Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 12, No. 1, pp. 43–54, January 2016

CONVEXITY FOR COMPOSITIONS OF SET-VALUED MAP AND MONOTONE SCALARIZING FUNCTION*

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Abstract: In this paper, we give some characterization on several types of cone-convexity and cone-concavity for compositions of a set-valued map and some kind of order-monotone scalarizing function for sets in a vector space.

Key words: *cone-convexity, cone-concavity, order-monotone mapping, scalarization method, set-relations, set-valued map.*

Mathematics Subject Classification: *49J53, 54C60, 90C29.*

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1 Introduction

Scalarization methods are used frequently in multiobjective optimization problems. In general, there are two typical types of scalarizing methods, which are linear and nonlinear ones. The most general nonlinear scalarizing functions for set-valued maps are proposed by Kuwano, Tanaka, and Yamada in [3]. Such kinds of functions are called unified types scalarizing functions for sets. One of important properties of them is monotonisity, which preserve a preference relationship between decision-space and outcome-space These are many studies by using them, for example, see [3, 4, 5, 6, 8, 9]. In [3], they study several types of inherited properties on the cone-convexity of parent set-valued map to the compositions of the set-valued map and unified types of scalarizing functions.

However, the inverse results, which are to drive convexity and concavity of set-valued map from those of the compositions of set-valued map and unified types scalarizing functions, have not been studied in detail. In addition, it is unclear that inherited properties on cone-convexity of parent set-valued map to the compositions of the set-valued map and any general order monotone function.

The aim of this paper is to show some kind of general cone convexity and concavity related to several types of compositions of set-valued map and monotone scalarizing function. At first, we characterize several types of cone-convexity and cone-concavity for set-valued map by compositions. Secondly, we show some essentiality about inherited properties of convexity of set-valued map in [3]. In other words, we show some cone-convexity and cone-concavity for compositions of an ordered monotone scalarizing function and a set-valued map.

*This work is supported by Grant-in-Aid for Scientific Research (C) (JSPS KAKENHI Grant Number 26400194) from Japan Society for the Promotion of Science (JSPS)

The organization of this paper is as follows. In Section 2, we recall some basic concepts in set optimization. In Section 3, we show two main results. At first, we show twelve statements about characterization of cone-convexity and cone-concavity for set-valued map by compositions. Second, we show general results of [3].

2 Preliminaries

Throughout this paper, let X be a real vector space, Y a real ordered topological vector space with the vector ordering $\leq_K$ induced by a nonempty proper closed convex cone K (that is, $K \neq Y$, $K + K = K$, and $\lambda K \subset K$ for all $\lambda \geq 0$) with $\operatorname{int} K \neq \emptyset$ as follow:

$$x \leq_K y \text{ if } y - x \in K \text{ for } x, y \in Y,$$

and F a set-valued map from X to $2^Y \setminus \{\emptyset\}$.

Definition 2.1. ([7]) Let A be a nonempty subset in Y. Then,

(i) A is said to be K-*convex* if $A + K$ is convex;

(ii) A is said to be K-*closed* if $A + K$ is closed;

(iii) A is said to be K-*proper* if $A + K$ is proper;

(iv) A is said to be K-*compact* if any cover of A of the form $\{U_\alpha + K \mid \alpha \in I, U_\alpha \text{ is open}\}$ admits a finite cover.

Definition 2.2. ([1]) Let $A, B \in 2^Y \setminus \{\emptyset\}$. Then, we denote

(i) $A \subset \bigcap_{b \in B}(b - K)$, equivalently $B \subset \cap_{a \in A}(a + K)$ by $A \leq_K^{(1)} B$;

(ii) $A \cap (\bigcap_{b \in B}(b - K)) \neq \emptyset$ by $A \leq_K^{(2)} B$;

(iii) $B \subset A + K$ by $A \leq_K^{(3)} B$;

(iv) $(\bigcap_{a \in A}(a + K)) \cap B \neq \emptyset$ by $A \leq_K^{(4)} B$;

(v) $A \subset (B - K)$ by $A \leq_K^{(5)} B$;

(vi) $A \cap (B - K) \neq \emptyset$, equivalently $(A + K) \cap B \neq \emptyset$ by $A \leq_K^{(6)} B$.

