m \lambda_i^j g_i(\bar{x})), j \in J$. Thus h^j is a SOS-convex polynomial, $h_j(\bar{x}) = 0$ and $\nabla h^j(\bar{x}) = 0$. It then follows by the definition of SOS-convexity that h^j is a sum of squares polynomial. For each $x \in \mathbb{R}^n$,

$$f_j(x) + \sum_{i=1}^{m} \lambda_i^j g_i(x) - \frac{\beta^j}{2} = h^j(x) + f_j(\bar{x}) + \sum_{i=1}^{m} \lambda_i^j g_i(\bar{x}) - \frac{\beta^j}{2}.$$

Let $\delta_j = \frac{\beta^j}{2}$ and $\sigma_j(x) = h^j(x) + f_j(\bar{x}) + \sum_{i=1}^m \lambda_i^j g_i(\bar{x}) - \delta_j$. Then for each $x \in \mathbb{R}^n$, $f_j(x) + \sum_{i=1}^m \lambda_i^j g_i(x) - \delta_j = \sigma_j(x)$. Since h^j is a sum of squares polynomial and $f_j(\bar{x}) + \sum_{i=1}^m \lambda_i^j g_i(\bar{x}) - \delta_j > 0$, $\sigma_j(x)$ is a sum of squares polynomial with degree at most d. Thus (ii) holds. □

Now we give an example illustrating Theorem 2.3.

Example 2.4. Consider the following convex semialgebraic sets $A := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 - 1 \leqq 0\}$ and $B := \{(x, y) \in \mathbb{R}^2 \mid -x-y+2 > 0, x-y+2 > 0, x+y+2 > 0, -x+y+2 > 0\}$. Clearly, we see from the diagram below that $A \subset B$. On the other hand, the following sum of squares conditions also hold.

$$\begin{aligned}
-x - y + 2 + 1 \cdot (x^2 + y^2 - 1) - \tfrac{1}{2} &= (x - \tfrac{1}{2})^2 + (y - \tfrac{1}{2})^2 \in \Sigma_2^2,\\
x - y + 2 + 1 \cdot (x^2 + y^2 - 1) - \tfrac{1}{2} &= (x + \tfrac{1}{2})^2 + (y - \tfrac{1}{2})^2 \in \Sigma_2^2,\\
x + y + 2 + 1 \cdot (x^2 + y^2 - 1) - \tfrac{1}{2} &= (x + \tfrac{1}{2})^2 + (y + \tfrac{1}{2})^2 \in \Sigma_2^2,\\
-x + y + 2 + 1 \cdot (x^2 + y^2 - 1) - \tfrac{1}{2} &= (x - \tfrac{1}{2})^2 + (y + \tfrac{1}{2})^2 \in \Sigma_2^2.
\end{aligned}$$

Hence, Theorem 2.3 is verified.

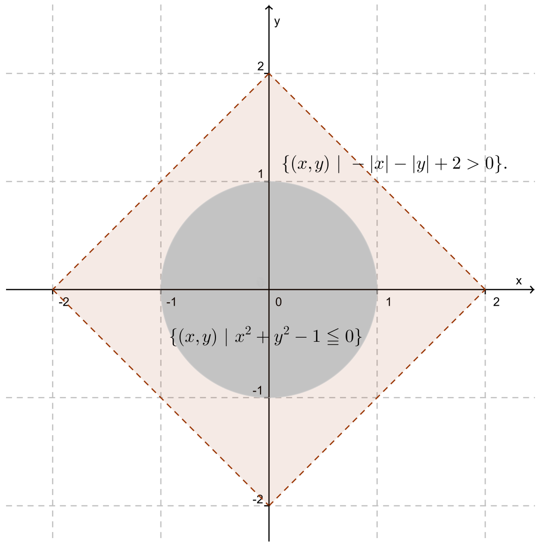

Figure 2.1: Containment of the convex semialgebraic set A in another convex semialgebraic set B.

Using Theorem 2.3, we can obtain the following generalized Farkas' lemma which holds without any qualifications.

Theorem 2.5 (Containment of a convex set in a closed reverse convex set I)**.** *Let f_j, $j = 1, \ldots, p$, be SOS-convex polynomials and let g_i, $i = 1, \ldots, m$, be SOS-convex polynomials. Assume that $K := \{x \in \mathbb{R}^n : g_i(x) \leqq 0, i = 1, \ldots, m\} \neq \emptyset$. Let $I = \{1, 2, \ldots, m\}$ and let $J = \{1, 2, \ldots, p\}$. Let $d = \max\{\deg f_1, \ldots, \deg f_p, \deg g_1, \ldots, \deg g_m\}$. Then following statements are equivalent:*

(i) $\{x \in \mathbb{R}^n : g_i(x) \leqq 0, i \in I\} \subset \{x \in \mathbb{R}^n : f_j(x) \geqq 0, j \in J\}$;

(ii) $(\forall j \in J)(\forall \epsilon_j \in \mathbb{R}_+ \backslash \{0\})(\exists \lambda^j \in \mathbb{R}^m_+)\ \ f_j + \sum_{i=1}^m \lambda_i^j g_i + \epsilon_j \in \Sigma^2_d$.

