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NEW CONSTRAINT QUALIFICATION AND OPTIMALITY FOR LINEAR SEMI-INFINITE PROGRAMMING

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Abstract: In this short paper, we present a new constraint qualification (CQ) for linear semi-infinite programming (LSIP) called directional Farkas-Minkowski CQ that characterizes the class of all LSIP problems for which a feasible solution is optimal if and only if it satisfies the classical optimality conditions. Furthermore, presented in terms of the new CQ, an new form of some recent optimality conditions without CQ for LSIP is obtained.

Key words: *linear semi-infinite programming, constraint qualification, optimality.*

Mathematics Subject Classification: *90C34, 90C46.*

1 Introduction

In the development of the optimality theory of linear semi-infinite programming (LSIP), there have been efforts in finding increasingly more general constraint qualifications (CQ) under which the classical optimality theorem holds. The CQs that have played an important role in the theoretical and computational study of LSIP can be found in [1, 2, 3, 5, 7, 8, 6, 9, 10] and [16]. Especially, one can find in [5] an account of the relations of these CQs. Worth mentioning here are the Slater CQ and the Farkas-Minkowski (FM) class of CQs. While the Slater CQ is useful for continuous LSIP which is one of the most studied classes of LSIP problems, it is much stronger than the FM CQs.

The classical KKT, complementary, and Lagrangian saddle point optimality conditions may fail to be satisfied at the optimal solution of some LSIP problems, and the classical optimality theorem fail to apply to all LSIP problems. At the same time, there have been attempts to obtain new optimality conditions without the requirement of a CQ. Interesting results have been obtained in [4] and [13]. However, the corresponding optimality conditions take asymptotic or sequential forms. The asymptotic nature of the optimality conditions without CQs makes them less convenient to apply compared to the classical optimality conditions. This is especially true when applications in numerical methods are concerned. Thus, finding a more general CQ is still interesting, both theoretically and practically.

On the other hand, there are LSIP problems which do not satisfy any of the known CQs, but the classical optimality conditions are satisfied at optimal solutions (see Example 3.2). It is thus natural to ask: does a weaker (possibly the weakest) CQ exist under which the classical optimality theorem still holds? This paper answers this question. We will propose CQs called directional Farkas-Minkowski CQs, and show that one of them characterizes all LSIP problems for which a feasible solution $\boldsymbol{x}^*$ is optimal if and only if one of the classical KKT, complementary, and Lagrangian saddle point conditions holds true at $\boldsymbol{x}^*$. The optimality theorem of this paper has found applications in [12] where the fundamental theorem of linear programming is extended to LSIP.

The outline of this paper is as follows. Following this introductory section, in Section 2, new Farkas-Minkowski type constraint qualifications are proposed. Some relations of the new CQs with the existing ones are discussed. A number of interesting properties involving the new and some existing CQs are given. In Section 3, the classical optimality theorem is discussed in relation to a number of constraint qualifications that are relevant to this paper. It is shown that the classical optimality theorem holds true under a more general CQ called locally directional FM CQ (or locally FM CQ in a given direction) proposed in Section 2, and that in order for the classical KKT, complementary, and Lagrangian saddle point conditions to remain equivalent to the optimality of a feasible point, this CQ cannot be further relaxed. These indicate that the locally directional FM CQ is the most general CQ for the classical optimality theorem to hold true. A few recent optimality results for LSIP without any constraint qualification, together with one obtained in this paper by applying the new optimality theorem under the LDFM CQ, are included for an updated picture of the optimality theory of LSIP. The paper is ended by some brief comments in Section 4.

2 The LSIP Problem and Some Associated CQs

For convenience, we use the following notations following [12, 13] and some earlier works cited therein. In case the elements, rows or columns of a matrix (or a vector) need to be specified, they will usually be listed in the text as demonstrated below. For example, the elements of 2×2 matrix $\mathbf{A} = [a_{ij}]_{2\times 2}$ can be specified as $\mathbf{A} = [a_{11}, a_{12}; a_{21}, a_{22}]$. If an $m \times n$ matrix $\mathbf{A}$ has rows $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m$ or columns $\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_n$, then $\mathbf{A}$ can be written as $\mathbf{A} = [\boldsymbol{a}_1; \boldsymbol{a}_2; \cdots; \boldsymbol{a}_m]$ or $\mathbf{A} = [\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_n]$, respectively. We note that in the above notation for matrices row entries are separated by commas and column entries are separated by semicolons. For clarity, we will use $\mathbb{R}^n$ and $\mathbb{R}_n$ to represent the n-column and the n-row vector spaces, respectively.