Proposition 2.3. ([3]) *For $A, B \in 2^Y \setminus \{\emptyset\}$ and a direction $e \in \operatorname{int} K$, the following statements hold:*

(i) *For each $j = 1, \ldots, 6$,*

$$A \leq_K^{(j)} (te + B) \text{ implies } A \leq_K^{(j)} (se + B) \text{ for any } s \geq t,$$
$$(te + B) \leq_K^{(j)} A \text{ implies } (se + B) \leq_K^{(j)} A \text{ for any } s \leq t;$$

(ii) *For each $j = 3, 5, 6$, $\leq_K^{(j)}$ is reflexive.*

Definition 2.4. ([1]) For each $j = 1, \ldots, 6$,

(i) A map F is said to be *type (j) K-convex* if for each $x_1, x_2 \in X$ and $\lambda \in (0, 1)$,

$$F(\lambda x_1 + (1 - \lambda)x_2) \leq_K^{(j)} \lambda F(x_1) + (1 - \lambda)F(x_2);$$

(ii) A map F is said to be *type* (j) *properly quasi* K*-convex* if for each $x_1, x_2 \in X$ and $\lambda \in (0,1)$,

$$F(\lambda x_1 + (1-\lambda)x_2) \leq_K^{(j)} F(x_1) \text{ or } F(\lambda x_1 + (1-\lambda)x_2) \leq_K^{(j)} F(x_2);$$

(iii) A map F is said to be *type* (j) *naturally quasi* K*-convex* if for each $x_1, x_2 \in X$ and $\lambda \in (0,1)$, there exists $\mu \in [0,1]$ such that

$$F(\lambda x_1 + (1-\lambda)x_2) \leq_K^{(j)} \mu F(x_1) + (1-\mu)F(x_2).$$

Definition 2.5. ([1]) For each $j = 1, \ldots, 3$, a map F is said to be *type* (j)*-lower quasiconvex* if for each $x_1, x_2 \in X$ and $\lambda \in (0,1)$,

$$F(\lambda x_1 + (1-\lambda)x_2) \leq_K^{(j)} (F(x_1) + K) \cap (F(x_2) + K).$$

Definition 2.6. ([1]) A map F is said to be *Ferro type* (-1) *quasiconvex* if for each $y \in Y$, $F^{-1}(y - K) := \{x \in X \mid F(x) \cap (y - K) \neq \emptyset\}$ is convex.

The concepts of cone-concavities are defined as well as cone-convexities.

Definition 2.7. ([4]) For each $j = 1, \ldots, 6$,

(i) A map F is said to be *type* (j) K*-concave* if for each $x_1, x_2 \in X$ and $\lambda \in (0,1)$,

$$\lambda F(x_1) + (1-\lambda)F(x_2) \leq_K^{(j)} F(\lambda x_1 + (1-\lambda)x_2);$$

(ii) A map F is said to be *type* (j) *properly quasi* K*-concave* if for each $x_1, x_2 \in X$ and $\lambda \in (0,1)$,

$$F(x_1) \leq_K^{(j)} F(\lambda x_1 + (1-\lambda)x_2) \text{ or } F(x_2) \leq_K^{(j)} F(\lambda x_1 + (1-\lambda)x_2);$$

(iii) A map F is said to be *type* (j) *naturally quasi* K*-concave* if for each $x_1, x_2 \in X$ and $\lambda \in (0,1)$, there exists $\mu \in [0,1]$ such that

$$\mu F(x_1) + (1-\mu)F(x_2) \leq_K^{(j)} F(\lambda x_1 + (1-\lambda)x_2).$$

Definition 2.8. For each $j = 1, 4, 5$, a map F is said to be *type* (j)*-lower quasiconcave* if for each $x_1, x_2 \in X$ and $\lambda \in (0,1)$,

$$(F(x_1) - K) \cap (F(x_2) - K) \leq_K^{(j)} F(\lambda x_1 + (1-\lambda)x_2).$$

Definition 2.9. A map F is said to be *Ferro type* (-1) *quasiconcave* if for each $y \in Y$, $F^{-1}(y + K) := \{x \in X \mid F(x) \cap (y + K) \neq \emptyset\}$ is convex.

Definition 2.10. ([3]) Let $A, B \in 2^Y \setminus \{\emptyset\}$ and a direction $e \in \operatorname{int} K$. For each $j = 1, \ldots, 6$, we define scalarizing functions $I_{e,B}^{(j)}$ and $S_{e,B}^{(j)}$ from $2^Y \setminus \{\emptyset\}$ to $\overline{\mathbb{R}}$ by

$$I_{e,B}^{(j)}(A) := \inf \{t \in \mathbb{R} \mid A \leq_K^{(j)} (te + B)\},\ S_{e,B}^{(j)}(A) := \sup \{t \in \mathbb{R} \mid (te + B) \leq_K^{(j)} A\}.$$

These functions are called *unified types of scalarizing functions* for sets.

We assume that $+\infty - \infty = +\infty$ and $\alpha(+\infty) = +\infty$, $\alpha(-\infty) = -\infty$ for $\alpha > 0$. In this thesis, let ψ be an extended real valued function from $2^Y \setminus \{\emptyset\}$ to $\overline{\mathbb{R}}$. In usual, a convex set is defined in a vector space and a convex function is defined as a map from a vector space to the real field or some real vector space. However, the family of sets 2^Y or its subset is not a vector space in the usual sense, but we would defined and treat some similarity of convexity of a family of sets $\mathscr{A} \subset 2^Y \setminus \{\emptyset\}$ and ψ in the same way.

