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EXACT PENALIZATION AND STRONG KARUSH-KUHN-TUCKER CONDITIONS FOR NONSMOOTH MULTIOBJECTIVE OPTIMIZATION PROBLEMS*

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Abstract: We study the strong Karush-Kuhn-Tucker conditions of multiobjective optimization problems with inequality, equality and set constraints, where the objective and constraint functions are locally Lipschitz. We introduce a calmness condition associated with the objective function and the constraint system, and demonstrate that it is equivalent to two classes of exact penalty functions. From the exact penalization results, strong Karush-Kuhn-Tucker conditions are obtained in terms of Clarke subdifferential. Finally, we obtain that the generalized Mangasarian-Fromovitz constraint qualification implies the calmness of multiobjective optimization problems in the case of smooth.

Key words: *multiobjective optimization, calmness condition, exact penalization, strong Karush-Kuhn-Tucker condition.*

Mathematics Subject Classification: *49J52, 90C29, 90C46.*

1 Introduction

Karush-Kuhn-Tucker (in short, KKT) conditions of multiobjective optimization problem is a significant topic in optimization, which play an important role in optimization theories and numerical algorithms of constrained extremum problems. Under the same constraint qualification used in nonlinear programming, the KKT conditions obtained can only guaranteed that the multiplier vector of the objective function is nonzero. If all the multipliers corresponding to the objective functions are positive, which we called strong KKT conditions, then all components of the objective function have effective influence on the optimum. Due to the importance of the strong KKT conditions in multiobjective optimization problems, it is essential to study the corresponding constraint qualifications.

During the past decades, there has been tremendous interest in studying the strong KKT conditions in the smooth and nonsmooth cases; see [1, 3, 4, 6–10, 12–17] and the references therein. Maeda [16] proposed a generalized Guignard constraint qualification, which was not only depends on the constraint system but also the objective function, and derived strong KKT necessary conditions in differentiable case. Later on, based on [16], Preda [17] established strong KKT conditions for the semidifferentiable multiobjective optimization

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problems. On the line of their work, strong KKT conditions under various generalized constraint qualifications were presented in [7, 8, 12] for locally Lipschitz multiobjective optimization problems. Burachik and Rizvi [4] obtained strong KKT necessary conditions by a new generalized Abadie constraint qualification in smooth case. In a word, the constraint qualifications presented in [1, 3, 4, 6–10, 12–17], which involve not just constraint system but also objective function, are extensions of constraint qualifications used in nonlinear programming.

As we all know, the penalization method is a very important and effective tool for dealing with optimization theories, see [2, 5, 11, 20, 21] and references therein. It is worth noting that the standard Mangasarian-Fromovitz constraint qualification and error bound condition for a nonlinear programming problem with equality and inequality constraints implies the calmness condition; see [19] for details. Recently, Zhu and Li [22] proposed a general multiobjective optimization problem with equilibrium constraints and showed two classes of multiobjective penalty problems are equivalent to the calmness condition, and obtained a Mordukhovich stationary necessary optimality condition. Inspired by the ideas reported in [17] and the set Q_i used in [3, 8, 12–14, 16, 17], the main purpose of this work is to study strong KKT conditions for nonsmooth multiobjective optimization problems (in short, MOP) via the penalization method. We introduce a (MOP)-calmness condition with order $\sigma > 0$ at a local efficient (weak efficient) solution associated with the objective function and the constraint system, and show that the (MOP)-calmness condition can be implied by a error bound condition of the parametric form of the set Q_i. Moreover, we establish some equivalent relationships between the exact penalization property with order $\sigma > 0$ and the (MOP)-calmness condition. Based on the (MOP)-calmness condition with order 1, we obtain strong KKT conditions for (MOP) in terms of the Clarke subdifferential. Finally, we obtain that the generalized Mangasarian-Fromovitz constraint qualification, which is considered by Golestani and Nobakhtian in [8] for multiobjective programming, implies the calmness of multiobjective optimization problems in the case of smooth.

The outline of this paper is as follows. In Section 2, we recall some notions and preliminary results. In Section 3, the (MOP)-calmness condition for (MOP) and some relationships between the exact penalization property and the (MOP)-calmness condition are presented. The strong KKT conditions for (MOP) under the (MOP)-calmness condition with order 1 are given in Section 4, and we also derive that the generalized Mangasarian-Fromovitz constraint qualification implies the calmness of multiobjective optimization problems in the case of smooth in this part.