In this paper, we consider the following general LSIP problem.
Problem $\mathrm{P}(\boldsymbol{c}; \boldsymbol{a}, b, T)$:

$$\inf \quad \boldsymbol{c}^T \boldsymbol{x} \tag{2.1}$$

$$\text{s. t.} \quad \boldsymbol{a}(t)\boldsymbol{x} \le b(t) \ \text{ for } \ t \in T, \tag{2.2}$$

where $\boldsymbol{x} = [x_1; x_2; \cdots; x_n] \in \mathbb{R}^n$ is the decision vector, $\boldsymbol{c} = [c_1; c_2; \cdots; c_n] \in \mathbb{R}^n$ is the objective vector, T is a given arbitrary index set, and

$$\boldsymbol{a}(t) = [a_1(t), a_2(t), \cdots, a_n(t)] \, : \, t \to \mathbb{R}_n$$

and

$$b(t) \, : \, t \to \mathbb{R}$$

are given functions defined on T, where $\boldsymbol{a}(t)$ satisfies $\|\boldsymbol{a}(t)\| = 1$ for all $t \in T$. The feasible region and the optimal set of problem $\mathrm{P}(\boldsymbol{c}; \boldsymbol{a}, b, T)$ are denoted by $\mathcal{F}$ and $\mathcal{F}^*$, respectively.

It is well known that the KKT, the complementary, and the Lagrangian saddle point conditions each characterizes the optimality of a feasible point for problems satisfying the locally Farkas Minkowski (LFM) CQ which is weaker than other main classes of CQs ([5]). We recall that a linear inequality $\boldsymbol{\alpha}^T \boldsymbol{x} \le \beta$, where $\boldsymbol{\alpha} \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$, is said to be a linear consequence of the constraint system of problem $\mathrm{P}(\boldsymbol{c};\boldsymbol{a},b,T)$ (or simply a linear consequence of problem $\mathrm{P}(\boldsymbol{c};\boldsymbol{a},b,T)$) if $\boldsymbol{\alpha}^T\boldsymbol{x} \le \beta$ is satisfied by all $\boldsymbol{x} \in \mathcal{F}$. A linear consequence of problem $\mathrm{P}(\boldsymbol{c};\boldsymbol{a},b,T)$ is said to be finite if it is a linear consequence of a finite constraint subsystem of problem $\mathrm{P}(\boldsymbol{c};\boldsymbol{a},b,T)$.

Problem $\mathrm{P}(\boldsymbol{c};\boldsymbol{a},b,T)$ (or its constraint system) is said to be LFM, if all of its linear consequences binding $\mathcal{F}$ at certain point of $\mathcal{F}$ are finite linear consequences. Problem $\mathrm{P}(\boldsymbol{c};\boldsymbol{a},b,T)$ (or its constraint system) is said to be LFM at $\boldsymbol{x}^* \in \mathcal{F}$, if every linear consequence binding $\mathcal{F}$ at $\boldsymbol{x}^*$ is finite.

Some new CQs of Farkas-Minkowski type are proposed in the following.

Definition 2.1. Let $\boldsymbol{\alpha} \in \mathbb{R}^n$ be a non-zero vector. Problem $\mathrm{P}(\boldsymbol{c};\boldsymbol{a},b,T)$ is said to be Farkas-Minkowski in the direction of $\boldsymbol{\alpha}$, or DFM in $\boldsymbol{\alpha}$ in brief, if every linear consequence of problem $\mathrm{P}(\boldsymbol{c};\boldsymbol{a},b,T)$ with normal vector $\boldsymbol{\alpha}$ is finite. It is said to be locally Farkas-Minkowski in the direction of $\boldsymbol{\alpha}$, or simply LDFM in $\boldsymbol{\alpha}$, if all linear consequences of the constraint system with normal vector $\boldsymbol{\alpha}$ that is binding $\mathcal{F}$ at some point of $\mathcal{F}$ are finite.

