Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 12, No. 1, pp. 87–100, January 2016

A CONSTRAINT QUALIFICATION CHARACTERIZING SURROGATE DUALITY FOR QUASICONVEX PROGRAMMING*

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Abstract: In this paper, we study a constraint qualification which completely characterizes surrogate duality for quasiconvex programming. We show that the closed cone constraint qualification for surrogate duality is a necessary and sufficient constraint qualification for surrogate strong duality and surrogate min-max duality via quasiconvex programming with convex constraints. Also, we compare our constraint qualification with previous ones for Lagrange duality and surrogate duality.

Key words: *surrogate strong duality, surrogate min-max duality, quasiconvex programming, constraint qualification.*

Mathematics Subject Classification: *90C26, 26B25.*

1 Introduction

In mathematical programming, constraint qualifications are essential elements for duality theory. The best-known constraint qualifications are the interior point conditions, also known as the Slater-type constraint qualifications. In convex programming, it is well known that the Slater condition assures the existence of Lagrange multipliers. However, such constraint qualifications are not always satisfied for problems that arise in applications. The lack of constraint qualifications can cause theoretical and numerical difficulties. Recently, some conditions have been investigated as necessary and sufficient constraint qualifications for Lagrange duality. For convex programming with vector-valued constraints, Jeyakumar, Dinh and Lee [10] developed the closed cone constraint qualification (CCCQ) involving epigraphs and extending the Slater-type conditions. Jeyakumar [9] demonstrated that CCCQ is a necessary and sufficient constraint qualification for Lagrange strong duality. Jeyakumar [9] introduced [CQ2] as a necessary and sufficient constraint qualification for Lagrange min-max duality. For convex programming with real-valued constraints, Goberna, Jeyakumar, and López [4] introduced Farkas Minkowski (FM) as a necessary and sufficient constraint qualification for Lagrange strong duality, and locally Farkas Minkowski (LFM) as a necessary and sufficient constraint qualification for Lagrange min-max duality. Li, Ng and Pong [11] established the basic constraint qualification (BCQ) as a necessary and sufficient constraint qualification for the optimality condition in convex programming. Finding a necessary and sufficient constraint qualification is one of the purposes of the research on optimization,

*This work was partially supported by JSPS KAKENHI Grant Numbers 15K17588, 25400205.

see [2, 4, 9, 10, 11, 16, 17, 18, 19, 20, 21]. In general, necessary and sufficient constraint qualifications for strong duality are different from necessary and sufficient constraint qualifications for min-max duality. Actually, CCCQ implies [CQ2], however, the converse is not always true. In other words, Lagrange strong duality and Lagrange min-max duality are not equivalent.

In mathematical programming, surrogate duality is widely studied by many authors, for example, see [1, 3, 5, 6, 7, 12, 13, 19, 21]. Surrogate duality enables one to replace the problem by a simpler problem, in which the constraint function is a scalar one. In [19], we investigated necessary and sufficient constraint qualifications for surrogate duality via quasiconvex programming. We introduced the closed cone constraint qualification for surrogate duality (S-CCCQ) as a necessary and sufficient constraint qualification for surrogate strong duality via quasiconvex programming with convex constraints. We investigated the basic constraint qualification for surrogate duality (S-BCQ) as a necessary and sufficient constraint qualification for surrogate min-max duality via convex programming. However, necessary and sufficient constraint qualifications for surrogate min-max duality via quasiconvex programming with convex constraints have not been investigated yet as far as we know. Also, relations between surrogate strong duality and surrogate min-max duality have not been studied yet.

In this paper, we study a constraint qualification which completely characterizes surrogate duality via quasiconvex programming. We show that S-CCCQ is a necessary and sufficient constraint qualification for surrogate strong duality and surrogate min-max duality via quasiconvex programming with convex constraints. Also, we compare our constraint qualification with previous ones for Lagrange duality and surrogate duality.

The remainder of the present paper is organized as follows. In Section 2, we introduce some preliminaries. In Section 3, we study a constraint qualification which completely characterizes surrogate duality via quasiconvex programming. In Section 4, we compare our constraint qualification with previous ones.

2 Preliminaries

Let X be a locally convex Hausdorff topological vector space, X^* the continuous dual space of X, and $\langle x^*, x\rangle$ the value of a functional $x^* \in X^*$ at $x \in X$. Given a set $A^* \subset X^*$, we denote the w^*-closure, the interior, the boundary, the convex hull, and the conical hull generated by A^*, by $\operatorname{cl} A^*$, $\operatorname{int} A^*$, $\operatorname{bd} A^*$, $\operatorname{co} A^*$, and $\operatorname{cone} A^*$, respectively. The indicator function δ_A of $A \subset X$ is defined by

$$\delta_A(x) := \begin{cases} 0, & x \in A, \\ \infty, & \textit{otherwise}. \end{cases}$$

The normal cone of A at $x \in A$ is $N_A(x) := \{v \in X^* \mid \forall y \in A, \langle v, y - x\rangle \leq 0\}$. Let f be a function from X to $\overline{\mathbb{R}} := [-\infty, \infty]$. A function f is said to be proper if for all $x \in X$, $f(x) > -\infty$ and there exists $x_0 \in X$ such that $f(x_0) \in \mathbb{R}$. We denote the domain of f by $\mathrm{dom} f := \{x \in X \mid f(x) < \infty\}$. The epigraph of f is $\mathrm{epi} f := \{(x, r) \in X \times \mathbb{R} \mid f(x) \leq r\}$, and f is said to be convex if $\mathrm{epi} f$ is convex. The subdifferential of f at x is defined as $\partial f(x) := \{v \in X^* \mid \forall y \in X, f(y) \geq f(x) + \langle v, y - x\rangle\}$. The Fenchel conjugate of f, $f^* : X^* \to \overline{\mathbb{R}}$, is defined as $f^*(v) := \sup_{x \in X}\{\langle v, x\rangle - f(x)\}$. Define level sets of f with respect to a binary relation $\diamond$ on $\overline{\mathbb{R}}$ as

$$L(f, \diamond, \beta) := \{x \in X \mid f(x) \diamond \beta\}$$

for each $\beta \in \mathbb{R}$. A function f is said to be quasiconvex if for each $\beta \in \mathbb{R}$, $L(f, \leq, \beta)$ is convex. Any convex function is quasiconvex, but the converse is not generally true. A function f is said to be quasiaffine if f and $-f$ are quasiconvex.