Proof. [(ii) ⇒ (i)] Suppose that for each $j \in J$, for any $\epsilon_j > 0$, there exist $\lambda_i^j \geqq 0$, $i \in I$ such that

$$f_j + \sum_{i=1}^m \lambda_i^j g_i + \epsilon_j \in \Sigma^2_d.$$

So, if $g_i(x) \leqq 0$, $i \in I$, then for any $x \in \mathbb{R}^n$ and for any $\epsilon_j > 0$,

$$0 \leqq f_j(x) + \sum_{i=1}^m \lambda_i^j g_i(x) + \epsilon_j \leqq f_j(x) + \epsilon_j.$$

Letting $\epsilon_j \to 0$, then we have $f_j(x) \geqq 0$. Thus (i) holds.
[(i) ⇒ (ii)] Assume that (i) holds. Then for each $j \in J$ and for any $\epsilon_j > 0$,

$$\{x \in \mathbb{R}^n : g_i(x) \leqq 0, i \in I\} \subset \{x \in \mathbb{R}^n : f_j(x) + \epsilon_j > 0, j \in J\}.$$

By Theorem 2.3, for each $j \in J$ and for any $\epsilon_j > 0$, there exist $\delta_j > 0$, $\lambda_i^j \geqq 0$ and $\sigma_j \in \Sigma_d^2$ such that $f_j + \epsilon_j + \sum_{i=1}^m \lambda_i^j g_i - \delta_j = \sigma_j$. Since $\sigma_j \in \Sigma_d^2$ and $\delta_j = (\sqrt{\delta_j})^2$,

$$f_j + \sum_{i=1}^m \lambda_i^j g_i + \epsilon_j \in \Sigma_d^2.$$

So, (ii) holds. □

Now we consider the quadratic case of Theorem 2.3, where the characterization is given in terms of linear matrix inequalities:

Corollary 2.6 (Containment of intersection of ellipsoids)**.** *Let $A_1, \dots, A_m$ be symmetric and positive semidefinite $n \times n$ matrices, $a_1, \dots, a_m \in \mathbb{R}^n$ and $\alpha_1, \dots, \alpha_m \in \mathbb{R}$. Let $B_1, \dots, B_p$ be symmetric and positive semidefinite $n \times n$ matrices, $b_1, \dots, b_p \in \mathbb{R}^n$ and $\beta_1, \dots, \beta_p \in \mathbb{R}$. Let $I = \{1, 2, \dots, m\}$ and let $J = \{1, 2, \dots, p\}$. Assume that $K := \{x : \frac{1}{2}\langle x, A_i x\rangle + \langle a_i, x\rangle + \alpha_i \leqq 0,\ i = 1, \dots, m\} \neq \emptyset$. Then the following statements are equivalent:*

(i) $\{x \in \mathbb{R}^n : \frac{1}{2}\langle x, A_i x\rangle + \langle a_i, x\rangle + \alpha_i \leqq 0,\ i \in I\} \subset \{x \in \mathbb{R}^n : \frac{1}{2}\langle x, B_j x\rangle + \langle b_j, x\rangle + \beta_j > 0,\ j \in J\}$;

(ii) $(\forall j \in J)(\exists (\lambda^j, \delta_j) \in \mathbb{R}_+^m \times \mathbb{R}_+\backslash\{0\}) \begin{pmatrix} B_j + \sum_{i=1}^m \lambda_i^j A_i & b_j + \sum_{i=1}^m \lambda_i^j a_i \\ (b_j + \sum_{i=1}^m \lambda_i^j a_i)^T & 2(\beta_j + \sum_{i=1}^m \lambda_i^j \alpha_i - \delta_j) \end{pmatrix} \succeq 0.$

Proof. [(ii) ⇒ (i)] Suppose that (ii) holds. Let $j \in J$. Then there exist $\lambda_i^j \geqq 0$ and $\delta_j \geqq 0$ such that

$$\begin{pmatrix} B_j + \sum_{i=1}^m \lambda_i^j A_i & b_j + \sum_{i=1}^m \lambda_i^j a_i \\ (b_j + \sum_{i=1}^m \lambda_i^j a_i)^T & 2(\beta_j + \sum_{i=1}^m \lambda_i^j \alpha_i - \delta_j) \end{pmatrix} \succeq 0.$$

Then for all $(x, \theta) \in \mathbb{R}^n \times \mathbb{R}$,

$$\begin{pmatrix} x \\ \theta \end{pmatrix}^T \begin{pmatrix} B_j + \sum_{i=1}^m \lambda_i^j A_i & b_j + \sum_{i=1}^m \lambda_i^j a_i \\ (b_j + \sum_{i=1}^m \lambda_i^j a_i)^T & 2(\beta_j + \sum_{i=1}^m \lambda_i^j \alpha_i - \delta_j) \end{pmatrix} \begin{pmatrix} x \\ \theta \end{pmatrix} \geqq 0.$$

In particular, for all $x \in \mathbb{R}^n$,

$$\begin{pmatrix} x \\ 1 \end{pmatrix}^T \begin{pmatrix} B_j + \sum_{i=1}^m \lambda_i^j A_i & b_j + \sum_{i=1}^m \lambda_i^j a_i) \\ (b_j + \sum_{i=1}^m \lambda_i^j a_i)^T & 2(\beta_j + \sum_{i=1}^m \lambda_i^j \alpha_i - \delta_j) \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} \geqq 0.$$