Lemma 2.2. *The following are simple properties regarding the directional FM CQs.*

(a) *Problem $P(\boldsymbol{c};\boldsymbol{a},b,T)$ has at most one binding linear consequence with normal vector in a given direction.*

(b) *If $\boldsymbol{x}^* \in \mathcal{F}^*$, then $-\boldsymbol{c}^T\boldsymbol{x} \le -\boldsymbol{c}^T\boldsymbol{x}^*$ is the only binding linear consequence. (Note that the only possible situation in which problem $P(\boldsymbol{c};\boldsymbol{a},b,T)$ is not LDFM in $-\boldsymbol{c}$ arises when this linear consequence is not a finite consequence.)*

(c) *For any non-zero vector $\boldsymbol{\alpha} \in \mathbb{R}^n$, if problem $P(\boldsymbol{c};\boldsymbol{a},b,T)$ is DFM in $\boldsymbol{\alpha}$ it must be LDFM in $\boldsymbol{\alpha}$.*

(d) *Problem $P(\boldsymbol{c};\boldsymbol{a},b,T)$ is LDFM in all directions if it has no optimal solution.*

(e) *If problem $P(\boldsymbol{c};\boldsymbol{a},b,T)$ has an optimal solution, then it is DFM in $-\boldsymbol{c}$ if and only if it is LDFM in $-\boldsymbol{c}$.*

Proof. Properties (a)-(d) are straightforward from the definitions of DFM in $\boldsymbol{\alpha}$ and LDFM in $\boldsymbol{\alpha}$. To show that (e) holds, let v^* be the optimal value of problem $\mathrm{P}(\boldsymbol{c};\boldsymbol{a},b,T)$. The only linear consequence binding $\mathcal{F}$ and having normal vector $-\boldsymbol{c}$ is $-\boldsymbol{c}^T\boldsymbol{x} \le v^*$. If problem $\mathrm{P}(\boldsymbol{c};\boldsymbol{a},b,T)$ is LDFM in $-\boldsymbol{c}$, that is, if $-\boldsymbol{c}^T\boldsymbol{x} \le v^*$ is a linear consequence of a finite constraint subsystem, then any other linear consequences with normal vector in the direction of $-\boldsymbol{c}$ are consequences of the same finite constraint subsystem and hence problem $\mathrm{P}(\boldsymbol{c};\boldsymbol{a},b,T)$ is DFM in $-\boldsymbol{c}$. Then, (e) follows from (c). □

The following example provides an LSIP problem that is LDFM in $-\boldsymbol{c}$ but not DFM in $-\boldsymbol{c}$.

Example 2.3. The problem is specified by $\boldsymbol{a}(t) = [1/\sqrt{1+t^4},\ -t^2/\sqrt{1+t^4}]$, $b(t) = -t/\sqrt{1+t^4}$, $\boldsymbol{c} = [0;\ 1]$, and $T = \{t \mid 0 < t < \infty\}$.

As shown in Figure 1, $\mathcal{F}$ is the region above the curve $x_2 = -\frac{1}{4x_1}$ for $-\infty < x_1 < 0$. The straight lines corresponding to $t = 1, 2, \cdots$ demonstrate the corresponding constraints. We see that this problem is feasible and bounded. It has optimal value $v^* = 0$ but has no optimal solution. The feasible region has no binding linear consequence with normal vector $-\boldsymbol{c}$. Thus, it is LDFM in $-\boldsymbol{c}$, but not DFM in $-\boldsymbol{c}$ as the inequality $-\boldsymbol{c}^T\boldsymbol{x} \le 0$ is a linear consequence but not a finite linear consequence.