Let Y be a locally convex Hausdorff topological vector space, partially ordered by a nonempty, closed, and convex cone $K \subset Y$, that is, for y, $z \in Y$, the notation $y \leq_K z$ will mean $z - y \in K$, Y^* the continuous dual space of Y, and g a function from X to Y. The positive polar cone of K is $K^+ := \{\lambda \in Y^* \mid \forall y \in K, \langle \lambda, y \rangle \geq 0\}$. A function g is said to be K-convex if for all x_1, $x_2 \in X$, and $\alpha \in [0,1]$, $(1-\alpha)g(x_1) + \alpha g(x_2) \in g((1-\alpha)x_1 + \alpha x_2) + K$. It is well known that g is K-convex if and only if $\lambda \circ g$ is convex for all $\lambda \in K^+$.

Let f be a function from X to $\overline{\mathbb{R}}$, g a continuous K-convex function from X to Y, C a closed convex subset of X, and $A = C \cap g^{-1}(-K)$. In convex programming, the following Lagrange strong duality has been studied mainly:

$$\inf_{x \in A} f(x) = \max_{\lambda \in K^+} \inf_{x \in C} \{f(x) + \lambda \circ g(x)\}.$$

Also, the following Lagrange min-max duality has attracted the attention of many researchers:

$$\min_{x \in A} f(x) = \max_{\lambda \in K^+} \inf_{x \in C} \{f(x) + \lambda \circ g(x)\}.$$

In quasiconvex programming, the following surrogate strong duality and surrogate min-max duality have been investigated:

$$\inf_{x \in A} f(x) = \max_{\lambda \in K^+} \inf\{f(x) \mid x \in C, \lambda \circ g(x) \leq 0\},$$

$$\min_{x \in A} f(x) = \max_{\lambda \in K^+} \inf\{f(x) \mid x \in C, \lambda \circ g(x) \leq 0\}.$$

We can easily check that the following inequality holds: for each $\lambda \in K^+$,

$$\inf_{x \in A} f(x) \geq \inf\{f(x) \mid x \in C, \lambda \circ g(x) \leq 0\} \geq \inf_{x \in C} \{f(x) + \lambda \circ g(x)\}.$$

Hence, if Lagrange duality holds, then surrogate duality also holds. Conversely, even if surrogate duality holds, Lagrange duality does not always hold, see Section 4 for more details. Recently, some conditions have been investigated as necessary and sufficient constraint qualifications for Lagrange strong duality and Lagrange min-max duality, see [2, 4, 9, 10, 11]. Since necessary and sufficient constraint qualifications for Lagrange strong duality are different from necessary and sufficient constraint qualifications for Lagrange min-max duality, Lagrange strong duality and Lagrange min-max duality are not equivalent. On the other hand, in [19], we investigate a necessary and sufficient constraint qualification for surrogate strong duality via quasiconvex programming with convex constraints and a necessary and sufficient constraint qualification for surrogate min-max duality via convex programming. However, necessary and sufficient constraint qualifications for surrogate min-max duality via quasiconvex programming with convex constraints have not been investigated yet as far as we know. Also, relations between surrogate strong duality and surrogate min-max duality have not been studied yet.

3 A Constraint Qualification Which Completely Characterizes Surrogate Duality

In this paper, we consider the following optimization problem:

$$\begin{cases} \text{minimize } f(x), \\ \text{subject to } x \in A = C \cap g^{-1}(-K), \end{cases}$$

where X and Y are locally convex Hausdorff topological vector spaces, $K \subset Y$ is a nonempty, closed, and convex cone, Y is partially ordered by K, f is a function from X to $\overline{\mathbb{R}}$, g is a continuous K-convex function from X to Y, and C is a closed convex subset of X. Assume that $A = C \cap g^{-1}(-K)$ is nonempty. Let $B_\lambda = C \cap L(\lambda \circ g, \leq, 0)$ for each $\lambda \in K^+$.

In this section, we study a constraint qualification which completely characterize surrogate duality via quasiconvex programming. We need the following lemmas.

Lemma 3.1 ([19]). *The following statements hold:*

(i) *for all* $\lambda \in K^+$, $\operatorname{epi} \delta^*_{B_\lambda} = \operatorname{cl}[\operatorname{cone} \operatorname{epi} (\lambda \circ g)^* + \operatorname{epi} \delta^*_C]$,

(ii) $\operatorname{epi} \delta^*_A = \operatorname{cl} \displaystyle\bigcup_{\lambda \in K^+} \operatorname{cl}[\operatorname{cone} \operatorname{epi} (\lambda \circ g)^* + \operatorname{epi} \delta^*_C]$.

An inequality system $\{g(x) \in -K \mid x \in C\}$ is said to satisfy the closed cone constraint qualification for surrogate duality (S-CCCQ) if

$$\bigcup_{\lambda \in K^+} \operatorname{cl}\Big[\operatorname{cone} \operatorname{epi} (\lambda \circ g)^* + \operatorname{epi} \delta^*_C\Big]$$

is w^*-closed. We can check easily that $\{g(x) \in -K \mid x \in C\}$ satisfies S-CCCQ if and only if

$$\operatorname{epi}\delta^*_A \subset \bigcup_{\lambda \in K^+} \operatorname{cl}\Big[\operatorname{cone} \operatorname{epi} (\lambda \circ g)^* + \operatorname{epi} \delta^*_C\Big],$$

in detail, see [19].