Expanding this inequality yields

$$\frac{1}{2}\langle x, B_j x\rangle + \langle b_j, x\rangle + \beta_j + \sum_{i=1}^m \lambda_i^j \Big(\frac{1}{2}\langle x, A_i x\rangle + \langle a_i, x\rangle + \alpha_i\Big) - \delta_j \geqq 0.$$

If $x \in K$, then we have $\frac{1}{2}\langle x, B_j x\rangle + \langle b_j, x\rangle + \beta_j > 0$. So, (i) holds.
[(i) ⇒ (ii)] Assume that (i) holds. Let $g_i(x) = \frac{1}{2}\langle x, A_i x\rangle + \langle a_i, x\rangle + \alpha_i$, $i \in I$, and $f_j(x) = \frac{1}{2}\langle x, B_j x\rangle + \langle b_j, x\rangle + \beta_j$, $j \in J$. Since A_i and B_j are symmetric and positive semidefinite, g_i and f_j are convex, and hence by Theorem 5.3 in [1], g_i and f_j are SOS-convex. Using Theorem 2.3, we see that (i) is equivalent to the condition that, for each $j = 1, \dots, p$,

$$(\exists \delta_j > 0, \lambda_i^j \geqq 0) \quad f_j + \sum_{i=1}^m \lambda_i^j g_i - \delta_j \in \Sigma_2^2.$$

As $f_j + \sum_{i=1}^m \lambda_i^j g_i - \delta_j$ is a quadratic function and $f_j + \sum_{i=1}^m \lambda_i^j g_i - \delta_j \geqq 0$ is equivalent to the semi-definite inequality

$$\begin{pmatrix} B_j + \sum_{i=1}^m \lambda_i^j A_i & b_j + \sum_{i=1}^m \lambda_i^j a_i \\ (b_j + \sum_{i=1}^m \lambda_i^j a_i)^T & 2(\beta_j + \sum_{i=1}^m \lambda_i^j \alpha_i - \delta_j) \end{pmatrix} \succeq 0,$$

and hence the conclusion follows. □

It is worth noting that the sum of squares set containment characterizations, given in Theorem 2.3 and Corollary 2.6, can be verified numerically because whether a polynomial is a sum of squares of polynomials or not can be verified by solving related semidefinite programs [22].

Under Slater's condition, we see that set containment characterization derived in Theorem 2.5 can be simplified to obtain an easily checkable set containment characterization result below, where the condition does not have to be checked for every $\epsilon > 0$.

Corollary 2.7 (Containment of a convex set in a reverse convex set under Slater's condition II)**.** *Let f_j, $j = 1, \ldots, p$, be SOS-convex polynomials and let g_i, $i = 1, \ldots, m$, be SOS-convex polynomials. Assume that there exists $x_0 \in \mathbb{R}^n$ such that $g_i(x_0) < 0, i = 1, \ldots, m$. Let $d = \max\{\deg f_1, \ldots, \deg f_p, \deg g_1, \ldots, \deg g_m\}$. Let $I = \{1, 2, \ldots, m\}$ and let $J = \{1, 2, \ldots, p\}$. Then the following statements are equivalent:*

(i) $\{x \in \mathbb{R}^n : g_i(x) \leqq 0, i \in I\} \subset \{x \in \mathbb{R}^n : f_j(x) \geqq 0, j \in J\}$;

(ii) $(\forall j \in J)\ \ (\exists \lambda^j \in \mathbb{R}^m_+)\ f_j + \sum_{i=1}^m \lambda_i^j g_i \in \Sigma^2_d$.

Proof. [(i) $\Rightarrow$ (ii)] Assume that (i) holds. Let $j \in J$. Then letting $f = f_j$, it follows from Theorem 2.5 that there exist $\lambda_i^{j,n} \geqq 0$, $i \in I$ such that

$$f_j + \sum_{i=1}^m \lambda_i^{j,n} g_i + \frac{1}{n} \in \Sigma^2_d. \tag{2.2}$$

If the sequence $\{\lambda_i^{j,n}\}$ is bounded for each $i \in \{1, \ldots, m\}$, then without loss of generality, the sequence $\{\lambda_i^{j,n}\}$ converges to λ_i^j for each $i \in I$, and hence from (2.2) for any $x \in \mathbb{R}^n$,

$$f_j(x) + \sum_{i=1}^m \lambda_i^j g_i(x) \geqq 0.$$

If the sequence $\{\lambda_i^{j,n}\}$ is not bounded for some $i_0 \in I$, then we may assume that $\lim_{n\to\infty}\{\lambda_{i_0}^{j,n}\} = +\infty$, and hence, by (2.2) and the assumption, we get that

$$0 \leqq f_j(x_0) + \sum_{i=1}^m \lambda_i^{j,n} g_i(x_0) + \frac{1}{n} \leqq f_j(x_0) + \lambda_{i_0}^j g_i(x_0) + \frac{1}{n} \to -\infty \text{ as } n \to \infty,$$

which is a contradiction. Hence, we may assume that for each $j \in J$, there exist $\lambda_i^j \geqq 0$, $i \in I$ such that $f_j(x) + \sum_{i=1}^m \lambda_i^j g_i(x) \geqq 0$ for any $x \in \mathbb{R}^n$. Since $f_j + \sum_{i=1}^m \lambda_i^j g_i$ is convex, it follows from Lemma 2.1, that there exist $x^j \in \mathbb{R}^n$, $j \in J$ such that

$$f_j(x) + \sum_{i=1}^m \lambda_i^j g_i(x) \geqq f_j(x^j) + \sum_{i=1}^m \lambda_i^j g_i(x^j) \text{ for any } x \in \mathbb{R}^n.$$