Definition 2.11. Let $\mathscr{A} \subset 2^Y \setminus \{\emptyset\}$. $\mathscr{A}$ is said to be *convex* if for each $A_1, A_2 \in \mathscr{A}$ and $\lambda \in (0,1)$,

$$\lambda A_1 + (1-\lambda)A_2 \in \mathscr{A}.$$

Definition 2.12.

(i) A function ψ is said to be *convex* if for each $A_1, A_2 \in 2^Y \setminus \{\emptyset\}$ and $\lambda \in (0,1)$,

$$\psi(\lambda A_1 + (1-\lambda)A_2) \leq \lambda\psi(A_1) + (1-\lambda)\psi(A_2).$$

(ii) A function ψ is said to be *concave* if for each $A_1, A_2 \in 2^Y \setminus \{\emptyset\}$ and $\lambda \in (0,1)$,

$$\psi(\lambda A_1 + (1-\lambda)A_2) \geq \lambda\psi(A_1) + (1-\lambda)\psi(A_2).$$

Definition 2.13. A function ψ is said to be *j-monotone with respect to* $\leq_K^{(j)}$ if

$$A \leq_K^{(j)} B \ \textit{ implies } \ \psi(A) \leq \psi(B).$$

In the cases of $\psi = I_{e,B}^{(j)}, S_{e,B}^{(j)}$, as in [3], the monotonicity of $I_{e,B}^{(j)}, S_{e,B}^{(j)}$ hold only in the case of $j = 1, \ldots, 5$. In other words, for $C, D \in 2^Y \setminus \{\emptyset\}$, $C \leq_K^{(j)} D$ implies $I_{e,B}^{(j)}(C) \leq I_{e,B}^{(j)}(D)$ and $S_{e,B}^{(j)}(C) \leq S_{e,B}^{(j)}(D)$.

Finally, we show some convexity and concavity of $I_{e,B}^{(j)}$ and $S_{e,B}^{(j)}$ in the sense of Definition 2.12.

Proposition 2.14. *For $B \in 2^Y \setminus \{\emptyset\}$ and a direction $e \in \operatorname{int} K$, the following statements hold*:

(i) *For each $j = 1, 2, 3$, $I_{e,B}^{(j)}$ is convex*;

(ii) *For each $j = 4, 5, 6$, if B is $(-K)$-convex, then $I_{e,B}^{(j)}$ is convex.*

Proof. We prove the case of $j = 5$ in (ii). Others in this Proposition are proved in a similar way. Suppose that B is $(-K)$-convex. Let $A_1, A_2 \in 2^Y \setminus \{\emptyset\}$, $\lambda \in (0,1)$, $\alpha_1 := I_{e,B}^{(5)}(A_1)$, and $\alpha_2 := I_{e,B}^{(5)}(A_2)$. We need to consider three cases: a) $\alpha_1 = +\infty$ or $\alpha_2 = +\infty$; b) $\alpha_1, \alpha_2 \in \mathbb{R}$; c) otherwise. We consider only b) because a) and c) can be proved easily. For any $s > 0$,

$$A_1 \subset (\alpha_1 + s)e + B - K, \ A_2 \subset (\alpha_2 + s)e + B - K.$$

Since B is $(-K)$-convex,

$$\begin{aligned}\lambda A_1 + (1-\lambda)A_2 &\subset \lambda\{(\alpha_1 + s)e + B - K\} + (1-\lambda)\{(\alpha_2 + s)e + B - K\} \\ &\Rightarrow \lambda A_1 + (1-\lambda)A_2 \subset (\lambda\alpha_1 + (1-\lambda)\alpha_2 + s)e + B - K.\end{aligned}$$

Therefore,

$$I_{e,B}^{(5)}(\lambda A_1 + (1-\lambda)A_2) \leq \lambda\alpha_1 + (1-\lambda)\alpha_2 + s.$$

As $s > 0$ is arbitrary,

$$I^{(5)}_{e,B}(\lambda A_1 + (1-\lambda)A_2) \leq \lambda\alpha_1 + (1-\lambda)\alpha_2.$$

□

Proposition 2.15. *For $B \in 2^Y \setminus \{\emptyset\}$ and a direction $e \in \operatorname{int} K$, the following statements hold:*

(i) *For each $j = 1, 4, 5$, $S^{(j)}_{e,B}$ is concave;*

(ii) *For each $j = 2, 3, 6$, if B is K-convex, then $S^{(j)}_{e,B}$ is concave.*

Proof. In similar ways in Proposition 2.14, the statements are proved. □

3 Main Results

In this section, we show two results about compositions of set-valued maps and monotone scalarizing functions.