Lemma 3.2 ([19]). *The following statements are equivalent:*

(i) $\{g(x) \in -K \mid x \in C\}$ *satisfies S-CCCQ,*

(ii) *for each usc quasiconvex function f from X to $\overline{\mathbb{R}}$,*

$$\inf_{x \in A} f(x) = \max_{\lambda \in K^+} \inf\{f(x) \mid x \in C, \lambda \circ g(x) \leq 0\},$$

(iii) *for each* $v \in X^*$,

$$\inf_{x \in A} v(x) = \max_{\lambda \in K^+} \inf\{v(x) \mid x \in C, \lambda \circ g(x) \leq 0\}.$$

In the following theorem, we show that S-CCCQ completely characterizes surrogate strong duality and surrogate min-max duality for quasiconvex programming with convex constraints. The proof is short and precise because the definition of f_0 is essential.

Theorem 3.3. *The following statements are equivalent:*

(i) $\{g(x) \in -K \mid x \in C\}$ *satisfies S-CCCQ,*

(ii) *for each usc quasiconvex function f from X to $\overline{\mathbb{R}}$,*

$$\inf_{x \in A} f(x) = \max_{\lambda \in K^+} \inf\{f(x) \mid x \in C, \lambda \circ g(x) \leq 0\},$$

(iii) *for each* $v \in X^*$,

$$\inf_{x \in A} v(x) = \max_{\lambda \in K^+} \inf\{v(x) \mid x \in C, \lambda \circ g(x) \leq 0\},$$

(iv) *for each usc quasiconvex function* f *from* X *to* $\overline{\mathbb{R}}$ *which attains its minimum on* A,

$$\min_{x \in A} f(x) = \max_{\lambda \in K^+} \inf\{f(x) \mid x \in C, \lambda \circ g(x) \leq 0\},$$

(v) *for each usc quasiaffine function* f *from* X *to* $\overline{\mathbb{R}}$ *which attains its minimum on* A,

$$\min_{x \in A} f(x) = \max_{\lambda \in K^+} \inf\{f(x) \mid x \in C, \lambda \circ g(x) \leq 0\},$$

(vi) *there exists* $x_0 \in A$ *such that for each usc quasiconvex function* f *from* X *to* $\overline{\mathbb{R}}$ *which attains its minimum on* A *at* x_0,

$$f(x_0) = \min_{x \in A} f(x) = \max_{\lambda \in K^+} \inf\{f(x) \mid x \in C, \lambda \circ g(x) \leq 0\}.$$

(vii) *there exists* $x_0 \in A$ *such that for each usc quasiaffine function* f *from* X *to* $\overline{\mathbb{R}}$ *which attains its minimum on* A *at* x_0,

$$f(x_0) = \min_{x \in A} f(x) = \max_{\lambda \in K^+} \inf\{f(x) \mid x \in C, \lambda \circ g(x) \leq 0\}.$$

Proof. By Lemma 3.2, (i), (ii), and (iii) are equivalent. Also, it is clear that

$$\text{(ii)} \Longrightarrow \text{(iv)} \Longrightarrow \text{(v)} \Longrightarrow \text{(vii), and (ii)} \Longrightarrow \text{(vi)} \Longrightarrow \text{(vii)}.$$

We show that (vii) implies (i). We only show that

$$\mathrm{epi}\delta_A^* \subset \bigcup_{\lambda \in K^+} \mathrm{cl}\,[\mathrm{cone}\ \mathrm{epi}\,(\lambda \circ g)^* + \mathrm{epi}\,\delta_C^*]\,.$$

Let $(x^*, \alpha) \in \mathrm{epi}\,\delta_A^*$. We define a function f_0 from X to $\mathbb{R}$ as follows:

$$f_0(x) = \begin{cases} -\langle x^*, x\rangle\,, & \langle x^*, x\rangle > \delta_A^*(x^*), \\ -\delta_A^*(x^*), & \langle x^*, x\rangle \leq \delta_A^*(x^*). \end{cases}$$

Let $\alpha \in \mathbb{R}$. If $\alpha \geq -\delta_A^*(x^*)$, then

$$L(f_0, \leq, \alpha) = X, \text{ and } L(f_0, \geq, \alpha) = \emptyset.$$

If $\alpha < -\delta_A^*(x^*)$, then

$$L(f_0, \leq, \alpha) = \{x \in X \mid x^*(x) \geq -\alpha\}, \text{ and } L(f_0, \geq, \alpha) = \{x \in X \mid x^*(x) \leq -\alpha\}.$$

This shows that for each $\alpha \in \mathbb{R}$, $L(f_0, \leq, \alpha)$ is convex, and $L(f_0, \geq, \alpha)$ is closed convex, that is, f_0 is usc quasiaffine. We can see that $f_0(x) = -\delta_A^*(x^*)$ for each $x \in A$. This means that f_0 is constant on A, that is, f_0 attains its minimum on A at $x_0 \in A$.