Let $h_j(x) = f_j(x) + \sum_{i=1}^m \lambda_i^j g_i(x) - f_j(x^j) - \sum_{i=1}^m \lambda_i^j g_i(x^j)$. Then h_j is SOS-convex. Since $h_j(x^j) = 0$ and $\nabla h_j(x^j) = 0$, by the definition of SOS-convexity, h_j is a sum of squares polynomial. Since $f_j(x^j) + \sum_{i=1}^m \lambda_i^j g_i(x^j) \geqq 0$, $f_j + \sum_{i=1}^m \lambda_i^j g_i$ is a sum of squares polynomial. This completes the proof as [(ii) $\Rightarrow$ (i)] follows easily. □

As a consequence of Theorem 2.5, we also obtain a set containment characterization involving max SOS-convex functions, extending the recent non-negativity characterization for max SOS-convex functions [16].

Corollary 2.8 (Containment and max SOS-convex functions). *Let p_j^l, $l = 1, \ldots, p$, $j = 1, \ldots, k_l$, be SOS-convex polynomials and let g_i, $i = 1, \ldots, m$, be SOS-convex polynomials. Assume that $K = \{x \in \mathbb{R}^n : g_i(x) \leqq 0,\ i = 1, \ldots, m\} \neq \emptyset$. Let $I = \{1, 2, \ldots, m\}$ and let $J = \{1, 2, \ldots, p\}$. Let $d = \max\{\deg p_1^1, \ldots, \deg p_{k_l}^l, \deg g_1, \ldots, \deg g_m\}$. Then the following statements are equivalent:*

(i) $\{x \in \mathbb{R}^n : g_i(x) \leqq 0, i \in I\} \subset \{x \in \mathbb{R}^n : \max_{j=1,\ldots,k_l} p_j^l(x) \geqq 0, l \in J\}$;

(ii) $(\forall l \in J)(\forall \epsilon^l > 0)$, $(\exists \lambda_i^l \geqq 0,\ \mu_j^l \geqq 0,\ \sum_{j=1}^{k_l} \mu_j^l = 1,\ \sigma^l \in \Sigma_d^2)\ \sum_{j=1}^{k_l} \mu_j^l p_j^l + \sum_{i=1}^m \lambda_i^l g_i + \epsilon^l = \sigma^l$.

Proof. The inclusion in (i) is equivalent to the following statement that, for each $l \in J$,

$$g_i(x) \leqq 0,\ i \in I,\ p_j^l(x) \leqq t,\ j = 1, \ldots, k_l \Rightarrow t \geqq 0.$$

So, by Theorem 2.5, (i) is equivalent to the condition that

$$\forall \epsilon^l > 0, (\exists \lambda_i^l,\ \mu_j^l \geqq 0,\ \sigma^l \in \Sigma_d^2)\Big(1 - \sum_{j=1}^{k_l} \mu_j^l\Big)t + \sum_{i=1}^m \lambda_i^l g_i(x) + \sum_{j=1}^{k_l} \mu_j^l p_j^l(x) + \epsilon^l = \sigma^l(x, t)$$

for any $(x, t) \in \mathbb{R}^{n+1}$. Here we can easily check that $\sum_{j=1}^{k_l} \mu_j^l = 1$. Thus letting $t = 1$, we see that (i) is equivalent to (ii) □

We note that Corollary 2.8 collapses to Theorem 2.2 of [16] whenever $J = \{1\}$.

3 Robust Set Containment Characterizations

In this section, we consider a robust set containment, where the robust counterpart of an uncertain convex semialgebraic set is contained in another reverse convex semialgebraic set.

Consider the semialgebraic set

$$K := \{x \in \mathbb{R}^n : g_i(x) \leqq 0,\ i = 1, \ldots, m\}.$$

The set K in the face of data uncertainty in the functions $g_i, i = 1, 2, \ldots, m$, can be captured by the following parameterized set

$$K' := \{x \in \mathbb{R}^n : g_i(x, v_i) \leqq 0,\ i = 1, \ldots, m\},$$

where $v_i's$ are uncertain parameters (or coefficients) and they belong to the specified convex and compact uncertainty sets $\mathcal{V}_i \subset \mathbb{R}^q$ and $g_i : \mathbb{R}^n \times \mathbb{R}^{q_i} \to \mathbb{R}$, $i = 1, \ldots, m$. The robust counterpart of K is given by

$$\bar{K} := \{x \in \mathbb{R}^n : g_i(x, v_i) \leqq 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1, \ldots, m\},$$

where the constraints are enforced for all possible uncertainties within $\mathcal{V}_i$, for $i = 1, 2, \ldots, m$. We establish characterizations for $\bar{K}$ to be contained in another reverse convex semialgebraic set. An example of a robust set containment is depicted in Figure 3.1 below. Related results for polyhedral set containment can be found in [13].