3.1 Characterization of compositions of set-valued map and unified types of scalarizing functions

At first, we show four properties of unified scalarizing functions.

Proposition 3.1. ([6]) *For $B \in 2^Y \setminus \{\emptyset\}$, the following statements hold:*

(i) *If B is K-proper, then $I^{(3)}_{e,B}(B) = 0$;*

(ii) *If B is $(-K)$-proper, then $I^{(5)}_{e,B}(B) = 0$.*

Proposition 3.2. ([4]) *For $A, B \in 2^Y \setminus \{\emptyset\}$ and $r \in \mathbb{R}$, the following statements hold:*

(i) *If A is K-closed, then $I^{(3)}_{e,B}(A) \leq r$ implies $A \leq^{(3)}_K re + B$;*

(ii) *If B is $(-K)$-closed, then $I^{(5)}_{e,B}(A) \leq r$ implies $A \leq^{(5)}_K re + B$.*

Using the same methods of proofs for Proposition 3.1 and 3.2, we can prove similar result for $S^{(3)}_{e,B}B$ and $S^{(5)}_{e,B}B$ as follows.

Proposition 3.3. *For $B \in 2^Y \setminus \{\emptyset\}$, the following statements hold:*

(i) *If B is K-proper, then $S^{(3)}_{e,B}(B) = 0$;*

(ii) *If B is $(-K)$-proper, then $S^{(5)}_{e,B}(B) = 0$.*

Proposition 3.4. *For $A, B \in 2^Y \setminus \{\emptyset\}$ and $r \in \mathbb{R}$, the following statements hold:*

(i) *If B is K-closed, then $S^{(3)}_{e,B}(A) \geq r$ implies $re + B \leq^{(3)}_K A$;*

(ii) *If A is $(-K)$-closed, then $S^{(5)}_{e,B}(A) \geq r$ implies $re + B \leq^{(5)}_K A$.*

In Proposition 3.2, $I_{e,B}^{(3)}(A) \le r$ and $I_{e,B}^{(5)}(A) \le r$ are characterized by $\le_K^{(3)}$ and $\le_K^{(5)}$, respectively. The inverse of the relation in Proposition 3.2 are clear by the definition of $I_{e,B}^{(3)}$ and $I_{e,B}^{(5)}$. However, it is unclear for the set-relation between A and $re+B$ in the case of $I_{e,B}^{(j)}(A) < r$. Such problems are discussed as the following.

Theorem 3.5. *For $A, B \in 2^Y \setminus \{\emptyset\}$ and $r \in \mathbb{R}$, $I_{e,B}^{(3)}(A) < r$ implies $A \le_{\mathrm{int}K}^{(3)} re + B$. The converse is true if B is K-compact.*

Proof. Since $I_{e,B}^{(3)}(A) < r$, there exists $\epsilon > 0$ such that $I_{e,B}^{(3)}(A) < r - \epsilon$. Then there exists $\overline{r} \in \mathbb{R}$ such that

$$I_{e,B}^{(3)}(A) < \overline{r} < r - \epsilon \text{ and } A \le_K^{(3)} \overline{r}e + B \ \ (\Leftrightarrow \overline{r}e + B \subset A + K).$$

By Proposition 2.3, we obtain $(\overline{r}+s)e + B \subset A + K$ for any $s > 0$. Therefore,

$$\begin{aligned} (r-\epsilon+s)e + B &= (\overline{r}+s)e + B + (r-\epsilon-\overline{r})e \\ &\subset A + K + \mathrm{int}\ K \\ &= A + \mathrm{int}\ K. \end{aligned}$$

We choose $s = \epsilon$, then $re + B \subset A + \mathrm{int}\ K$, that is, $A \le_{\mathrm{int}K}^{(3)} re + B$.

In the case that B is K-compact. We show that $A \le_{\mathrm{int}K}^{(3)} re + B$ implies $I_{e,B}^{(3)}(A) < r$. Since $A \le_{\mathrm{int}K}^{(3)} re + B \Leftrightarrow re + B \subset A + \mathrm{int}\ K$, for any $b \in B$, there exists $a_b \in B$ such that

$$re + b \in a_b + \mathrm{int}\ K.$$

As each $a_b + \mathrm{int}\ K$ is an open set, there exists $\epsilon_b > 0$ such that

$$b \in a_b - (r - \epsilon_b)e + K.$$

Let $\alpha_b \in (0, \epsilon_b)$. Then

$$b \in a_b - (r - \epsilon_b + \alpha_b)e + \alpha_b e + K \subset A - (r - \epsilon_b + \alpha_b)e + \mathrm{int}\ K + K.$$