Hence, by the statement (vii), there exists $\bar{\lambda} \in K^+$ such that

$$-\delta_A^*(x^*) = f_0(x_0) = \min_{x \in A} f_0(x) = \inf\{f_0(x) \mid x \in C, \bar{\lambda} \circ g(x) \leq 0\}.$$

Let $x \in B_{\bar{\lambda}}$. We show that $\langle x^*, x\rangle \leq \delta_A^*(x^*)$. Actually, if $\langle x^*, x\rangle > \delta_A^*(x^*)$, then $f_0(x) = -\langle x^*, x\rangle$, and

$$\langle x^*, x\rangle = -f_0(x) \leq \delta_A^*(x^*).$$

This is a contradiction. This implies that $\delta_{B_{\bar{\lambda}}}^*(x^*) \leq \delta_A^*(x^*) \leq \alpha$. Therefore,

$$(x^*, \alpha) \in \operatorname{epi} \delta_{B_{\bar{\lambda}}}^* = \operatorname{cl}[\operatorname{cone} \operatorname{epi} (\bar{\lambda} \circ g)^* + \operatorname{epi} \delta_C^*]$$

because of Lemma 3.1. This shows that (i) holds. □

Remark 3.4. Let $\mathcal{F}$ be a subset of $\{f : X \to \overline{\mathbb{R}}\}$. If

$$X^* \subset \mathcal{F} \subset \{f : X \to \overline{\mathbb{R}}, \text{ usc quasiconvex}\},$$

then S-CCCQ is equivalent that for each $f \in \mathcal{F}$, surrogate strong duality holds. Also, if

$$\{f : X \to \overline{\mathbb{R}}, \text{ usc quasiaffine}\} \subset \mathcal{F} \subset \{f : X \to \overline{\mathbb{R}}, \text{ usc quasiconvex}\},$$

then S-CCCQ is equivalent that for each $f \in \mathcal{F}$, surrogate min-max duality holds. S-CCCQ implies that for each $v \in X^*$, surrogate min-max duality holds. However, the converse is not generally true, since

$$\{f : X \to \overline{\mathbb{R}}, \text{ usc quasiaffine}\} \not\subset X^*.$$

4 Comparisons

In this section, we compare S-CCCQ with previous constraint qualifications for Lagrange duality and surrogate duality.

At first, we introduce a necessary and sufficient constraint qualification for surrogate min-max duality via convex programming. An inequality system $\{g(x) \in -K \mid x \in C\}$ is said to satisfy the basic constraint qualification for surrogate duality (S-BCQ) at $x_0 \in A$ if

$$N_A(x_0) \subset \bigcup_{\lambda \in K^+} \left\{ x^* \in X^* \,\middle|\, (x^*, \langle x^*, x_0\rangle) \in \operatorname{cl}[\operatorname{cone} \operatorname{epi} (\lambda \circ g)^* + \operatorname{epi} \delta_C^*] \right\}.$$

$\{g(x) \in -K \mid x \in C\}$ is said to satisfy S-BCQ if for all $y \in A$, $\{g(x) \in -K \mid x \in C\}$ satisfies S-BCQ at y.

In the following theorem, we show that S-BCQ is a necessary and sufficient constraint qualification for surrogate min-max duality via convex programming.

Theorem 4.1 ([19])**.** *The following statements are equivalent:*

(i) $\{g(x) \in -K \mid x \in C\}$ *satisfies S-BCQ,*

(ii) *for each real-valued continuous convex function f on X which attains its minimum on A,*

$$\min_{x \in A} f(x) = \max_{\lambda \in K^+} \inf\{f(x) \mid x \in C, \lambda \circ g(x) \leq 0\}.$$

(iii) *for each $v \in X^*$ which attains its minimum on A,*

$$\min_{x \in A} f(x) = \max_{\lambda \in K^+} \inf\{f(x) \mid x \in C, \lambda \circ g(x) \leq 0\}.$$

Remark 4.2. Let $\mathcal{F}$ be a subset of $\{f : X \to \overline{\mathbb{R}}\}$ satisfying

$$X^* \subset \mathcal{F} \subset \{f : X \to \mathbb{R}, \text{ continuous convex}\}.$$

Then, S-BCQ is equivalent that for each $f \in \mathcal{F}$, surrogate min-max duality holds. By Theorem 3.3 and Theorem 4.1, we can prove that S-CCCQ implies S-BCQ. However, the converse is not generally true.

In convex programming, S-BCQ characterizes surrogate min-max duality. It is clear that S-BCQ implies S-BCQ at $x_0 \in A$. However, the converse is not always true. Hence, the following two statements are not equivalent:

(i) for each real-valued continuous convex function f from X to $\overline{\mathbb{R}}$ which attains its minimum on A,
$$\min_{x\in A} f(x) = \max_{\lambda\in K^+} \inf\{f(x) \mid x \in C, \lambda \circ g(x) \leq 0\},$$

(ii) there exists $x_0 \in A$ such that for each real-valued continuous convex function f which attains its minimum at x_0,
$$f(x_0) = \min_{x\in A} f(x) = \max_{\lambda\in K^+} \inf\{f(x) \mid x \in C, \lambda \circ g(x) \leq 0\}.$$

On the other hand, surprisingly, the statements (iv) and (vi) in Theorem 3.3 are equivalent. S-CCCQ is a necessary and sufficient constraint qualification for three types of surrogate duality via quasiconvex programming with convex constraints.

Next, we introduce some constraint qualifications for convex constraints.

Definition 4.3 ([2, 9, 10])**.** Let g be a continuous K-convex function from X to Y, and C a closed convex subset of X. Assume that $A = C \cap g^{-1}(-K)$ is nonempty.

(i) $\{g(x) \in -K \mid x \in C\}$ is said to satisfy the closed cone constraint qualification (CCCQ) if
$$\bigcup_{\lambda\in K^+} \operatorname{epi}(\lambda \circ g)^* + \operatorname{epi}\delta_C^*$$
is w^*-closed,

(ii) $\{g(x) \in -K \mid x \in C\}$ is said to satisfy [CQ2] if
$$N_A(x_0) \subset N_C(x_0) + \left\{ x^* \in X^* \,\middle|\, (x^*, \langle x^*, x_0\rangle) \in \bigcup_{\lambda\in K^+} \operatorname{epi}(\lambda \circ g)^* \right\}$$
for all $x_0 \in A$.