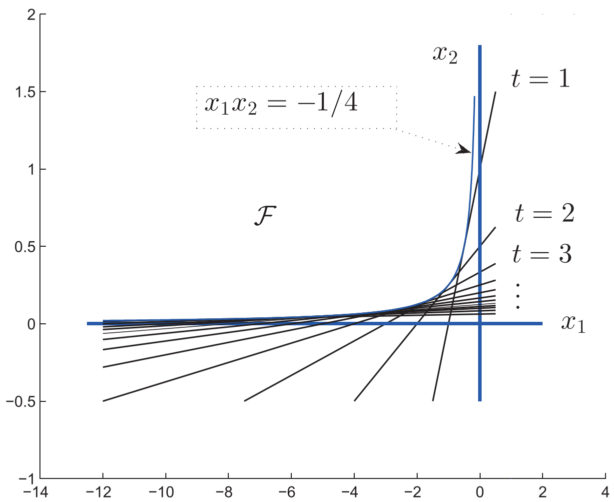

Figure 1: Constraints and Feasible Region for Example 2.3

Lemma 2.4. *The following are properties on the relation between the directional FM CQs and some well-known CQs.*

(a) *Problem $P(\boldsymbol{c}; \boldsymbol{a}, b, T)$ is FM if and only if it is DFM in all directions.*

(b) *Problem $P(\boldsymbol{c}; \boldsymbol{a}, b, T)$ is LFM if and only if it is LDFM in all directions.*

(c) *For $\boldsymbol{x}^* \in \mathcal{F}^*$, if the constraint system of problem $P(\boldsymbol{c}; \boldsymbol{a}, b, T)$ is LFM at $\boldsymbol{x}^*$, then problem $P(\boldsymbol{c}; \boldsymbol{a}, b, T)$ must be LDFM in $-\boldsymbol{c}$.*

Proof. Properties (a) and (b) are straightforward from the definitions of the corresponding CQs. To prove (c), we note that if problem $\mathrm{P}(\boldsymbol{c}; \boldsymbol{a}, b, T)$ is not LDFM in $-\boldsymbol{c}$, $-\boldsymbol{c}^T\boldsymbol{x} \le -\boldsymbol{c}^T\boldsymbol{x}^*$, which in this case is the unique linear consequence binding $\mathcal{F}$ (at $\boldsymbol{x}^*$), must not be a finite consequence of the constraint system and hence problem $\mathrm{P}(\boldsymbol{c}; \boldsymbol{a}, b, T)$ is not LFM at $\boldsymbol{x}^*$. □

3 Optimality

A function $\lambda : T \to \mathbb{R}$ is said to be a generalized finite sequence on T if its support given by

$$\mathrm{supp}(\lambda) = \{t \in T \mid \lambda(t) \neq 0\}$$

is a finite set. Let $\mathbb{R}^{(T)}$ denote the linear space of all generalized finite sequences on T, and $\mathbb{R}^{(T)}_+$ the positive cone of $\mathbb{R}^{(T)}$. For given $f : T \to \mathbb{R}^m$ and $\lambda \in \mathbb{R}^{(T)}$ with $\mathrm{supp}(\lambda) = \{t_1, t_2, \cdots, t_p\}$, we define

$$\sum_{t\in T} \lambda(t) f(t) = \begin{cases} \displaystyle\sum_{1\le i\le p} \lambda(t_i) f(t_i)\,, & \text{if } p \ge 1, \\ 0\,, & \text{if } p = 0 \ \ (\text{i.e. } \mathrm{supp}(\lambda) = \phi). \end{cases}$$

The Lagrangian on $\mathbb{R}^n \times \mathbb{R}^{(T)}$ is defined by

$$L(\boldsymbol{x}, \lambda) = \boldsymbol{c}^T\boldsymbol{x} + \sum_{t\in T} \lambda(t)(\boldsymbol{a}(t)\boldsymbol{x} - b(t)).$$

The optimality theorem under the LFM CQ, according to [16], can be stated as follows:

Theorem 3.1. *Let the constraint system of problem $P(\boldsymbol{c}; \boldsymbol{a}, b, T)$ be LFM at $\boldsymbol{x}^* \in \mathcal{F}$. Then, the following statements are equivalent:*

(i) $\boldsymbol{x}^* \in \mathcal{F}^*$.

(ii) *(KKT condition)* $-\boldsymbol{c} \in \mathcal{A}(\boldsymbol{x}^*)$, *where*

$$\mathcal{A}(\boldsymbol{x}^*) = cone\left\{\boldsymbol{a}(t)^T \mid t \in T \ and \ \boldsymbol{a}(t)\boldsymbol{x}^* = b(t)\right\}$$

is the active cone at $\boldsymbol{x}^*$.