2 Preliminaries

Let $\mathbb{R}^l$ be the l-dimensional Euclidean space. For $\forall x = (x_1, \dots, x_l)$ and $y = (y_1, \dots, y_l) \in \mathbb{R}^l$, we use the following notations

$$\begin{gathered} x = y, \text{ if } x_i = y_i, \text{ for all } i, \\ x \leqq y, \text{ if } x_i \le y_i, \text{ for all } i, \\ x < y, \text{ if } x_i < y_i, \text{ for all } i, \\ x \le y, \text{ if } x \leqq y, \text{ and } x \ne y. \end{gathered}$$

Since all norms on finite dimensional spaces are equivalent, we take specially the sum norm on $\mathbb{R}^n$ and the product space $\mathbb{R}^n \times \mathbb{R}^m$ for simplicity, that is, for all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we write $\|x\| = |x_1| + |x_2| + \cdots + |x_n|$, and for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, $\|(x, y)\| = \|x\| + \|y\|$. As usual, we denote by $x^\top$ the transposition of x, and $\langle x, y\rangle := x^\top y$ the inner product of vectors x and y. In general, all vectors are viewed as column vectors and we denote by $\mathcal{B}_{\mathbb{R}^n}$ the closed unit ball in $\mathbb{R}^n$, and $\mathbb{B}(\bar{x}, r)$ the open ball with center at $\bar{x}$ and radius $r > 0$ for any $\bar{x} \in \mathbb{R}^n$. For a point $\bar{x}$ and a set C, the distance between them is denoted by $d(\bar{x}, C) = \inf_{c \in C} \|\bar{x} - c\|$. It is said that $G : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz around $\bar{x}$ iff there exist $L > 0$ and $\delta > 0$ such that $|G(x) - G(y)| \leq L\|x - y\|, \forall x, y \in \mathbb{B}(\bar{x}, \delta)$, and G is locally Lipschitz on $A \subseteq \mathbb{R}^n$ if and only if G is Lipschitz around each $\bar{x} \in A$. For $n, p, m, s \in \mathbb{N}$, we consider the following multiobjective optimization problem:

$$\begin{array}{lll} (\textbf{MOP}) & \min & f(x) \\ & s.t. & g(x) \leqq 0, \\ & & h(x) = 0, \\ & & x \in K, \end{array}$$

where $f : \mathbb{R}^n \to \mathbb{R}^p, f(x) = (f_1(x), f_2(x), \ldots, f_p(x)), g : \mathbb{R}^n \to \mathbb{R}^m, g(x) = (g_1(x), g_2(x), \ldots, g_m(x))$, $h : \mathbb{R}^n \to \mathbb{R}^s, h(x) = (h_1(x), h_2(x), \ldots, h_s(x))$ are vector-valued maps and K is a nonempty and closed subset of $\mathbb{R}^n$.

Throughout this paper, we assume that the feasible set of (MOP), $X := \{x \in \mathbb{R}^n \mid g(x) \leqq 0, h(x) = 0, x \in K\}$ is nonempty and $f_i\, (i \in \{1, 2, \ldots, p\}), g_j\, (j \in \{1, 2, \ldots, m\}), h_k\, (k \in \{1, 2, \ldots, s\})$ are locally Lipschitz on X. Obviously, X is a closed subset of $\mathbb{R}^n$, note $\mathfrak{J}(\bar{x}) := \{i \in \{1, 2, \ldots, m\} \mid g_i(\bar{x}) = 0\}$ is the index set of active constraints of g at $\bar{x}$.

In the context of multiobjective optimization problems, an optimal solution that simultaneously minimizes all the objectives is usually not possible, so solutions are often interchanged by efficient solutions and weak efficient solutions, now we give the definitions as below.

Definition 2.1. A point $\bar{x} \in X$ is said to be efficient for problem (MOP) iff there is no $x \in X$ such that $f(x) \leq f(\bar{x})$. A point $\bar{x}$ is said to be local efficient for problem (MOP) iff there exists $r > 0$ such that there is no $x \in X \cap \mathbb{B}(\bar{x}, r)$ such that $f(x) \leq f(\bar{x})$.

Definition 2.2. A point $\bar{x} \in X$ is said to be weak efficient for problem (MOP) iff there is no $x \in X$ such that $f(x) < f(\bar{x})$. A point $\bar{x}$ is said to be local weak efficient for problem (MOP) iff there exists $r > 0$ such that there is no $x \in X \cap \mathbb{B}(\bar{x}, r)$ such that $f(x) < f(\bar{x})$.

Given a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ local Lipschitz around $\bar{x}$, we define its Clarke directional derivative at $\bar{x}$ in the direction d by:

$$\varphi^\circ(\bar{x}; d) := \limsup_{u \to \bar{x}, t \downarrow 0} \frac{\varphi(u + td) - \varphi(u)}{t},$$

and the Clarke subdifferential of φ at $\bar{x}$ is defined by:

$$\partial_C \varphi(\bar{x}) := \{\xi \in \mathbb{R}^n \mid \varphi^\circ(\bar{x}; d) \geq \langle \xi, d \rangle, \forall d \in \mathbb{R}^n\}.$$

The Clarke subdifferential of φ is always convex, nonempty and compact.

Let A be a subset of $\mathbb{R}^n$ and $\bar{x} \in A$, we introduce the Clarke tangent cone and the Clarke normal cone to A at $\bar{x}$:

The Clarke tangent cone to A at $\bar{x}$ is

$$T_C(A, \bar{x}) := \{v \in \mathbb{R}^n \mid \forall t_n \downarrow 0, \forall x_n \to \bar{x} \text{ with } x_n \in A, \exists v_n \to v; x_n + t_n v_n \in A, \forall n\}.$$

The Clarke normal cone to A at $\bar{x}$ is

$$N(A, \bar{x}) := \{v \in \mathbb{R}^n \mid \langle w, v \rangle \leq 0, \forall w \in T_C(A, \bar{x})\}.$$

Propositions 2.3 and 2.4 summarize some well-known properties of the Clarke subdifferential, which presented in [5].