Definition 4.4 ([2, 4, 11])**.** Let I be an index set, g_i proper lsc convex functions from X to $\overline{\mathbb{R}}$, C a closed convex subset of X, and $S = \cap_{i\in I} L(g_i, \leq, 0)$. Assume that $A = C \cap S$ is nonempty.

(i) $\{g_i(x) \leq 0 \mid i \in I, x \in C\}$ is said to be Farkas-Minkowski (FM) if
$$\operatorname{epi}\delta_A^* = \operatorname{cone}\operatorname{co} \bigcup_{i\in I} \operatorname{epi} g_i^* + \operatorname{epi}\delta_C^*,$$

(ii) $\{g_i(x) \leq 0 \mid i \in I\}$ is said to satisfy the basic constraint qualification (BCQ) relative to C at $x \in A$ if

$$N_{C\cap S}(x) \subset N_C(x) + \operatorname{cone}\operatorname{co} \bigcup_{i\in I(x)} \partial g_i(x).$$

(iii) $\{g_i(x) \leq 0 \mid i \in I, x \in C\}$ is said to be locally Farkas-Minkowski (LFM) if $\{g_i(x) \leq 0 \mid i \in I\}$ satisfies BCQ relative to C at for each $x \in A$.

In convex programming, CCCQ, FM, and similar constraint qualifications involving an epigraph, are studied as necessary and sufficient constraint qualifications for Lagrange strong duality. Also, [CQ2], BCQ, LFM, and similar constraint qualifications involving a normal cone, are introduced as necessary and sufficient constraint qualifications for Lagrange min-max duality. Similarly, in [19, 21], we investigate necessary and sufficient constraint qualifications for surrogate duality. S-CCCQ, which is a condition involving an epigraph, is a necessary and sufficient constraint qualification for surrogate strong duality via quasiconvex programming with convex constraints. S-BCQ at $x \in A$, which is a condition involving a normal cone of A at x, is a necessary and sufficient constraint qualification for surrogate min-max duality via convex programming. In general, necessary and sufficient constraint qualifications for strong duality are different from necessary and sufficient constraint qualifications for min-max duality. In convex programming, it is well known that CCCQ implies [CQ2], and the converse is not generally true. Also, we can easily show that [CQ2] at a fixed $x \in A$ is weaker than [CQ2]. On the other hand, surprisingly, S-CCCQ is a necessary and sufficient constraint qualification for three types of surrogate duality, see Theorem 3.3. This result indicates that S-CCCQ completely characterizes surrogate duality for quasiconvex programming with convex constraints. Such a constraint qualification have not been introduced yet as far as we know. The following Table 1 summarizes our results.

duality	objective	constraint	strong	min-max	min-max at x_0
Lagrange	convex	real	FM [4]	LFM [4] BCQ	LFM at x_0 BCQ at x_0
Lagrange	convex	vector	CCCQ [9]	[CQ2] [9]	[CQ2] at x_0
surrogate	convex	vector	S-CCCQ	S-BCQ	S-BCQ at x_0 [19]
surrogate	quasiconvex	vector	S-CCCQ [19]	S-CCCQ	S-CCCQ

Table 1 Necessary and sufficient constraint qualifications

Strictly speaking, BCQ is a necessary and sufficient constraint qualification for the optimality condition, and LFM at x_0 and [CQ2] at x_0 have not been investigated yet in the previous literatures. However, we can check easily that these conditions are necessary and sufficient constraint qualifications for these duality theorems.

Next, we study relations between constraint qualifications for convex constraints. For simplicity, we consider constraint qualifications for a real-valued convex inequality system. Let I be an index set, g_i a real-valued continuous convex function on X for each $i \in I$, and g a function from X to $\mathbb{R}^I$ such that $g(x) = (g_i(x))_{i\in I}$. Then, we can easily see that g is $\mathbb{R}^I_+$-convex. We define the following constraint qualifications. $\{g_i(x) \leq 0 \mid i \in I\}$ is said to satisfy S-CCCQ if $\{g(x) \in -\mathbb{R}^I_+\}$ satisfies S-CCCQ, that is,

$$\operatorname{epi}\delta_A^* \subset \bigcup_{\lambda\in\mathbb{R}^{(\mathrm{i})}_+} \operatorname{cl}\left[\operatorname{cone}\operatorname{epi}\left(\sum_{i\in I}\lambda_i g_i\right)^* + \operatorname{epi}\delta_C^*\right],$$

where $\mathbb{R}_+^{(\mathrm{i})} := \{\lambda \in \mathbb{R}^I \mid \forall i \in I, \lambda_i \geq 0, \{i \in I \mid \lambda_i \neq 0\} \text{ is finite}\}$. Similarly, $\{g_i(x) \leq 0 \mid i \in I\}$ is said to satisfy S-BCQ at $x_0 \in A$ if $\{g(x) \in -\mathbb{R}_+^I\}$ satisfies S-BCQ at x_0, that is,

$$N_A(x_0) \subset \bigcup_{\lambda \in \mathbb{R}_+^{(\mathrm{i})}} \left\{ x^* \in X^* \,\middle|\, (x^*, \langle x^*, x_0 \rangle) \in \mathrm{cl}\left[\text{cone epi}\left(\sum_{i \in I} \lambda_i g_i\right)^* + \text{epi}\,\delta_C^*\right] \right\},$$

We show the following nine examples concerned with constraint qualifications for convex inequality constraints.