(iii) *(complementarity condition) There exist* $t_j \in T$ *and* $\lambda_j \geq 0$, $j = 1, 2, \cdots, k$, *such that*

$$-\boldsymbol{c} = \sum_{j=1}^{k} \lambda_j \boldsymbol{a}(t_j)^T$$

and

$$\lambda_j(\boldsymbol{a}(t_j)\boldsymbol{x}^* - b(t_j)) = 0, \ j = 1, 2, \cdots, k;$$

(iv) *(Lagrangian saddle point condition) There exists* $\lambda^* \in \mathbb{R}_+^{(T)}$ *such that the Lagrangian* $L(\boldsymbol{x}, \lambda)$ *satisfies*

$$L(\boldsymbol{x}^*, \lambda) \leq L(\boldsymbol{x}^*, \lambda^*) \leq L(\boldsymbol{x}, \lambda^*), \ \text{for all } \boldsymbol{x} \in \mathbb{R}^n \text{ and all } \lambda \in \mathbb{R}_+^{(T)}.$$

The above theorem says that, for a given feasible point $\boldsymbol{x}^*$, under the condition that the problem is LFM at $\boldsymbol{x}^*$, the classical optimality conditions (ii)-(iv) are equivalent to the optimality of $\boldsymbol{x}^*$. However, the converse is not true, as shown by the following example where the conditions (ii)-(iv) are equivalent to the optimality of $x^* \in \mathcal{F}$, but the problem is not LFM at $\boldsymbol{x}^*$.

Example 3.2. The LSIP problem is specified by $\boldsymbol{a}(t) = [\sin(t), \ -\cos(t)]$, $b(t) = 0$, $\boldsymbol{c} = [0; \ -1]$, and $T = \{t \in \mathbb{R} \mid -\pi/4 \leq t < \pi/4\}$.

For this example, the only optimal solution to the LSIP problem is $\boldsymbol{x}^* = [0; \ 0]$. It is easy to see that the optimality of any feasible point $\bar{\boldsymbol{x}}$ is equivalent to the satisfaction of any of the conditions (ii)-(iv) in Theorem 3.1 at $\bar{\boldsymbol{x}}$, as they all hold at $\boldsymbol{x}^* = [0; \ 0]$. However, the problem is not LFM at $\boldsymbol{x}^*$, as the inequality

$$x_1 - x_2 \leq 0$$

is a linear consequence of the constraints binding the feasible region, but it is not a finite consequence.

There have been efforts to further relax the LFM CQ. Especially, the optimality conditions for LSIP problems without any CQ stated in the following theorem are obtained in [13] (conditions (ii)-(iv)) and [4] (condition (v)). As we can see, the conditions (ii)-(iv) in Theorem 2 are generalizations of the corresponding conditions in Theorem 1.

Theorem 3.3. *For* $\boldsymbol{x}^* \in \mathcal{F}$, *the following statements are equivalent:*

(i) $\boldsymbol{x}^* \in \mathcal{F}^*$;

(ii) *(generalized KKT condition) there exist sequences* $\{\boldsymbol{c}^i\}$ *and* $\{\boldsymbol{x}^i\}$ *in* $\mathbb{R}^n$ *such that*

$$\boldsymbol{c}^i \to \boldsymbol{c} \ (i \to \infty), \tag{3.1}$$

$$\boldsymbol{x}^i \to \boldsymbol{x}^* \ (i \to \infty), \tag{3.2}$$

and

$$-\boldsymbol{c}^i \in \mathcal{A}_i(\boldsymbol{x}^i), \ i = 1, 2, \cdots ; \tag{3.3}$$

where $\mathcal{A}_i(\boldsymbol{x}^i)$ *is the active cone of problem* $P(\boldsymbol{c}^i; \boldsymbol{a}, b, T(\boldsymbol{x}^i))$ *at* $\boldsymbol{x}^i$ *given by*

$$\mathcal{A}_i\left(\boldsymbol{x}^i\right) = cone\left\{\boldsymbol{a}(t)^T \,|\, t \in T(\boldsymbol{x}^i) \ and \ \boldsymbol{a}(t)\boldsymbol{x}^i = b(t)\right\}$$

in which $T(\boldsymbol{x}^i) = \{t \,|\, t \in T, \ \boldsymbol{a}(t)\boldsymbol{x}^i \le b(t)\}$.