Proposition 2.3. *If* $\varphi : \mathbb{R}^n \to \mathbb{R}$ *is locally Lipschitz around* $\bar{x}$, *then*

(i) *For any* $\sigma \in \mathbb{R}, \partial_C(\sigma\varphi)(\bar{x}) = \sigma\partial_C\varphi(\bar{x})$.

(ii) *If* $\bar{x}$ *is a local minimizer of* φ *on* $D \subset \mathbb{R}^n$, *then* $0 \in \partial_C\varphi(\bar{x}) + N(D, \bar{x})$.

Proposition 2.4. *For all* $i \in \{1, 2, \dots, n\}, \varphi_i : \mathbb{R}^n \to \mathbb{R}$ *be a locally Lipschitz function, then*

(i) $\partial_C(\Sigma_{i=1}^n \varphi_i)(\bar{x}) \subseteq \Sigma_{i=1}^n \partial_C\varphi_i(\bar{x})$.

(ii) *For the maximum functional* $\varphi : \mathbb{R}^n \to \mathbb{R}$ *defined by* $\varphi(x) := \max\{\varphi_i(x) \mid i = 1, 2, \dots, n\}, x \in \mathbb{R}^n$. *Then*

$$\partial_C\varphi(\bar{x}) \subseteq co\{\partial_C\varphi_i(\bar{x}) \mid i \in I(\bar{x})\},$$

where $I(\bar{x}) = \{i \in \{1, 2, \dots, n\} \mid \varphi_i(\bar{x}) = \varphi(\bar{x})\}$.

3 Exact Penalization, Calmness Condition for MOP

In this section, we focus our attention on establishing some equivalent properties between a single objective exact penalization and a calmness condition, called (MOP)-calmness, for (MOP). Simultaneously, we present that a local error bound condition associated with an extension of the constraint system of (MOP), that is, a calmness condition of the parametric form of the set Q_i, which used in [8, 16, 17], implies the (MOP)-calmness condition.

Fixed $\bar{x} \in X$, for every $i \in \{1, 2, \dots, p\}$, define Q_i, which is called the extension of the constraint system of (MOP), as below:

$$Q_i := \{x \in K \mid g(x) \leqq 0, h(x) = 0, f_j(x) \leq f_j(\bar{x}), j = 1, 2, \dots, p \text{ and } j \neq i\}.$$

Consider the following parametric form of the set Q_i with parameter $(u, v, y^i) \in \mathbb{R}^{m+s+p-1}$:

$$\begin{aligned} &g(x) + u \leqq 0,\\ &h(x) + v = 0,\\ &f_j(x) + y^i_j \leq f_j(\bar{x}), j = 1, 2, \dots, p \text{ and } j \neq i,\\ &y^i = (y^i_1, y^i_2, \dots, y^i_{i-1}, y^i_{i+1}, \dots, y^i_p). \end{aligned}$$

Denote the corresponding feasible set by

$$Q_i(u, v, y^i) := \{x \in K \mid g(x)+u \leqq 0, h(x)+v = 0, f_j(x)+y^i_j \leq f_j(\bar{x}), j = 1, 2, \dots, p \text{ and } j \neq i\}. \tag{3.1}$$

(MOP)-calmness plays an important role in this paper, which is crucial for the strong KKT conditions of multiobjective optimization problems.

Definition 3.1. Given $\sigma > 0$ and $\bar{x} \in X$ being a local efficient (resp. local weak efficient) solution for (MOP), then (MOP) is said to be (MOP)-calm with order σ at $\bar{x}$ iff there exist $\delta > 0$ and $M > 0$ such that for every $i \in \{1, 2, \dots, p\}$, all $(u, v, y^i) \in \mathbb{B}(0_{\mathbb{R}^{m+s+p-1}}, \delta)$ and all $x \in Q_i(u, v, y^i) \cap \mathbb{B}(\bar{x}, \delta)$, one has:

$$f_i(x) + M\|(u, v, y^i)\|^\sigma \geq f_i(\bar{x}).$$

Remark 3.2. Given $\sigma > 0$ and $\bar{x} \in X$ being a local efficient (resp. local weak efficient) solution for (MOP), we can also characterize the (MOP)-calmness condition by means of sequences. (MOP) is (MOP)-calm with order σ at $\bar{x}$ if and only if there exists $M > 0$ such that for every $i \in \{1, 2, \dots, p\}$, every sequence $\{(u_k, v_k, y^{i(k)})\} \subset \mathbb{R}^{m+s+p-1}$ with $(u_k, v_k, y^{i(k)}) \to 0_{\mathbb{R}^{m+s+p-1}}$ and every sequence $\{x_k\} \subset K$ satisfying $g(x_k) + u_k \leqq 0, h(x_k) + v_k = 0, f_j(x_k) + y_j^{i(k)} \leq f_j(\bar{x}), j = 1, 2, \dots, p, j \neq i$ and $x_k \to \bar{x}$, it holds that

$$f_i(x_k) + M\|(u_k, v_k, y^{i(k)})\|^\sigma \geq f_i(\bar{x}),$$

where $y^{i(k)} = (y_1^{i(k)}, y_2^{i(k)}, \dots, y_{i-1}^{i(k)}, y_{i+1}^{i(k)}, \dots, y_p^{i(k)})$.