Example 4.5 (FM)**.** Let $g_1(x) = x^2 - 2x$. Then, $A = \{x \in \mathbb{R} \mid g_1(x) \leq 0\} = [0, 2]$ and $g_1^*(v) = \frac{v^2}{4} + v + 1$. Hence

$$\text{epi}\delta_A^* = \{(x, \alpha) \in \mathbb{R}^2 \mid \alpha \geq \max\{2x, 0\}\} = \text{cone epi}g_1^*,$$

that is, FM is satisfied.

Example 4.6 (S-CCCQ, LFM and not FM)**.** Let $I = [0, 1]$, $w_i = (1-i, i)$, for each $i \in (0, 1)$,

$$g_i(x) = \langle w_i, x \rangle + 2\sqrt{i(1-i)},$$

and for each $i \in \{0, 1\}$,

$$g_i(x) = \begin{cases} (\langle w_1, x \rangle)^2 & \langle w_1, x \rangle \geq 0, \\ 0 & \textit{otherwise}. \end{cases}$$

Then $A = \{x \in \mathbb{R}^2 \mid g_i(x) \leq 0, \forall i \in I\} = \{x \in \mathbb{R}^2 \mid x_1 x_2 \geq 1, x_1 < 0\}$. We can check that Lagrange strong duality does not always hold. Actually, let $f(x) = -x_1$ then $\inf_{x \in A} f(x) = 0$, but f does not attain its minimum. For each $\lambda \in \mathbb{R}_+^{(\mathrm{i})}$,

$$\inf_{x \in A} f(x) = 0 > \inf_{x \in \mathbb{R}^n} \left\{ f(x) + \sum_{i \in I} \lambda_i g_i(x) \right\}.$$

This shows that Lagrange strong duality does not hold for f. Since FM is a necessary and sufficient constraint qualification for Lagrange strong duality, FM is not satisfied.

On the other hand, Lagrange min-max duality always holds. Actually, let f be a real-valued convex function which attains its minimum on A. If a solution $x \in \text{int}A$, then

$$\inf_{x \in A} f(x) = \inf_{x \in \mathbb{R}^2} f(x) = \inf_{x \in \mathbb{R}^2} \left\{ f(x) + \sum_{i \in I} 0 g_i(x) \right\}.$$

If a solution $x \in \text{bd}A$, then $x_1 x_2 = 1$. Then, there exists $i_0 \in (0, 1)$ such that $\langle w_{i_0}, x \rangle = -2\sqrt{i_0(1-i_0)}$. Hence, $N_A(x) = \text{cone}\{w_{i_0}\}$, and

$$0 \in \partial f(x) + N_A(x) = \partial f(x) + \text{cone}\{w_{i_0}\}.$$

This shows that there exists $\lambda_{i_0} \geq 0$ such that

$$\inf_{x \in A} f(x) = \inf_{x \in \mathbb{R}^2} \{f(x) + \lambda_{i_0} g_{i_0}(x)\}.$$

Since LFM is a necessary and sufficient constraint qualification for Lagrange min-max duality, LFM is satisfied.

Surrogate strong duality always holds. Let $v \in \mathbb{R}^2$. If v attains its minimum on A, then Lagrange min-max duality holds. Hence,

$$\min_{x\in A} v(x) = \max_{\lambda\in\mathbb{R}_+^{(\mathrm{i})}} \inf_{x\in\mathbb{R}^2} \left\{ v(x) + \sum_{i\in I} \lambda_i g_i(x) \right\}.$$

We can check easily that for each $\lambda \in \mathbb{R}_+^{(\mathrm{i})}$,

$$\inf_{x\in A} v(x) \geq \inf \left\{ v(x) \;\middle|\; \sum_{i\in I} \lambda_i g_i(x) \leq 0 \right\} \geq \inf_{x\in\mathbb{R}^2} \left\{ v(x) + \sum_{i\in I} \lambda_i g_i(x) \right\}.$$

This means that surrogate strong duality holds. If v does not attain its minimum on A, then $v \in \{w \in \mathbb{R}^2 \mid w_1 > 0 \text{ or } w_2 > 0\} \cup \operatorname{cone}\{(-1,0),(0,-1)\}$. When $v \in \{w \in \mathbb{R}^2 \mid w_1 > 0 \text{ or } w_2 > 0\}$, then $\inf_{x\in A} v(x) = -\infty$ and surrogate strong duality holds for each $\lambda \in \mathbb{R}_+^{(\mathrm{i})}$. If $v \in \operatorname{cone}\{(-1,0)\}$, then

$$\inf_{x\in A} v(x) = 0 = \inf\{v(x) \mid g_0(x) \leq 0\}.$$

Also, if $v \in \operatorname{cone}\{(0,-1)\}$, then

$$\inf_{x\in A} v(x) = 0 = \inf\{v(x) \mid g_1(x) \leq 0\}.$$

This shows that surrogate strong duality holds for each $v \in \mathbb{R}^2$. Hence, by Lemma 3.2, S-CCCQ is satisfied.

Example 4.7 (LFM and not S-CCCQ)**.** Let $I = (0,1)$, $w_i = (1-i, i)$, and

$$g_i(x) = \langle w_i, x\rangle + 2\sqrt{i(1-i)},$$

Then $A = \{x \in \mathbb{R}^2 \mid g_i(x) \leq 0, \forall i \in I\} = \{x \in \mathbb{R}^2 \mid x_1x_2 \geq 1, x_1 < 0\}$. By the similar way in Example 4.6, we can show that LFM is satisfied.

Let $v = (-1,0)$, then $\inf_{x\in A} v(x) = 0$, and we can check that for each $\lambda \in \mathbb{R}_+^{(\mathrm{i})}$,

$$\inf_{x\in A} v(x) = 0 > \inf \left\{ v(x) \;\middle|\; \sum_{i\in I} \lambda_i g_i(x) \leq 0 \right\}.$$

This means that S-CCCQ is not satisfied.