Theorem 3.1 (Robust convex set containment). *Let $f_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \ldots, p$, be SOS-convex polynomials and let $g_i : \mathbb{R}^n \times \mathbb{R}^{q_i} \to \mathbb{R}$, $i = 1, \ldots, m$, be functions such that for each $v_i \in \mathbb{R}^{q_i}$, $g_i(\cdot, v_i)$ is a SOS-convex polynomial and for each $x \in \mathbb{R}^n$, $g_i(x, \cdot)$ is a concave function. Let $\mathcal{V}_i \subset \mathbb{R}^{q_i}$, $i = 1, \ldots, m$ be compact and convex. Assume that $\bar{K} := \{x \in \mathbb{R}^n : g_i(x, v_i) \leqq 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1, \ldots, m\} \neq \emptyset$. Let $I = \{1, 2, \ldots, m\}$ and let $J = \{1, 2, \ldots, p\}$. Then the following statements are equivalent:*

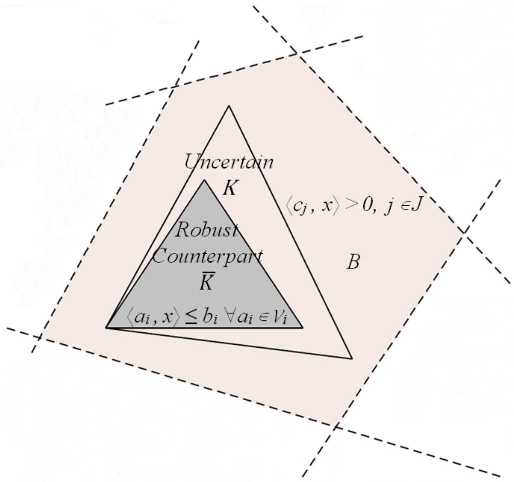

Figure 3.1: Given an uncertain convex semialgebraic set $K = \{x : \langle a_i, x\rangle \leq b_i,\ i = 1, \ldots, m\}$, its robust counterpart $\bar{K} = \{x : \langle a_i, x\rangle \leq b_i, \forall a_i \in \mathcal{V}_i,\ i = 1, \ldots, m\} \subset B = \{x \mid \langle c_j, x\rangle > 0,\ j \in J\}$. In robust counterpart $\bar{K}$, the constraints of K are enforced for every possible value of a_i within the prescribed set $\mathcal{V}_i$.

(i) $\{x \in \mathbb{R}^n : g_i(x, v_i) \leqq 0,\ \forall v_i \in \mathcal{V}_i,\ i \in I\} \subset \{x \in \mathbb{R}^n : f_j(x) > 0, j \in J\}$;

(ii) $(\forall j \in J)\ (\exists \lambda_i^j \geqq 0,\ v_i^j \in \mathcal{V}_i\ \delta_j > 0, \sigma_j \in \Sigma^2)\ f_j + \sum_{i=1}^m \lambda_i^j g_i(\cdot, v_i^j) - \delta_j = \sigma_j$.

Proof. [(ii) $\Rightarrow$ (i)] Suppose that (ii) holds. Let $j \in J$. Then by (ii), there exist $\lambda_i^j \geqq 0,\ v_i^j \in \mathcal{V}_i,\ \delta_j > 0,\ \sigma_j \in \Sigma^2$ such that

$$f_j(x) + \sum_{i=1}^{m} \lambda_i^j g_i(x, v_i^j) - \delta_j = \sigma_j.$$

Hence, for any $x \in \bar{K}$ and each $j \in J$, $f_j(x) > 0$. Thus (i) holds.
[(i) $\Rightarrow$ (ii)] Assume that (i) holds. Let $j \in J$. By (i), we have

$$g_i(x, v_i) \leqq 0,\ \forall v_i \in \mathcal{V}_i,\ i \in I \Rightarrow f_j(x) > 0, j \in J. \tag{3.1}$$

Since f_j is SOS-convex and it follows from (3.1) that $\inf_{x \in \bar{K}} f_j(x) \geqq 0$, by Lemma 2.1, there exists $x^* \in \bar{K}$ such that $f_j(x^*) = \inf_{x \in \bar{K}} f_j(x)$. Let $\beta_j = f_j(x^*)$. Then it follows from (3.1) that $\beta_j > 0$. Let $\tilde{g}_i(x) = \max_{v_i \in \mathcal{V}_i} g_i(x, v_i)$, $i \in I$. Consider the following set

$$C_j = \{(y_j, z) \in \mathbb{R} \times \mathbb{R}^m \mid \exists x \in \mathbb{R}^n\ s.t.\ f_j(x) \leqq y_j,\ \tilde{g}_i(x) \leqq z_i,\ i \in I\}.$$

Using the same line of arguments as in the proof of Theorem 2.3, we can prove that there exist $\lambda_i^j \geqq 0$, $i \in I$ such that

$$f_j(x) + \sum_{i=1}^{m} \lambda_i^j \tilde{g}_i(x) - \frac{\beta_j}{2} > 0.$$

So, we have

$$\begin{aligned}\frac{\beta_j}{2} &\leqq \inf_{x\in\mathbb{R}^n}\Big\{f_j(x)+\sum_{i=1}^m \lambda_i^j \tilde{g}_i(x)\Big\}\\ &= \inf_{x\in\mathbb{R}^n}\Big\{f_j(x)+\sum_{i=1}^m \lambda_i^j \max_{v_i\in\mathcal{V}_i} g_i(x,v_i)\Big\}\\ &= \inf_{x\in\mathbb{R}^n}\max_{v_i\in\mathcal{V}_i}\Big\{f_j(x)+\sum_{i=1}^m \lambda_i^j g_i(x,v_i)\Big\}.\end{aligned}$$