According to Definition 3.1, (MOP)-calmness condition is not only depend on the objective function but also the constraint system. Now we propose the following local error bound notion for (MOP) associated with the extension of the constraint system of (MOP).

Definition 3.3. Given $\sigma > 0$ and $\bar{x} \in X$, the extension of the constraint system of (MOP) is said to be have a local error bound with order σ at $\bar{x}$ iff there exist $\delta > 0$ and $M > 0$ such that for every $i \in \{1, 2, \dots, p\}$, all $(u, v, y^i) \in \mathbb{B}(0_{\mathbb{R}^{m+s+p-1}}, \delta)\backslash\{0\}$ and all $x \in Q_i(u, v, y^i) \cap \mathbb{B}(\bar{x}, \delta)$, one has:

$$d(x, Q_i) < M\|(u, v, y^i)\|^\sigma.$$

Remark 3.4. If $p = 1$, the Definition 3.3 reduces to that there exist $\delta > 0$ and $M > 0$ such that for all $(u, v) \in \mathbb{B}(0_{\mathbb{R}^{m+s}}, \delta)\backslash\{0\}$ and all $x \in Q(u, v) \cap \mathbb{B}(\bar{x}, \delta)$, one has $d(x, Q) < M\|(u, v)\|^\sigma$, where

$$Q := \{x \in K \mid g(x) \leqq 0, h(x) = 0\},$$
$$Q(u, v, y) := \{x \in K \mid g(x) + u \leqq 0, h(x) + v = 0\}.$$

Now we verify that the calmness of (MOP) can be implied by the error boundness of the extension of the constraint system of (MOP).

Theorem 3.5. *Let $\bar{x} \in X$ be a local efficient solution for* (MOP)*, if the extension of the constraint system of* (MOP) *has a local error bound with order σ at $\bar{x}$, then* (MOP) *is* (MOP)*-calm with order σ at $\bar{x}$.*

Proof. We consider two cases, respectively.

Case 1: $(u, v, y^i) = 0_{\mathbb{R}^{m+s+p-1}}$. Since $\bar{x} \in X$ is a local efficient solution for (MOP), then for each $i \in \{1, 2, \dots, p\}$, for all $x \in Q_i(u, v, y^i) \cap \mathbb{B}(\bar{x}, \delta)$ and $\delta > 0$ sufficiently small, we have

$$f_i(x) + M\|(u, v, y^i)\|^\sigma = f_i(x) \geq f_i(\bar{x}).$$

Case 2: $(u, v, y^i) \neq 0_{\mathbb{R}^{m+s+p-1}}$. We assume that (MOP) is not (MOP)-calm with order σ at $\bar{x}$. Then, there exists $i \in \{1, 2, \dots, p\}$, such that for every $k \in \mathbb{N}$, there exist $(u_k, v_k, y^{i(k)}) \in \mathbb{B}(0, \frac{1}{k})\backslash\{0_{\mathbb{R}^{m+s+p-1}}\}$ and $x_k \in Q_i(u_k, v_k, y^{i(k)}) \cap \mathbb{B}(\bar{x}, \frac{1}{k})$ satisfying

$$f_i(x_k) + k\|(u_k, v_k, y^{i(k)})\|^\sigma - f_i(\bar{x}) < 0. \tag{3.2}$$

Since Q_i is nonempty and closed, hence there exists a projection $P(x_k, Q_i)$ of x_k onto Q_i such that $d(x_k, Q_i) = \|x_k - P(x_k, Q_i)\|$ for all $k \in \mathbb{N}$. As $(u_k, v_k, y^{i(k)}) \to (0_{\mathbb{R}^m}, 0_{\mathbb{R}^s}, 0_{\mathbb{R}^{p-1}})$, $x_k \in Q_i(u_k, v_k, y^{i(k)})$, then $d(x_k, Q_i) \to 0$. Together with $x_k \to \bar{x}$, it follows that

$$\|P(x_k, Q_i) - \bar{x}\| \leq \|P(x_k, Q_i) - x_k\| + \|x_k - \bar{x}\| = d(x_k, Q_i) + \|x_k - \bar{x}\| \to 0.$$

Combining with $\bar{x} \in X$ is a local efficient solution for (MOP), there exists $N_1 \in \mathbb{N}$, such that

$$f_i(P(x_k, Q_i)) - f_i(\bar{x}) \geq 0, \qquad \forall k \geq N_1. \tag{3.3}$$

Moreover as f_i is locally Lipschitz, there exist a constant $L > 0$ and $N_2 \in \mathbb{N}$, such that

$$|f_i(x_k) - f_i(P(x_k, Q_i))| \leq L\|x_k - P(x_k, Q_i)\|, \qquad \forall k \geq N_2. \tag{3.4}$$