Example 4.8 (S-CCCQ, LFM for some $x_0 \in \mathrm{bd}A$ and not LFM)**.** Let $I = [0,1]$, $w_i = (-i, i-1)$, for each $i \in (0,1]$,

$$g_i(x) = \begin{cases} (\langle w_i, x\rangle)^2 & \langle w_i, x\rangle > 0, \\ 0 & \textit{otherwise}, \end{cases}$$

and

$$g_0(x) = \begin{cases} \langle w_0, x\rangle & \langle w_0, x\rangle > 0, \\ 0 & \textit{otherwise}, \end{cases}$$

Then, $A = \{x \in \mathbb{R}^2 \mid g_i(x) \leq 0, \forall i \in I\} = \mathbb{R}_+^2$. Let $x_0 = (1,0)$. Then, LFM at x_0 is satisfied. Actually, $\langle w_0, x_0\rangle = 0 = g_0(x_0)$ and

$$N_A(x_0) = \{(v_1,v_2) \in \mathbb{R}^2 \mid v_1 = 0, v_2 \leq 0\} = \operatorname{cone} \partial g_0(x_0).$$

However, let $x = (0,0)$, then LFM at x is not satisfied. Actually,

$$N_A(x) = -\mathbb{R}^2_+ \supsetneq \{(v_1, v_2) \in \mathbb{R}^2 \mid v_1 = 0, v_2 \leq 0\} = \text{cone co} \bigcup_{i \in I(x)} \partial g_i(x).$$

We show that surrogate strong duality always holds. Let $v \in \mathbb{R}^2$. If v attains its minimum on A, then $v \in \mathbb{R}^2_+$ and $\inf_{x\in A} v(x) = 0$. We can check that there exists $i_0 \in I$ such that $v \in -\text{cone}\{w_{i_0}\}$ and

$$\inf_{x \in A} v(x) = 0 = \inf\{v(x) \mid g_{i_0}(x) \leq 0\}.$$

If v does not attain its minimum on A, then $v \in \{w \in \mathbb{R}^2 \mid w_1 < 0 \text{ or } w_2 < 0\}$. Since $\inf_{x\in A} v(x) = -\infty$, surrogate strong duality holds for each $\lambda \in \mathbb{R}^{(\mathrm{i})}_+$. By Lemma 3.2, S-CCCQ is satisfied.

Example 4.9 (S-BCQ, LFM for some $x_0 \in \text{bd}A$, not S-CCCQ, and not LFM)**.** Let $I = (0,1)$, $w_i = (1-i, i)$ for each $i \in (0,1)$,

$$g_i(x) = \begin{cases} \left(\langle w_i, x\rangle + 2\sqrt{i(1-i)}\right)^2 & \langle w_i, x\rangle > -2\sqrt{i(1-i)}, \\ 0 & \textit{otherwise} \end{cases}$$

for each $i \in (0,1) \setminus \{\frac{1}{2}\}$, and

$$g_{\frac{1}{2}}(x) = \left\langle w_{\frac{1}{2}}, x \right\rangle + 1.$$

Then $A = \{x \in \mathbb{R}^2 \mid g_i(x) \leq 0, \forall i \in I\} = \{x \in \mathbb{R}^2 \mid x_1 x_2 \geq 1, x_1 < 0\}$.

For each $x \in A$, S-BCQ at x is satisfied. Actually, if $x \in \text{int}A$, it is clear that

$$N_A(x) = \{0\} \subset \bigcup_{\lambda \in \mathbb{R}^{(\mathrm{i})}_+} \left\{ x^* \in \mathbb{R}^2 \,\middle|\, (x^*, \langle x^*, x\rangle) \in \text{cl}\left[\text{cone epi}\left(\sum_{i\in I} \lambda_i g_i\right)^*\right]\right\}.$$

It $x \in \text{bd}A$, then there exists $i_0 \in I$ such that $N_A(x) = \text{cone}\{w_{i_0}\}$. This shows that

$$\begin{aligned} N_A(x) &= \text{cone}\{w_{i_0}\} \\ &\subset \left\{x^* \in \mathbb{R}^2 \mid (x^*, \langle x^*, x\rangle) \in \text{cl cone epi } g^*_{i_0}\right\} \\ &\subset \bigcup_{\lambda \in \mathbb{R}^{(\mathrm{i})}_+} \left\{ x^* \in \mathbb{R}^2 \,\middle|\, (x^*, \langle x^*, x\rangle) \in \text{cl}\left[\text{cone epi}\left(\sum_{i\in I} \lambda_i g_i\right)^*\right]\right\}. \end{aligned}$$

We can easily show that LFM at $x_0 = (-1,-1) \in A$ is satisfied since $g_{\frac{1}{2}}(x_0) = 0$ and $\partial g_{\frac{1}{2}}(x_0) = \{w_{\frac{1}{2}}\}$. However, LFM at $x = \left(-\frac{1}{\sqrt{3}}, -\sqrt{3}\right)$ is not satisfied since $g_{\frac{1}{4}}(x) = 0$ and $\partial g_{\frac{1}{4}}(x) = \{0\}$.

Let $v = (-1, 0)$, then $\inf_{x\in A} v(x) = 0$, and we can check that for each $\lambda \in \mathbb{R}^{(\mathrm{i})}_+$,

$$\inf_{x\in A} v(x) = 0 > \inf\left\{ v(x) \,\middle|\, \sum_{i\in I} \lambda_i g_i(x) \leq 0 \right\}.$$

This means that S-CCCQ is not satisfied.