(iii) *(generalized complementarity condition) there exist sequences* $\left\{\boldsymbol{c}^i\right\}$ *and* $\{\boldsymbol{x}^i\}$ *satisfying (3.1) and (3.2), and* $t^i_j \in T(\boldsymbol{x}^i)$, $1 \le j \le k$, *such that*

$$\left\{\boldsymbol{a}(t^i_j) \mid 1 \le j \le k\right\}$$

is linearly independent,

$$\boldsymbol{a}(t^i_j)\boldsymbol{x}^i = b(t^i_j), \quad 1 \le j \le k, \ i = 1, 2, \cdots$$

and

$$-\boldsymbol{c}^i = \sum_{j=1}^{k} \lambda^i_j \boldsymbol{a}(t^i_j)^T, \quad \lambda^i_j > 0, \ j = 1, 2, \cdots, k, \ i = 1, 2, \cdots ;$$

(iv) *(generalized Lagrange saddle point condition) there exist sequences* $\left\{\boldsymbol{c}^i\right\}$ *and* $\left\{\boldsymbol{x}^i\right\}$ *satisfying (3.1) and (3.2), and* $\lambda_i \in \mathbb{R}^{(T(\boldsymbol{x}^i))}_+$, $i = 1, 2, \cdots$, *such that*

$$L(\boldsymbol{x}^i, \lambda) \le L(\boldsymbol{x}^i, \lambda_i) \le L(\boldsymbol{x}, \lambda_i), \ i = 1, 2, \cdots$$

for all $\boldsymbol{x} \in \mathcal{F}_i = \left\{\boldsymbol{x} \mid \boldsymbol{a}(t)\boldsymbol{x} \le b(t) \ for \ all \ t \in T(\boldsymbol{x}^i)\right\}$ *and all* $\lambda \in \mathbb{R}^{(T(\boldsymbol{x}^i))}_+$.

(v) *There exists a sequence* $\{(\lambda_i, \varepsilon_i)\} \subset \mathbb{R}^{(T)}_+ \times \mathbb{R}$ *such that*

$$\sum_{t \in T} \lambda_i(t) b\left(t\right) \le \varepsilon_i - \boldsymbol{c}^T \boldsymbol{x}^*, \quad for \ i = 1, 2, \cdots, \tag{3.4}$$

and

$$(\sum_{t \in T} \lambda_i(t) \boldsymbol{a}\left(t\right)^T, \varepsilon_i) \to (-\boldsymbol{c}, 0_+). \tag{3.5}$$

We note that all conditions (ii)-(v) in Theorem 3.3 are in asymptotic form which is not preferred in computational applications as they are difficult to check when applied to numerical algorithms. At the same time, a drawback of Theorem 3.1 is the fact that its CQ depends on the optimal solution $\boldsymbol{x}^*$ which is usually not known, at least not directly from the given data defining the problem. Furthermore, the CQ is a bit too strong to cover all problems whose optimal solutions are characterized by any of the classical optimality conditions. All these highlight the advantages of the CQ we propose here, as one will see from the remaining of this section.

The following lemma is obvious.

Lemma 3.4. *Consider problem $P(\boldsymbol{c};\boldsymbol{a},b,T)$. If any of the conditions (ii)-(iv) in Theorem 3.1 is satisfied at any $\boldsymbol{x}^*\in\mathcal{F}^*$, then all of them are satisfied at all optimal solutions.*

The following example shows that LDFM in $-\boldsymbol{c}$ doesn't necessarily imply that the constraint system of problem $\mathrm{P}(\boldsymbol{c};\boldsymbol{a},b,T)$ is LFM at an optimal solution, even if optimal solution exists.

Example 3.5. The problem is specified by $\boldsymbol{c}=[0;\,1]$, $\boldsymbol{a}(t)=[\sin t,\,-1]$, $b(t)=0$, and $T=\{t\,|\,-\frac{\pi}{2}<t<\frac{\pi}{2}\}$.