Combining with (3.2) and (3.3), we have that for all $k \geq N_1$

$$f_i(P(x_k, Q_i)) - f_i(x_k) = f_i(P(x_k, Q_i)) - f_i(\bar{x}) + f_i(\bar{x}) - f_i(x_k) > k\|u_k, v_k, y^{i(k)}\|^\sigma > 0.$$

Further, in view of $d(x_k, Q_i) = \|x_k - P(x_k, Q_i)\|$ and (3.4), we have

$$\begin{aligned} d(x_k, Q_i) = \|x_k - P(x_k, Q_i)\| &\geq \frac{1}{L}|f_i(x_k) - f_i(P(x_k, Q_i))| \\ &> \frac{k}{L}\|(u_k, v_k, y^{i(k)})\|^\sigma, \ \forall k \geq \max\{N_1, N_2\}. \end{aligned}$$

This is a contradiction to that (MOP) has a local error bound with order σ at $\bar{x}$ since $\frac{K}{L} \to +\infty, (u_k, v_k, y^{i(k)}) \neq 0_{\mathbb{R}^{m+s+p-1}}, (u_k, v_k, y^{i(k)}) \to 0_{\mathbb{R}^{m+s+p-1}}, x_k \in Q_i(u_k, v_k, y^{i(k)})$ and $x_k \to \bar{x}$. □

Now we give an example to illustrate that the converse of Theorem 3.5 may not true.

Example 3.6. For $n = 2, p = 2, m = s = 1, K = [-2, 2] \times [-2, 2]$,

$$f_1(x) = \begin{cases} |x_1| & -2 \leq x_1 < -1.5 \text{ or } 1.5 < x_1 \leq 2 \\ 1.5 & -1.5 \leq x_1 \leq 1.5 \end{cases}$$

and

$$f_2(x) = \begin{cases} 2|x_2| & -2 \leq x_2 < -0.5 \text{ or } 0.5 < x_2 \leq 2 \\ 1 & -0.5 \leq x_2 \leq 0.5. \end{cases}$$

Consider the following multiobjective programming:

$$\begin{aligned} \min \quad & f(x) = (f_1(x), f_2(x)) \\ s.t. \quad & g(x) = |x_1| + x_2 \leq 0, \\ & h(x) = x_1 + 2|x_2| = 0, \\ & x \in K. \end{aligned}$$

Obviously, $f_1(x), f_2(x), g(x)$ and $h(x)$ are locally Lipschitz maps. The set of all efficient solutions is given as $S = \{(x_1, x_2) \mid x_1 = 2x_2, -0.5 \leq x_2 \leq 0\}$. Now we choose $\bar{x} = (0, 0)$ and $\sigma = 2$, by the definition of Q_i, we have

$$\begin{aligned} Q_1(u, v, y^1) &= \{x \in K \mid |x_1| + x_2 + u \leq 0, x_1 + 2|x_2| + v = 0, f_2(x) + y^1 \leq 1\}, \\ Q_2(u, v, y^2) &= \{x \in K \mid |x_1| + x_2 + u \leq 0, x_1 + 2|x_2| + v = 0, f_1(x) + y^2 \leq 1.5\}. \end{aligned}$$

For any $0 < \delta \leq 0.5$ and $M > 0$, for each $i \in \{1, 2\}$ and all $(u, v, y^i) \in \mathbb{B}(0, \delta)$ and all $x \in Q_i(u, v, y^i) \cap \mathbb{B}(\bar{x}, \delta)$, we have

$$f_i(x) + M\|(u, v, y^i)\|^2 \geq f_i(\bar{x}),$$

hence, (MOP)-calmness condition is satisfied for $\bar{x} = (0, 0)$.

For $i = 1$ and every $k \in \mathbb{N}$, we just choose $(u_k, v_k, y^{1(k)}) = (-\frac{1}{9k}, -\frac{1}{9k}, -\frac{1}{9k}) \in \mathbb{B}(0_{\mathbb{R}^3}, \frac{1}{k})$ and $\tilde{x} = (\tilde{x_1}, \tilde{x_2}) = (\frac{1}{9k}, 0) \in \mathbb{B}(\bar{x}, \frac{1}{k})$, then

$$|\tilde{x_1}| + \tilde{x_2} + u_k \leq 0, \quad \tilde{x_1} + 2|\tilde{x_2}| + v_k = 0, \quad f_2(\tilde{x}) + y^{1(k)} \leq 1$$

are satisfied, thus $\tilde{x} \in Q_1(u_k, v_k, y^{1(k)}) \cap \mathbb{B}(\bar{x}, \frac{1}{k})$. As

$$Q_1 = \{x \in K \mid |x_1| + x_2 \leq 0, x_1 + 2|x_2| = 0, f_2(x) \leq 1\} = \{(0, 0)\},$$

then $d(\tilde{x}, Q_1) = \frac{1}{9k}$, together with $k\|(u_k, v_k, y^{1(k)}\|^2 = \frac{1}{9k}$ and Definition 3.3, the (MOP) does not have a local error bound with order 2 at $\bar{x}$.