Therefore,

$$B \subset \bigcup_{b \in B} \{A - (r - \epsilon_b + \alpha_b)e + \mathrm{int}\ K + K\},$$

that is, $\{A - (r - \epsilon_b + \alpha_b)e + \mathrm{int}\ K + K \mid b \in B\}$ is a cover of B and each $A - (r - \epsilon_b + \alpha_b)e + \mathrm{int}\ K + K$ is an open set. Since B is K-compact, there exists $\{b_1, \dots, b_n\} \subset B$ such that

$$B \subset \bigcup_{i=1}^{n} \{A - (r - \epsilon_{b_i} + \alpha_{b_i})e + \mathrm{int}\ K + K\}.$$

Let $j \in \{1, \dots, n\}$ be the one such that

$$r - \epsilon_{b_j} + \alpha_{b_j} = \max\{r - \epsilon_{b_i} + \alpha_{b_i} \mid i = 1, \dots, n\}.$$

Therefore, $B \subset A - (r - \epsilon_{b_j} + \alpha_{b_j})e + \mathrm{int}\ K + K \subset A - (r - \epsilon_{b_j} + \alpha_{b_j})e + K$. That is, $B + (r - \epsilon_{b_j} + \alpha_{b_j})e \subset A + K$. As a result,

$$I_{e,B}^{(3)}(A) \le (r - \epsilon_{b_j} + \alpha_{b_j}) < r.$$

□

We have similar statements, which are Theorem 3.6 and Theorem 3.7, as well as Theorem 3.5.

Theorem 3.6. *For $A, B \in 2^Y \setminus \{\emptyset\}$ and $r \in \mathbb{R}$, the following statements hold*:

(i) $I^{(5)}_{e,B}(A) < r$ *implies* $A \leq^{(5)}_{\mathrm{int}K} re + B$. *The converse is true if A is $(-K)$-compact ;*

(ii) $S^{(3)}_{e,B}(A) > r$ *implies* $re + B \leq^{(3)}_{\mathrm{int}K} A$. *The converse is true if A is K-compact ;*

(iii) $S^{(5)}_{e,B}(A) > r$ *implies* $re + B \leq^{(5)}_{\mathrm{int}K} A$. *The converse is true if A is $(-K)$-compact.*

Proof. In a similar way in Theorem 3.5, the statements are proved. □

Theorem 3.7. *For $A, B \in 2^Y \setminus \{\emptyset\}$ and $r \in \mathbb{R}$, the following statements hold*: *For $j = 1, 2, 4, 6$,*

(i) *If* $I^{(j)}_{e,B}(A) < r$, *then* $A \leq^{(j)}_{\mathrm{int}K} re + B$;

(ii) *If* $S^{(j)}_{e,B}(A) > r$, *then* $re + B \leq^{(j)}_{\mathrm{int}K} A$.

Proof. In a similar way in Theorem 3.5, the statements are proved. □

In [3, 4, 9], the authors study the inherited properties of convexity and continuity of $I^{(j)}_{e,B} \circ F$ and $S^{(j)}_{e,B} \circ F$ from those of F. In the below, we show several inverse results, that is, we drive convexities and concavities of F from those of $I^{(j)}_{e,B} \circ F$ and $S^{(j)}_{e,B} \circ F$.

Theorem 3.8. *For $B \in 2^Y \setminus \{\emptyset\}$, the following statements hold*:

(i) *If F is type (3)-lower quasiconvex, then $I^{(3)}_{e,B} \circ F$ is quasiconvex for any $B \in 2^Y \setminus \{\emptyset\}$. The converse is true if F is K-closed-valued*;

(ii) *Let F be K-closed-valued, K-convex-valued, and K-proper-valued. If $S^{(3)}_{e,B} \circ F$ is convex for any K-closed, K-convex, and K-proper set $B \in 2^Y \setminus \{\emptyset\}$, then F is type (3) K-convex*;

(iii) *Let F be K-closed-valued and K-proper-valued. If $S^{(3)}_{e,B} \circ F$ is quasiconvex for any $B \in 2^Y \setminus \{\emptyset\}$, then F is type (3) properly quasi K-convex*;

(iv) *Let F be $(-K)$-closed-valued and $(-K)$-proper-valued. If $S^{(5)}_{e,B} \circ F$ is quasiconvex for any $B \in 2^Y \setminus \{\emptyset\}$, then F is type (5) properly quasi K-convex*;

(v) *Let F be $(-K)$-closed-valued and cone-valued. If $I^{(5)}_{e,B} \circ F$ is quasiconvex for any $B \in 2^Y \setminus \{\emptyset\}$, then F is type (5) K-convex*;

(vi) *Let F be $(-K)$-closed-valued and cone-valued. If $S^{(5)}_{e,B} \circ F$ is convex for $(-K)$-closed cone $B \in 2^Y \setminus \{\emptyset\}$, then F is type (5) K-convex.*