It can be seen that $\boldsymbol{x}^*$ is the only optimal solution to the problem in Example 3.5. The linear inequality $-\boldsymbol{c}^T\boldsymbol{x}\leq 0$, which is the only linear consequence having normal vector $-\boldsymbol{c}$ and binding the feasible region, is itself a constraint of the constraint system. Thus, the problem is LDFM in $-\boldsymbol{c}$. On the other hand, both $x_1-x_2\leq 0$ and $-x_1-x_2\leq 0$ are linear consequences binding $\mathcal{F}$ but none of them is a linear consequence of any finite constraint subsystem.

Theorem 3.6. *Suppose that the constraint system of problem $P(\boldsymbol{c};\boldsymbol{a},b,T)$ is LDFM in $-\boldsymbol{c}$. Then, for any $\boldsymbol{x}^*\in\mathcal{F}$, the following statements are equivalent:*

(i) $\boldsymbol{x}^*\in\mathcal{F}^*$.

(ii) *(KKT condition) $-\boldsymbol{c}\in\mathcal{A}(\boldsymbol{x}^*)$, where $\mathcal{A}(\boldsymbol{x}^*)$ is the same as in Theorem 3.1.*

(iii) *(complementarity condition) There exist $t_j\in T$ and $\lambda_j\geq 0$, $j=1,2,\cdots,k$, such that*

$$-\boldsymbol{c}=\sum_{j=1}^{k}\lambda_j\boldsymbol{a}(t_j)^T$$

and

$$\lambda_j(\boldsymbol{a}(t_j)\boldsymbol{x}^*-b(t_j))=0,\ \ j=1,2,\cdots,k;$$

(iv) *(Lagrangian saddle point condition) There exists $\lambda^*\in\mathbb{R}^{(T)}_+$ such that*

$$L(\boldsymbol{x}^*,\lambda)\leq L(\boldsymbol{x}^*,\lambda^*)\leq L(\boldsymbol{x},\lambda^*),\ \ \textit{for all } \boldsymbol{x}\in\mathbb{R}^n \textit{ and all } \lambda\in\mathbb{R}^{(T)}_+.$$

Proof. From Lemma 3.4, it suffices to prove that (i) and (ii) are equivalent. It is well-known that (ii) implies (i). So, we need only to prove that (i) implies (ii).

Let (i) holds. The linear inequality $-\boldsymbol{c}^T\boldsymbol{x}\leq-\boldsymbol{c}^T\boldsymbol{x}^*$ is a (the only) linear consequence of the constraint system, that is binding $\mathcal{F}$ in direction $-\boldsymbol{c}$. Since the constraint system of problem $\mathrm{P}(\boldsymbol{c};\boldsymbol{a},b,T)$ is LDFM in $-\boldsymbol{c}$, $-\boldsymbol{c}^T\boldsymbol{x}\leq-\boldsymbol{c}^T\boldsymbol{x}^*$ is a linear consequence of a finite constraint subsystem

$$\boldsymbol{a}(t_i)\boldsymbol{x}\leq b(t_i),\ \ i=1,2,\cdots,K,$$

where $t_i\in T$ for $1\leq i\leq K$. This means that $\boldsymbol{x}^*$ is an optimal solution to the LP problem **Problem** LP:

$$\begin{array}{ll}\inf & \boldsymbol{c}^T\boldsymbol{x}\\ \text{s. t.} & \boldsymbol{a}(t_i)\boldsymbol{x}\leq b(t_i)\ \ i=1,2,\cdots,K\end{array}$$

Therefore, there are t_{i_j}, with $1 \le i_j \le K$, $j = 1, 2, \cdots, l$, such that

$$-\boldsymbol{c} = \sum_{j=1}^{l} \lambda_j \boldsymbol{a}(t_{i_j})^T,$$
$$\boldsymbol{a}(t_{i_j})\boldsymbol{x}^* = b(t_{i_j})$$

for some $\lambda_j > 0$, $j = 1, 2, \cdots, l$. These imply $-\boldsymbol{c} \in \mathcal{A}(\boldsymbol{x}^*)$. □

In fact, LDFM in $-\boldsymbol{c}$ is the weakest constraint qualification for the classical optimality theorem to hold. In other words, the inverse of Theorem 3.6 is true, as shown by the following result.