Recall that a set-valued map $\Psi : \mathbb{R}^n \rightrightarrows \mathbb{R}^s$ is said to be calm with order $\sigma > 0$ at $(\hat{x}, \hat{y}) \in \text{gph}\Psi = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^s \mid y \in \Psi(x)\}$ iff there exist neighborhoods U of $\hat{x}$ and V of $\hat{y}$, and a real number $\ell > 0$ such that

$$\Psi(x) \cap V \subset \Psi(\hat{x}) + \ell\|x - \hat{x}\|^\sigma \mathfrak{B}_{\mathbb{R}^s}, \qquad \forall x \in U.$$

In the following proposition, we obtain two equivalent enumerates of the local error bounds of the extension constraint system of (MOP).

Proposition 3.7. *Suppose $\sigma > 0$ and $\bar{x} \in X$. Then the following assertions are equivalent:*

(i) *The extension constraint system of* (MOP) *has a local error bound with order σ at $\bar{x}$.*

(ii) *For every $i \in \{1, 2, \ldots, p\}$, the set-valued map $Q_i : \mathbb{R}^{m+s+p-1} \rightrightarrows \mathbb{R}^n$, defined in* (3.1), *is calm with order σ at $(0_{\mathbb{R}^{m+s+p-1}}, \bar{x})$.*

If $\sigma = 1$, then (i) and (ii) are also equivalent to (iii).

(iii) *For every $i \in \{1, 2, \ldots, p\}$, there exist $M > 0, \tilde{\varepsilon} > 0$, for any $x \in \mathbb{B}(\bar{x}, \tilde{\varepsilon})$, $d(x, Q_i) < Md((g(x), h(x), p_i(x), x), \mathbb{R}^m_- \times \{0\}_{\mathbb{R}^s} \times \mathbb{R}^{p-1}_- \times K)$ holds, where*

$$p_i(x) = (f_1(x) - f_1(\bar{x}), \ldots, f_{i-1}(x) - f_{i-1}(\bar{x}), f_{i+1}(x) - f_{i+1}(\bar{x}), \ldots, f_p(x) - f_p(\bar{x})).$$

Proof. (ii) $\Rightarrow$ (i). For every $i \in \{1, 2, \ldots, p\}$, by virtue of (ii), there exist $\mathbb{B}(0_{\mathbb{R}^{m+s+p-1}}, \delta)$, a neighborhood of $0_{\mathbb{R}^{m+s+p-1}}$ and $\mathbb{B}(\bar{x}, \delta)$, a neighborhood of $\bar{x}$ and $h > 0$, for $\forall (u, v, y^i) \in \mathbb{B}(0_{\mathbb{R}^{m+s+p-1}}, \delta)$, we have that

$$Q_i(u, v, y^i) \cap \mathbb{B}(\bar{x}, \delta) \subset Q_i + h\|(u, v, y^i)\|^\sigma \mathfrak{B}_{\mathbb{R}_n},$$

thus for $x \in \mathbb{B}(\bar{x}, \delta) \cap Q_i(u, v, y^i)$, we obtain

$$d(x, Q_i) < h\|(u, v, y^i)\|^\sigma.$$

Thus (i) follows.

(i) $\Rightarrow$ (ii). It is easy to verify that this proof is reversed of the former.

If $\sigma = 1$, let $p(x) := (g(x), h(x), p_i(x), x)$. Now we choose $0 < \tilde{\varepsilon} < \varepsilon$ such that $\|p(x) - p(\bar{x})\| \leq \frac{\varepsilon}{2}$ for all $x \in \mathbb{B}(\bar{x}, \tilde{\varepsilon})$, where ε refers to (iii). For arbitrary $\eta \in (0, \frac{\varepsilon}{2})$, there is some $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R}^m_- \times \{0\}_{\mathbb{R}^s} \times \mathbb{R}^{p-1}_- \times K$ such that

$$\|p(x) - (\lambda_1, \lambda_2, \lambda_3, \lambda_4)\| \leq d(p(x), \mathbb{R}^m_- \times \{0\}_{\mathbb{R}^s} \times \mathbb{R}^{p-1}_- \times K) + \eta \leq \|p(x) - p(\bar{x})\| + \frac{\varepsilon}{2} \leq \varepsilon.$$

(i) $\Rightarrow$ (iii). For any $x \in \mathbb{B}(\bar{x}, \tilde{\varepsilon})$, since $x \in Q_i((\lambda_1, \lambda_2, \lambda_3, \lambda_4) - p(x))$ and $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) - p(x) \in \mathbb{B}(0, \varepsilon)$, by (i), there exists $M > 0$ such that

$$d(x, Q_i) < M\|(\lambda_1, \lambda_2, \lambda_3, \lambda_4) - p(x)\| \leq Md(p(x), \mathbb{R}^m_- \times \{0\}_{\mathbb{R}^s} \times \mathbb{R}^{p-1}_- \times K) + M\eta.$$

Taking into account that η is arbitrary, (iii) is obtained.