By min-max theorem ([25, Corollary 37.3.2]), $\frac{\beta_j}{2} \leqq \max_{v_i\in\mathcal{V}_i}\inf_{x\in\mathbb{R}^n}\{f_j(x)+\sum_{i=1}^m \lambda_i^j g_i(x,v_i)\}$. Hence there exist $v_i^j \in \mathcal{V}_i$, $i \in I$ such that $\inf_{x\in\mathbb{R}^n}\{f_j(x)+\sum_{i=1}^m \lambda_i^j g_i(x,v_i^j)\} \geqq \frac{\beta_j}{2}$. So, $f_j(x)+\sum_{i=1}^m \lambda_i^j g_i(x,v_i^j) \geqq \frac{\beta_j}{2}$ for any $x \in \mathbb{R}^n$. Since $f_j(x)+\sum_{i=1}^m \lambda_i^j g_i(\cdot,v_i^j)$ is a bounded below SOS-convex polynomial, it follows from Lemma 2.1 that $\bar{x} \in \mathbb{R}^n$ such that for any $x \in \mathbb{R}^n$,

$$f_j(\bar{x})+\sum_{i=1}^m \lambda_i^j g_i(\bar{x},v_i^j) \leqq f_j(x)+\sum_{i=1}^m \lambda_i^j g_i(x,v_i^j).$$

Let $\Psi_j(x) = f_j(x)+\sum_{i=1}^m \lambda_i^j g_i(x,v_i^j) - (f(\bar{x})+\sum_{i=1}^m \lambda_i^j g_i(\bar{x},v_i^j))$. Then Ψ_j is a SOS-convex polynomial, that is, $f_j(x)+\sum_{i=1}^m \lambda_i^j g_i(x,v_i^j) - (f(\bar{x})+\sum_{i=1}^m \lambda_i^j g_i(\bar{x},v_i^j)) \in \Sigma^2$. Let $\delta_j = f_j(\bar{x})+\sum_{i=1}^m \lambda_i^j g_i(\bar{x},v_i^j)$. Then $\delta_j \geqq \frac{\beta_j}{2} > 0$. Hence $f_j+\sum_{i=1}^m \lambda_i^j g_i(\cdot,v_i^j) - \delta_j \in \Sigma^2$. So, (ii) holds. □

Remark 3.2. It is worth noting that in Theorem 3.1, the degree of SOS-convex polynomial $g_i(\cdot,v_i)$ may be different when $v_i \in \mathcal{V}_i$ is changing.

In passing note that if the uncertainty sets in Theorem 3.1 are singleton, i.e. $\mathcal{V}_i = \{v_i\}$, then Theorem 3.1 collapses to Theorem 2.3 of the previous Section.

Now, using the results of the previous section, in the following, we derive robust set containment characterizations for special classes of commonly used uncertainty sets of robust optimization [4, 13]. We first obtain a characterization for robust containment of intersection of ellipsoids in another ellipsoid under spectral norm uncertainty. Recall that the spectral norm of $\Delta \in S^n$, denoted by $\|\Delta\|_{\text{spec}}$, is the square root of the largest eigenvalue of the matrix $\Delta^T\Delta$.

Corollary 3.3 (Robust containment of ellipsoids under spectral norm uncertainty)**.** *Let $a_i \in \mathbb{R}^n$ and $\alpha_i \in \mathbb{R}$, $i = 1,\ldots,m$. Let B_j, $j = 1\ldots,p$, be symmetric and positive semidefinite $n \times n$ matrices, $b_j \in \mathbb{R}^n$ and $\beta_j \in \mathbb{R}$, $j = 1,\ldots,p$. Let $\mathcal{V}_i = \{A_0^i + M_i : M_i \in S^n,\ M_i \succeq 0,\ \|M_i\|_{\text{spec}} \leqq \rho_i\}$, $i = 1,\ldots,m$, where A_0^i is symmetric and positive semidefinite $n \times n$ matrices and $\rho_i > 0$, $i = 1,\ldots,m$. Assume that $\bar{K} := \{x : \frac{1}{2}\langle x, A_i x\rangle + \langle a_i, x\rangle + \alpha_i \leqq 0,\ \forall A_i \in \mathcal{V}_i,\ i = 1,\ldots,m\} \neq \emptyset$. Let $I = \{1,2,\ldots,m\}$ and let $J = \{1,2,\ldots,p\}$. Then following statements are equivalent:*

(i) $\{x : \frac{1}{2}\langle x, A_i x\rangle + \langle a_i, x\rangle + \alpha_i \leqq 0,\ \forall A_i \in \mathcal{V}_i,\ i \in I\} \subset \{x : \frac{1}{2}\langle x, B_j x\rangle + \langle b_j, x\rangle + \beta_j > 0,\ j \in J\}$;

(ii) $(\forall j \in J)$ $(\exists \lambda_i^j \geqq 0,\ \delta_j > 0, i = 1,2,\ldots,m)$

$$\begin{pmatrix} B_j + \sum_{i=1}^m \lambda_i^j (A_0^i + \rho_i I_n) & b_j + \sum_{i=1}^m \lambda_i^j a_i \\ (b_j + \sum_{i=1}^m \lambda_i^j a_i)^T & 2(\beta_j + \sum_{i=1}^m \lambda_i^j \alpha_i - \delta_j) \end{pmatrix} \succeq 0.$$