Corollary 3.7. *Suppose $\mathcal{F}^* \neq \phi$. Then, the following statements hold.*

(a) *If problem $P(\boldsymbol{c}; \boldsymbol{a}, b, T)$ is LDFM in $-\boldsymbol{c}$, then (ii)-(iv) in Theorem 3.6 are satisfied at all $\boldsymbol{x}^* \in \mathcal{F}^*$.*

(b) *If any one of (ii)-(iv) in Theorem 3.6 is satisfied at any $\boldsymbol{x}^* \in \mathcal{F}^*$, the problem $P(\boldsymbol{c}; \boldsymbol{a}, b, T)$ is LDFM in $-\boldsymbol{c}$.*

Proof. The sufficiency is given by Theorem 3.6. The necessity is seen by observing the fact that the only binding linear consequence $-\boldsymbol{c}^T\boldsymbol{x} \le \boldsymbol{c}^T\boldsymbol{x}^*$ is a consequence of the finite constraint subsystem

$$\{\boldsymbol{a}(t_j)\boldsymbol{x} \le b(t_j) \,|\, j = 1, 2, \cdots, k\}$$

corresponds to those appear in the complementarity condition (iii). □

The following is an immediate consequence of Corollary 3.7.

Corollary 3.8. *The following statements are equivalent:*

(a) *Problem $P(\boldsymbol{c}; \boldsymbol{a}, b, T)$ is LDFM in $-\boldsymbol{c}$.*

(b) *Conditions (i)-(iv) are equivalent at all optimal solutions.*

The last corollary shows that in order for the classical optimality conditions hold at all optimal solutions, the requirement that problem $\mathrm{P}(\boldsymbol{c}; \boldsymbol{a}, b, T)$ be LDFM in $-\boldsymbol{c}$ cannot be further relaxed.

The application of Theorem 3.6 to Theorem 3.3 leads to an optimality result without CQ, as given in the following. We note that (3.3) is the KKT condition of problem $\mathrm{P}(\boldsymbol{c}^i, \boldsymbol{a}, b, T(\boldsymbol{x}^i))$ at its feasible solution $\boldsymbol{x}^i$. The following theorem is then implied by Theorems 3.3 and 3.6.

Theorem 3.9. *Consider problem $P(\boldsymbol{c}, \boldsymbol{a}, b, T)$. A feasible solution $\boldsymbol{x}^* \in \mathcal{F}$ is optimal if and only if there exist sequences $\{\boldsymbol{c}^i\}$ and $\{\boldsymbol{x}^i\}$ in $\mathbb{R}^n$ such that*

$$\boldsymbol{c}^i \to \boldsymbol{c} \ (i \to \infty),$$
$$\boldsymbol{x}^i \to \boldsymbol{x}^* \ (i \to \infty),$$

and for each $i = 1, 2, \cdots$, problem $P(\boldsymbol{c}^i, \boldsymbol{a}, b, T(\boldsymbol{x}^i))$ is LDFM in $-\boldsymbol{c}^i$ and has optimal solution $\boldsymbol{x}^i$.

Optimality theorems for various subclasses of semi-infinite programming problems can be found in, for example, [11, 14, 15, 17] and the references cited therein.

4 Comments

This paper provides a simple but non-trivial constraint qualification that characterizes the class of all LSIP problems for which the classical KKT, complementary, and Lagrangian saddle point conditions at a feasible point are equivalent to the optimality of that point. This result is an improvement over the most general result in the existing literature. It is shown that the satisfaction of the classical KKT, or complementary, or Lagrangian saddle point conditions at a feasible point is equivalent to its optimality if and only if the problem is locally directional Farkas-Minkowski in the negative direction of the objective vector, given that the LSIP problem is posed as a minimization problem. This, together with the recent development of LSIP optimality theorems without constraint qualifications, makes the LSIP optimality theory fairly complete in the sense that we are able to characterize all LSIP problems for which the classical optimality theorem applies, and that we have developed new optimality conditions in cases where the classical optimality theorem fails to apply. In addition, the new result has found an application in generalizing the fundamental theorem of linear programming to linear semi-infinite programming.

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