(iii) $\Rightarrow$ (ii). For every $x \in \mathbb{B}(\bar{x}, \tilde{\varepsilon}) \cap Q_i(u, v, y^i)$, then $g(x) + u \leqq 0, h(x) + v = 0, f_j(x) + y^i_j \leq f_j(\bar{x}), j = 1, 2, \ldots, p, j \neq i, x + 0_{\mathbb{R}^n} \in K$, thus $d(p(x), \mathbb{R}^m_- \times \{0\}_{\mathbb{R}^s} \times \mathbb{R}^{p-1}_- \times K) \leq \|(u, v, y^i, 0_{\mathbb{R}^n})\| = \|(u, v, y^i)\|$, so (ii) follows.

In view of (i) $\Leftrightarrow$ (ii), the proof is complete. □

Now we give the equivalent characterizations of two classes of penalty problems and the (MOP)-calmness condition in the following theorem.

Theorem 3.8. *Let $\bar{x} \in X$ be a local efficient (resp. local weak efficient) solution for* (MOP), *then the following assertions are equivalent:*

(i) (MOP) *is* (MOP)*-calm with order $\sigma > 0$ at $\bar{x}$.*

(ii) *For each $i \in \{1, 2, \ldots, p\}$, there exist some $\hat{\rho} > 0$ such that for any $\rho \geq \hat{\rho}$, $(\bar{x}, 0_{\mathbb{R}^{p-1}})$ is a local efficient (resp. local weak efficient) solution for the following penalty problem with order σ:*

$$\begin{array}{lll} (\mathbf{I}) & \min & f_i(x) + \rho[\|g_+(x)\| + \|h(x)\| + \|y^i\|]^\sigma \\ & s.t. & f_j(x) + y^i_j \leq f_j(\bar{x}), j = 1, 2, \ldots, p, j \neq i, \\ & & x \in K, \end{array}$$

where

$$g_+(x) := (\max\{g_1(x), 0\}, \max\{g_2(x), 0\}, \ldots, \max\{g_m(x), 0\}),$$
$$y^i = (y^i_1, \ldots, y^i_{i-1}, y^i_{i+1}, \ldots, y^i_p).$$

(iii) *For each $i \in \{1, 2, \ldots, p\}$, there exist some $\hat{\mu} > 0$ such that for any $\mu \geq \hat{\mu}$, $\bar{x}$ is a local efficient (resp. local weak efficient) solution for the following penalty problem with order σ:*

$$\begin{array}{lll} (\mathbf{II}) & \min & f_i(x) + \mu[\|g_+(x)\| + \|h(x)\| + \displaystyle\sum_{j=1, j\neq i}^{p} |(f_j(x) - f_j(\bar{x}))_+|]^\sigma \\ & s.t. & x \in K. \end{array}$$

Proof. We only prove the case for $\bar{x}$ being a local weak efficient solution since the proof of the case for $\bar{x}$ being a local efficient solution is similar.

(i) $\Rightarrow$ (ii). Assume to the contrary that for every $k \in \mathbb{N}$, there exists $i \in \{1, 2, \ldots, p\}$, $(x_k, y^{i(k)}) \in \mathbb{B}((\bar{x}, 0_{\mathbb{R}^{p-1}}), \frac{1}{k})$ with $x_k \in K$ and $f_j(x_k) + y^{i(k)}_j \leq f_j(\bar{x}), j = 1, 2, \ldots, p, j \neq i$ such that

$$f_i(x_k) + k[\|g_+(x_k)\| + \|h(x_k)\| + \|y^{i(k)}\|]^\sigma < f_i(\bar{x}). \tag{3.5}$$

Taking $u_k = -g_+(x_k)$ and $v_k = -h(x_k)$, then it follows that $g(x_k) + u_k \leqq 0, h(x_k) + v_k = 0, f_j(x_k) + y^{i(k)}_j \leq f_j(\bar{x}), j = 1, 2, \ldots, p, j \neq i$, $\|g_+(x_k)\| = \|u_k\|$ and $\|h(x_k)\| = \|v_k\|$, thus $x_k \in Q_i(u_k, v_k, y^{i(k)})$ for all $k \in \mathbb{N}$. And by (3.5), we have

$$f_i(x_k) + k\|(u_k, v_k, y^{i(k)})\|^\sigma < f_i(\bar{x}).$$

As $x_k \to \bar{x}, g(\bar{x}) \leqq 0, h(\bar{x}) = 0, (x_k, y^{i(k)}) \in \mathbb{B}((\bar{x}, 0_{\mathbb{R}^{p-1}}), \frac{1}{k})$, g and h are locally Lipschitz, we have $(u_k, v_k, y^{i(k)}) \to 0_{\mathbb{R}^{m+s+p-1}}$, $x_k \in \mathbb{B}(\bar{x}, \frac{1}{k}) \cap Q_i(u_k, v_k, y^{i(k)})$, this is a contradiction to (MOP)-calmness with order $\sigma > 0$ of (MOP) at $\bar{x}$.