Proof. Let $j \in J$. Since $\max_{A_i \in \mathcal{V}_i}\{\frac{1}{2}\langle x, A_i x\rangle + \langle a_i, x\rangle + \alpha_i\} = \max_{\|M_i\|_{\text{spec}} \leqq \rho_i}\{\frac{1}{2}\langle x, (A_0^i + M_i)x\rangle + \langle a_i, x\rangle + \alpha_i\} = \frac{1}{2}\langle x, (A_0^i + \rho_i I_n)x\rangle + \langle a_i, x\rangle + \alpha_i$, (i) is equivalent to the condition that

$$\Big\{x : \frac{1}{2}\langle x, (A_0^i + \rho_i I_n)x\rangle + \langle a_i, x\rangle + \alpha_i \leqq 0, \quad i \in I\Big\} \subset \Big\{x : \frac{1}{2}\langle x, B_j x\rangle + \langle b_j, x\rangle + \beta_j > 0,\ j \in J\Big\}.$$

So, it follows from Corollary 2.6 that (i) is equivalent to the condition that there exist $\lambda_i^j \geqq 0$ and $\delta_j > 0$ such that

$$\begin{pmatrix} B_j + \sum_{i=1}^m \lambda_i^j (A_0^i + \rho_i I_n) & b_j + \sum_{i=1}^m \lambda_i^j a_i \\ (b_j + \sum_{i=1}^m \lambda_i^j a_i)^T & 2(\beta_j + \sum_{i=1}^m \lambda_i^j \alpha_i - \delta_j) \end{pmatrix} \succeq 0.$$

□

Corollary 3.4 (Robust containment under scenario uncertainty)**.** *Let $f_i(\cdot, v_i)$ be SOS-convex polynomial for each $v_i \in \mathcal{V}_i := \{v_i^1, \ldots, v_i^{s_i}\}$ and each $i \in \{0, 1, 2, \ldots, k\}$. Let $d = \max\{\deg f_0(\cdot, v_0^1), \ldots, \deg f_0(\cdot, v_0^{s_0}), \ldots, \deg f_k(\cdot, v_k^1)\ , \ldots, f_k(\cdot, v_k^{s_k})\}$. Then the following statements are equivalent:*

(i) $\{x \in \mathbb{R}^n : f_i(x, v_i) \leqq 0\ \forall v_i \in \mathcal{V}_i, i = 1, \ldots, k\} \subset \{x \in \mathbb{R}^n : \max_{v_0 \in \mathcal{V}_0} f_0(x, v_0) \geqq 0\}$*;*

(ii) $\forall \epsilon > 0$, $(\exists \lambda_i^j \geqq 0,\ \mu_l \geqq 0\ (\sum_{l=1}^{s_0} \mu_l = 1), \sigma \in \Sigma_d^2)$

$$\sum_{l=1}^{s_0} \mu_l f_0(\cdot, v_0^l) + \sum_{i=1}^{k}\sum_{j=1}^{s_i} \lambda_i^j f_i(\cdot, v_i^j) + \epsilon = \sigma.$$

Proof. (i) is equivalent to the inclusion

$$\{x \in \mathbb{R}^n : f_i(x, v_i^j) \leqq 0, j = 1, \ldots, s_i,\ i = 1, \ldots, k\} \subset \{x \in \mathbb{R}^n : \max_{i=1,\ldots,s_0} f_0(x, v_0^i) \geqq 0\}.$$

Then by Corollary 2.8, (i) is equivalent to (ii). □

Corollary 3.5 (Robust containment under polytopic uncertainty)**.** *Let $f_i(\cdot, v_i)$, $i \in \{0, 1, \ldots, k\}$ be SOS-convex polynomial for each $v_i \in \mathcal{V}_i := \text{co}\{v_i^1, \ldots, v_i^{s_i}\}$ and let $f_i(x, \cdot)$, $i \in \{0, 1, \ldots, k\}$ be affine for each $x \in \mathbb{R}^n$. Let $d = \max\{\deg f_0(\cdot, v_0^1), \ldots, \deg f_0(\cdot, v_0^{s_0}), \ldots, \deg f_k(\cdot, v_k^1)\ , \ldots, f_k(\cdot, v_k^{s_k})\}$. Then the following statements are equivalent:*

(i) $\{x \in \mathbb{R}^n : f_i(x, v_i) \leqq 0\ \forall v_i \in \mathcal{V}_i, i = 1, \ldots, k\} \subset \{x \in \mathbb{R}^n : \max_{v_0 \in \mathcal{V}_0} f_0(x, v_0) \geqq 0\}$*;*

(ii) $\forall \epsilon > 0$, $(\exists \lambda_i^j \geqq 0,\ \mu_l \geqq 0\ (\sum_{l=1}^{s_0} \mu_l = 1), \sigma \in \Sigma_d^2)$

$$\sum_{l=1}^{s_0} \mu_l f_0(\cdot, v_o^l) + \sum_{i=1}^{k}\sum_{j=1}^{s_i} \lambda_i^j f_i(\cdot, v_i^j) + \epsilon = \sigma.$$

Proof. Since $f_i(x, \cdot)$, $i = 1, \ldots, k$ are affine, $f_i(x, v_i) \leqq 0\ \forall v_i \in \mathcal{V}_i$ if and only if $f_i(x, v_i^j) \leqq 0$, $v_i^j \in \mathcal{V}_i$, $i = 1, \ldots, k$, $j = 1, \ldots, s_i$. Moreover, $\max_{v_0 \in \mathcal{V}_0} f_0(x, v_0) = \max_{j=1,\ldots,s_0} f_0(x, v_0^j)$. So, by Corollary 2.8, (i) is equivalent to (ii). □

Finally, as an application of Theorem 3.1 we derive a zero duality gap result for a SOS-convex program in the face of parameter uncertainty, where the dual problem can be represented by a semidefinite program which can be easily solved by interior-point methods. Related strong duality results for robust convex programs are given in [15] and for a general robust minimax convex program under a constraint qualification are provided in [17].