Proof. (i) Assume that F is type (3)-lower quasiconvex. We prove that $\mathrm{lev}(I^{(3)}_{e,B} \circ F) := \{x \in X \mid (I^{(3)}_{e,B} \circ F)(x) \leq \alpha\}$ is convex for any $\alpha \in \mathbb{R}$. Let $x_1, x_2 \in \mathrm{lev}(I^{(3)}_{e,B} \circ F, \leq, \alpha)$ and $\lambda \in (0, 1)$. We need to consider two cases: a) $(I^{(3)}_{e,B} \circ F)(x_1) \in \mathbb{R}$ or $(I^{(3)}_{e,B} \circ F)(x_2) \in \mathbb{R}$; b) $(I^{(3)}_{e,B} \circ F)(x_1) = -\infty$ and $(I^{(3)}_{e,B} \circ F)(x_2) = -\infty$. We only consider a) because b) can be proved in an analogous argument of a). In the case of a), for $i = 1, 2$, $(I^{(3)}_{e,B} \circ F)(x_i) < \alpha + s$ for any $s > 0$. By Theorem 3.5, $(\alpha + s)e + B \subset F(x_i) + \text{int } K \subset F(x_i) + K$. Therefore,

$$(\alpha + s)e + B \subset (F(x_1) + K) \cap (F(x_2) + K).$$

Since F is type (3)-lower quasiconvex, we have

$$(\alpha + s)e + B \subset F(\lambda x_1 + (1-\lambda)x_2) + K.$$

That is, $I^{(3)}_{e,B}(F(\lambda x_1 + (1-\lambda)x_2)) \le \alpha + s$. As $s > 0$ is arbitrary,

$$(I^{(3)}_{e,B} \circ F)(\lambda x_1 + (1-\lambda)x_2) \le \alpha.$$

Conversely, We show that if $I^{(3)}_{e,B} \circ F$ is quasiconvex for any $B \in 2^Y \setminus \{\emptyset\}$ and F is K-closed-valued, then F is type (3)-lower quasiconvex. Since type (3)-lower quasiconvex is equivalent to Ferro type (-1)-quasiconvex, we prove that F is Ferro type (-1)-quasiconvex. Let $y \in Y, x_1, x_2 \in F^{-1}(y - K), \lambda \in (0,1)$. For $i = 1, 2$, $y \in F(x_i) + K$, that is, $F(x_i) \le^{(3)}_K \{y\}$. Due to the monotonicity of $I^{(3)}_{e,B}$, $I^{(3)}_{e,B}(F(x_i)) \le I^{(3)}_{e,\{y\}}(\{y\})$. Since K is proper, $\{y\}$ is K-proper. Therefore, by Proposition 3.1, $I^{(3)}_{e,\{y\}}(\{y\}) = 0$. As a result, $I^{(3)}_{e,\{y\}}(F(x_i)) \le 0$. Since $I^{(3)}_{e,\{y\}} \circ F$ is quasiconvex,

$$(I^{(3)}_{e,\{y\}} \circ F)(\lambda x_1 + (1-\lambda)x_2) \le 0.$$

By Proposition 3.2,

$$F(\lambda x_1 + (1-\lambda)x_2) \le^{(3)}_K \{y\},$$

that is, $y \in F(\lambda x_1 + (1-\lambda)x_2) + K$.

(ii) Let $x_1, x_2 \in X, \lambda \in (0,1)$, and $B := F(\lambda x_1 + (1-\lambda)x_2)$. Then B is K-closed, K-convex, and K-proper set. Therefore,

$$\begin{aligned} 0 &= S^{(3)}_{e,B}(B) \\ &= S^{(3)}_{e,B}(F(\lambda x_1 + (1-\lambda)x_2)) \\ &\le \lambda(S^{(3)}_{e,B} \circ F)(x_1) + (1-\lambda)(S^{(3)}_{e,B} \circ F)(x_2) \\ &\le S^{(3)}_{e,B}(\lambda F(x_1) + (1-\lambda)F(x_2)) \end{aligned}$$

By Proposition 3.4, $F(\lambda x_1 + (1-\lambda)x_2) = B \le^{(3)}_K \lambda F(x_1) + (1-\lambda)F(x_2)$.

(iii) We give the proof by the method of contradiction. We assume that F is not type (3) properly quasi K-convex. Then, there exist $x_1, x_2 \in X$ and $\lambda \in (0,1)$ such that

$$F(\lambda x_1 + (1-\lambda)x_2) \not\le^{(3)}_K F(x_1) \text{ and } F(\lambda x_1 + (1-\lambda)x_2) \not\le^{(3)}_K F(x_2).$$

Let $B := F(\lambda x_1 + (1-\lambda)x_2)$. By the contraposition of Proposition 3.4, $S^{(3)}_{e,B}(F(x_i)) < 0$ for $i = 1, 2$. Since $S^{(3)}_{e,B} \circ F$ is quasiconvex,

$$0 = S^{(3)}_{e,B}(B) = (S^{(3)}_{e,B} \circ F)(F(\lambda x_1 + (1-\lambda)x_2)) < 0.$$

But this is contradiction.