Example 4.10 (S-CCCQ and not LFM for each $x \in \mathrm{bd}A$)**.** Let $I = [0,1]$, $w_i = (-i, i-1)$, for each $i \in I$,

$$g_i(x) = \begin{cases} (\langle w_i, x\rangle)^2 & \langle w_i, x\rangle > 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Then, $A = \{x \in \mathbb{R}^2 \mid g_i(x) \leq 0, \forall i \in I\} = \mathbb{R}^2_+$. We can show that S-CCCQ is satisfied by the similar way in Example 4.8. Also, since $\partial g_i(x) = \{0\}$ for each $i \in I$ and $x \in A$, LFM is not satisfied for each $x \in \mathrm{bd}A$.

Example 4.11 (S-BCQ, not S-CCCQ and not LFM for each $x \in \mathrm{bd}A$)**.** Let $I = (0,1)$, $w_i = (1-i, i)$, for each $i \in (0,1)$,

$$g_i(x) = \begin{cases} \left(\langle w_i, x\rangle + 2\sqrt{i(1-i)}\right)^2 & \langle w_i, x\rangle > -2\sqrt{i(1-i)}, \\ 0 & \textit{otherwise.} \end{cases}$$

Then $A = \{x \in \mathbb{R}^2 \mid g_i(x) \leq 0, \forall i \in I\} = \{x \in \mathbb{R}^2 \mid x_1 x_2 \geq 1, x_1 < 0\}$. By the similar way in Example 4.9, we can show that S-BCQ is satisfied, S-CCCQ is not satisfied. Also, by the similar way in Example 4.10, we can show that LFM at for each $x \in A$ are not satisfied.

Example 4.12 (LFM for some $x_0 \in \mathrm{bd}A$ and not S-BCQ)**.** Let $I = (0,1]$, $w_i = (-i, i-1)$, g_i be a function as follows:

$$g_i(t) = \begin{cases} \langle w_i, x\rangle & \langle w_i, x\rangle > 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Then, $A = \{x \in \mathbb{R}^2 \mid g_i(x) \leq 0, \forall i \in I\} = \mathbb{R}^2_+$. LFM at $x_0 = (0,1)$ is satisfied since $g_1(x) = 0$ and $N_A(x_0) = \{(v_1, 0) \mid v_1 \leq 0\} = \mathrm{cone}\, \partial g_1(x_0)$. However, let $x = (1,0)$, then for each $i \in I$, $g_i(x) < 0$. This means that LFM at x is not satisfied, that is, LFM is not satisfied.

We show that S-BCQ at $x = (1,0)$ is not satisfied. Actually, let $v = (0,1)$, then v is a linear function on $\mathbb{R}^2$ and $v(x) = 0 = \inf_{y \in A} v(y)$. However, for each $\lambda \in \mathbb{R}^{(\mathrm{i})}_+$,

$$v(x) = 0 > \inf\left\{ v(y) \,\middle|\, \sum_{i \in I} \lambda_i g_i(y) \right\}.$$

This shows that S-BCQ at x is not satisfied, that is, S-BCQ is not satisfied.

Example 4.13 (S-BCQ for some $x_0 \in \mathrm{bd}A$, not S-BCQ, and not LFM for each $x \in \mathrm{bd}A$)**.** Let $I = (0,1) \setminus \{\frac{1}{2}\}$, $w_i = (1-i, i)$, for each $i \in I$,

$$g_i(x) = \begin{cases} \left(\langle w_i, x\rangle + 2\sqrt{i(1-i)}\right)^2 & \langle w_i, x\rangle > -2\sqrt{i(1-i)}, \\ 0 & \textit{otherwise.} \end{cases}$$

Then $A = \{x \in \mathbb{R}^2 \mid g_i(x) \leq 0, \forall i \in I\} = \{x \in \mathbb{R}^2 \mid x_1 x_2 \geq 1, x_1 < 0\}$. By the similar way in Example 4.12, we can show that S-BCQ at $x = (-1,-1)$ is not satisfied. By the similar way in Example 4.9, we can show that S-BCQ at for each $x \in \mathrm{bd}A \setminus \{(-1,-1)\}$ is satisfied. Also, by the similar way in Example 4.10, LFM is not satisfied at for each $x \in \mathrm{bd}A$.

The following Venn diagram of constraint qualifications summarizes the results illustrated by the examples.

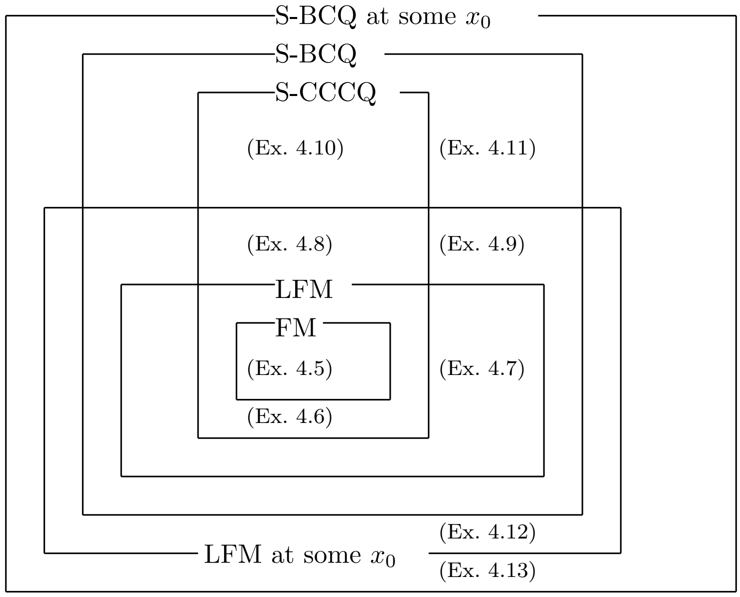

Fig. 1 Venn Diagram of constraint qualifications

Acknowledgements

The authors are grateful to anonymous referees for many comments and suggestions improved the quality of the paper.

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Manuscript received 12 December 2014
revised 11 March 2014
accepted for publication 11 March 2014

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