Corollary 3.6 (Zero duality gap). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a SOS-convex polynomial and let $g_i : \mathbb{R}^n \times \mathbb{R}^{q_i} \to \mathbb{R}$, $i = 1, \ldots, m$, be functions such that for each $v_i \in \mathbb{R}^{q_i}$, $g_i(\cdot, v_i)$ is a SOS-convex polynomial and for each $x \in \mathbb{R}^n$, $g_i(x, \cdot)$ is a concave function. Let $\mathcal{V}_i \subset \mathbb{R}^{q_i}$, $i = 1, \ldots, m$, be compact and convex. Assume that $\bar{K} := \{x \in \mathbb{R}^n : g_i(x, v_i) \leqq 0, \ \forall v_i \in \mathcal{V}_i, \ i = 1, \ldots, m\} \neq \emptyset$. Let $I = \{1, \ldots, m\}$. Then*

$$\inf_{x \in \mathbb{R}^n} \{f(x) \mid g_i(x, v_i) \leqq 0, \ \forall v_i \in \mathcal{V}_i, \ i \in I\} = \sup_{\substack{\mu \in \mathbb{R}, \lambda_i \geqq 0 \\ v_i \in \mathcal{V}_i}} \Big\{\mu \mid f + \sum_{i=1}^m \lambda_i g_i(\cdot, v_i) - \mu \in \Sigma^2\Big\}.$$

Proof. Let $\alpha = \inf_{x \in \mathbb{R}^n} \{f(x) \mid g_i(x, v_i) \leqq 0, \ \forall v_i \in \mathcal{V}_i, \ i \in I\}$. Then $\alpha = -\infty$ or $\alpha \in \mathbb{R}$. If $\alpha = -\infty$, then the conclusion always holds. So, we assume that $\alpha \in \mathbb{R}$. Let $\epsilon > 0$. Then

$$\{x \in \mathbb{R}^n \mid g_i(x, v_i) \leqq 0, \ \forall v_i \in \mathcal{V}_i, \ i \in I\} \subset \{x \in \mathbb{R}^n \mid f(x) > \alpha - \epsilon\}.$$

By Theorem 3.1, there exist $\delta > 0$, $\lambda_i \geqq 0$, $v_i \in \mathcal{V}_i$, $i \in I$ and $\sigma \in \Sigma^2$ such that

$$f + \sum_{i=1}^m \lambda_i g_i(\cdot, v_i) - \delta - \alpha + \epsilon = \sigma.$$

Since $\delta > 0$, $f + \sum_{i=1}^m \lambda_i g_i(\cdot, v_i) - \alpha + \epsilon \in \Sigma^2$. Thus $\sup_{\mu \in \mathbb{R}, \lambda_i \geqq 0, v_i \in \mathcal{V}_i} \{\mu \mid f + \sum_{i=1}^m \lambda_i g_i(\cdot, v_i) - \mu \in \Sigma^2\} \geqq \alpha - \epsilon$. Since ϵ is arbitrary, $\sup_{\mu \in \mathbb{R}, \lambda_i \geqq 0, v_i \in \mathcal{V}_i} \{\mu \mid f + \sum_{i=1}^m \lambda_i g_i(\cdot, v_i) - \mu \in \Sigma^2\} \geqq \alpha$.

On the other hand, if $f + \sum_{i=1}^m \lambda_i g_i(\cdot, v_i) - \beta \in \Sigma^2$, then we have for any $x \in \mathbb{R}^n$, $f(x) + \sum_{i=1}^m \lambda_i g_i(x, v_i) - \beta \geqq 0$, that is, $f(x) + \sum_{i=1}^m \lambda_i g_i(x, v_i) \geqq \beta$. If $g_i(x, v_i) \leqq 0$, $\forall v_i \in \mathcal{V}_i$, $i = 1, \ldots, m$, then we have $f(x) \geqq f(x) + \sum_{i=1}^m \lambda_i g_i(x, v_i) \geqq \beta$. Hence $\beta \leqq \alpha$. So, $\sup_{\mu \in \mathbb{R}, \lambda_i \geqq 0, v_i \in \mathcal{V}_i} \{\mu \mid f + \sum_{i=1}^m \lambda_i g_i(\cdot, v_i) - \mu \in \Sigma^2\} \leqq \alpha$. Hence $\alpha = \sup_{\mu \in \mathbb{R}, \lambda_i \geqq 0, v_i \in \mathcal{V}_i} \{\mu \mid f + \sum_{i=1}^m \lambda_i g_i(\cdot, v_i) - \mu \in \Sigma^2\}$. □

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