(iv) In a similar way of (iii), this statement is proved.

(v) Let $x_1, x_2 \in X$, $\lambda \in (0,1)$, and $B := F(x_1) \cup F(x_2)$. For $i = 1, 2$, $F(x_i) \subset 0e + B - K$. By the definition of $I^{(5)}_{e,B}$, $I^{(5)}_{e,B}(F(x_i)) \leq 0$. Since $I^{(5)}_{e,B} \circ F$ is quasiconvex,

$$(I^{(5)}_{e,B} \circ F)(\lambda x_1 + (1-\lambda)x_2)) \leq 0.$$

By Proposition 3.2, $F(\lambda x_1 + (1-\lambda)x_2)) \leq^{(5)}_K B$. As F is cone-valued,

$$B \subset \lambda F(x_1) + (1-\lambda)F(x_2) - K.$$

As a result, $F(\lambda x_1 + (1-\lambda)x_2)) \leq^{(5)}_K \lambda F(x_1) + (1-\lambda)F(x_2)$.

(vi) In a similar way of (v), this statement is proved. □

Theorem 3.9. *For $B \in 2^Y \setminus \{\emptyset\}$, the following statements hold:*

(i) *If F is type (5)-lower quasiconcave, then $S^{(5)}_{e,B} \circ F$ is quasiconvex. The converse is true if F is $(-K)$-closed-valued and $S^{(5)}_{e,\{y\}} \circ F$ is quasiconcave for any $y \in Y$;*

(ii) *Let F be $(-K)$-closed-valued and cone-valued. If $S^{(5)}_{e,B} \circ F$ is convex for any $(-K)$-closed cone $B \in 2^Y \setminus \{\emptyset\}$, then F is type (5) K-convex;*

(iii) *Let F be $(-K)$-closed-valued and $(-K)$-proper-valued. If $I^{(5)}_{e,B} \circ F$ is quasiconvex for any $B \in 2^Y \setminus \{\emptyset\}$, then F is type (5) properly quasi K-concave;*

(iv) *Let F be K-closed-valued and K-proper-valued. If $I^{(3)}_{e,B} \circ F$ is quasiconcave for any $B \in 2^Y \setminus \{\emptyset\}$, then F is type (3) properly quasi K-concave;*

(v) *Let F be K-closed-valued and cone-valued. If $S^{(3)}_{e,B} \circ F$ is quasiconcave for any K-closed cone $B \in 2^Y \setminus \{\emptyset\}$, then F is type (3) K-concave;*

(vi) *Let F be K-closed-valued and cone-valued. If $I^{(3)}_{e,B} \circ F$ is concave for any K-closed cone $B \in 2^Y \setminus \{\emptyset\}$, then F is type (3) K-convex.*

Proof. By similar ways to the proof of theorem 3.8, the statements are proved. □

3.2 Convexity properties for compositions of set-valued map and monotone scalarizing function

$I^{(j)}_{e,B}$ and $S^{(j)}_{e,B}$ have convexity and concavity in the sense of Definition 2.12, respectively. Moreover, they have monotonicity (see [3]). Therefore, we show a certain essentiality of the results of [3] by investigating convexity properties for compositions of set-valued map and monotone scalarizing function.

At first, we define quasiconvexity and quasiconcavity of ψ.

Definition 3.10.

(i) A function ψ is said to be *quasiconvex* if for any $\alpha \in \mathbb{R}$, $\mathrm{lev}(\psi, \leq, \alpha) := \{A \in 2^Y \setminus \{\emptyset\} \mid \psi(A) \leq \alpha\}$ is convex.

(ii) A function ψ is said to be *quasiconcave* if for any $\alpha \in \mathbb{R}$, $\mathrm{lev}(\psi, \geq, \alpha) := \{A \in 2^Y \setminus \{\emptyset\} \mid \psi(A) \geq \alpha\}$ is convex.

Next, we show convexity and concavity properties for compositions of F and ψ. The following statements are proved by similar ways, hence we prove only Theorem 3.12 and Theorem 3.15.

Theorem 3.11. *Let ψ be j-monotone with respect to $\leq_K^{(j)}$ and convex. Then the following statements hold*:

(i) *If F is type (j) K-convex, then $\psi \circ F$ is convex*;

(ii) *If F is type (j) naturally quasi K-convex, then $\psi \circ F$ is quasiconvex.*

Theorem 3.12. *Let ψ be j-monotone with respect to $\leq_K^{(j)}$ and quasiconvex. Then the following statements hold*:

(i) *If F is type (j) K-convex, then $\psi \circ F$ is quasiconvex*;

(ii) *If F is type (j) naturally quasi K-convex, then $\psi \circ F$ is quasiconvex.*

Proof. We prove only (ii) because (i) is proved in a similar way. Suppose that ψ is quasiconvex, j-monotone with respect to $\leq_K^{(j)}$ and F is type (j) naturally quasi K-convex. Let $\alpha \in \mathbb{R}$, $x_1, x_2 \in \operatorname{lev}(\psi \circ \mathrm{F}, \leq, \alpha)$, and $\lambda \in (0,1)$. Since $\psi \circ F(x_1), \psi \circ F(x_2) \leq \alpha$, $F(x_1), F(x_2) \in \operatorname{lev}(\psi, \leq, \alpha)$. As F is type (j) naturally quasi K-convex, there exists $\mu \in [0,1]$ such that
$$F(\lambda x_1 + (1-\lambda)x_2) \leq_K^{(j)} \mu F(x_1) + (1-\mu)F(x_2).$$
Since ψ is quasicovex, for this μ,
$$\mu F(x_1) + (1-\mu)F(x_2) \in \operatorname{lev}(\psi, \leq, \alpha).$$
As ψ is j-monotone with respect to $\leq_K^{(j)}$,
$$\begin{aligned} \psi \circ F(\lambda x_1 + (1-\lambda)x_2) &\leq \psi(\mu F(x_1) + (1-\mu)F(x_2)) \\ &\leq \alpha. \end{aligned}$$
Therefore, $\lambda x_1 + (1-\lambda)x_2 \in \operatorname{lev}(\psi \circ F,\ \leq,\ \alpha)$, that is, $\psi \circ F$ is quasiconvex. □

Theorem 3.13. *Let ψ be j-monotone with respect to $\leq_K^{(j)}$ and concave. Then the following statements hold*:

(i) *If F is type (j) K-concave, then $\psi \circ F$ is concave*;

(ii) *If F is type (j) naturally quasi K-concave, then $\psi \circ F$ is quasiconcave.*

Theorem 3.14. *Let ψ be j-monotone with respect to $\leq_K^{(j)}$ and quasiconcave. Then the following statements hold*:

(i) *If F is type (j) K-concave, then $\psi \circ F$ is quasiconcave*;

(ii) *If F is type (j) naturally quasi K-concave, then $\psi \circ F$ is quasiconcave.*

Theorem 3.15. *Let ψ be j-monotone with respect to $\leq_K^{(j)}$. Then the following statements hold*:

(i) *If F is type (j) properly quasi K-convex, then $\psi \circ F$ is quasiconvex*;

(ii) *If F is type (j) properly quasi K-concave, then $\psi \circ F$ is quasiconcave.*

Proof. We prove only (i) because (ii) is proved in a similar way. Suppose that ψ is j-monotone with respect to $\leq_K^{(j)}$ and F is type (j) properly quasi K-convex. Let $\alpha \in \mathbb{R}$, $x_1, x_2 \in X$, and $\lambda \in (0, 1)$. Since F is type (j) properly quasi K-convex,

$$F(\lambda x_1 + (1-\lambda)x_2) \leq_K^{(j)} F(x_1) \text{ or } F(\lambda x_1 + (1-\lambda)x_2) \leq_K^{(j)} F(x_2),$$

and that ψ is j-monotone with respect to $\leq_K^{(j)}$,

$$\psi \circ F(\lambda x_1 + (1-\lambda)x_2) \leq \max\{\psi \circ F(x_1), \psi \circ F(x_2)\},$$

that is, $\psi \circ F$ is quasiconvex. □

The results as mentioned above are collected as below. We denote "convex", "quasiconvex", "concave", and "quasiconvex" by "cv", "qcv", "cc", and "qcc" for short in table, respectively.

Table 1: Summary about convexity and concavity of a composite function

Assumption			Conclusion
ψ		F	$\psi \circ F$
j-monotone	cv	type (j) K-convex	cv
	cv	type (j) naturally quasi K-convex	qcv
	qcv	type (j) K-convex	
		type (j) naturally quasi K-convex	
	cc	type (j) K-concave	cc
	cc	type (j) naturally quasi K-concave	qcc
	qcc	type (j) K-concave	
		type (j) naturally quasi K-concave	
j-monotone		type (j) properly quasi K-convex	qcv
		type (j) properly quasi K-concave	qcc

Acknowledgment. The authors would like to express their sincere gratitude to the referees and the editors for a lot of valuable suggestions. The authors thank Professor Xuefeng Liu and Mr. Juan Luis Rodenas Pedregosa for his helpful comments on the paper.

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Manuscript received 1 March 2015
revised 16 April 2015
accepted for publication 18 August 2015

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