(ii) $\Rightarrow$ (i). Suppose that (MOP) is not (MOP)-calm with order $\sigma > 0$ at $\bar{x}$. Then for every $k \in \mathbb{N}$, there exists $i \in \{1, 2, \dots, p\}, (u_k, v_k, y^{i(k)}) \in \mathbb{B}(0_{\mathbb{R}^{m+s+p-1}}, \frac{1}{k})$ and $x_k \in Q_i(u_k, v_k, y^{i(k)}) \cap \mathbb{B}(\bar{x}, \frac{1}{k})$ such that

$$f_i(x_k) + k\|(u_k, v_k, y^{i(k)})\|^\sigma - f_i(\bar{x}) < 0. \tag{3.6}$$

As $g(x_k) + u_k \leqq 0, h(x_k) + v_k = 0$, we have

$$\begin{aligned} \|g_+(x_k)\| + \|h(x_k)\| &\leq \|g(x_k) - (g(x_k) + u_k)\| + \|h(x_k) - (h(x_k) + v_k)\| \\ &= \|u_k\| + \|v_k\|. \end{aligned}$$

Combining with (3.6), we obtain

$$\begin{aligned} & f_i(x_k) + k[\|g_+(x_k)\| + \|h(x_k)\| + \|y^{i(k)}\|]^\sigma \\ = \ & f_i(x_k) + k\|(u_k, v_k, y^{i(k)})\|^\sigma + k[(\|g_+(x_k)\| + \|h(x_k)\| + \|y^{i(k)}\|)^\sigma - \|(u_k, v_k, y^{i(k)})\|^\sigma] \\ \leq \ & f_i(x_k) + k\|(u_k, v_k, y^{i(k)})\|^\sigma < f_i(\bar{x}). \end{aligned}$$

This shows that the penalty problem with order σ does not admit a local exact penalization at $(\bar{x}, 0_{\mathbb{R}^{p-1}})$ since $x_k \in Q_i(u_k, v_k, y^{i(k)}) \cap \mathbb{B}(\bar{x}, \frac{1}{k})$ and $(x_k, y^{i(k)}) \to (\bar{x}, 0_{\mathbb{R}^{p-1}})$.

(i)$\Rightarrow$ (iii). Assume that for every $k \in \mathbb{N}$, there exist $i \in \{1, 2, \dots, p\}, a > 0$ and $x_k \in K \cap \mathbb{B}(\bar{x}, \frac{1}{ak})$, such that

$$f_i(x_k) + k[\|g_+(x_k)\| + \|h(x_k)\| + \sum_{j=1, j\neq i}^{p} |(f_j(x_k) - f_j(\bar{x}))_+|]^\sigma < f_i(\bar{x}), \tag{3.7}$$

and $\bar{x}$ is a weak efficient solution for (MOP) in $\mathbb{B}(\bar{x}, \frac{1}{ak})$. Taking $u_k = -g_+(x_k)$ and $v_k = -h(x_k)$. If $f_j(x_k) > f_j(\bar{x})$, take $y_j^{i(k)} = f_j(\bar{x}) - f_j(x_k)$, otherwise take $y_j^{i(k)} = 0$. As $g(x_k) + u_k \leqq 0, h(x_k) + v_k = 0, f_j(x_k) + y_j^{i(k)} \leq f_j(\bar{x}), j = 1, 2, \dots, p, j \neq i$ and $x_k \in K$, thus $x_k \in Q_i(u_k, v_k, y^{i(k)})$ for all $k \in \mathbb{N}$ and $|y_j^{i(k)}| = |f_j(x_k) - f_j(\bar{x})| = |(f_j(x_k) - f_j(\bar{x}))_+|, j = 1, 2, \dots, p, j \neq i$. By (3.7), we have

$$f_i(x_k) + k\|(u_k, v_k, y^{i(k)})\|^\sigma < f_i(\bar{x}). \tag{3.8}$$

As $x_k \to \bar{x}, g(\bar{x}) \leqq 0, h(\bar{x}) = 0$, g and h are locally Lipschitz, we have $(u_k, v_k, y^{i(k)}) \to 0$, together with $k \to +\infty, x_k \in Q_i(u_k, v_k, y^{i(k)}), x_k \to \bar{x}$ and (3.8), then this is a contradiction to (MOP)-calmness with order $\sigma > 0$ of (MOP) at $\bar{x}$.

(iii) $\Rightarrow$ (i). Suppose that (MOP) is not (MOP)-calm with order $\sigma > 0$ at $\bar{x}$. Then for every $k \in \mathbb{N}$, there exist $i \in \{1, 2, \dots, p\}, a > 0, (u_k, v_k, y^{i(k)}) \in \mathbb{B}(0, \frac{1}{ak})$ and $x_k \in Q_i(u_k, v_k, y^{i(k)}) \cap \mathbb{B}(\bar{x}, \frac{1}{ak})$ such that (3.6) holds and $\bar{x}$ is a weak efficient solution for (MOP) in $\mathbb{B}(\bar{x}, \frac{1}{ak})$. As $y_j^{i(k)} \leq f_j(\bar{x}) - f_j(x_k)$, we have $|y_j^{i(k)}| \geq |(f_j(x_k) - f_j(\bar{x}))_+|$, together with $\|g_+(x_k)\| \leq \|g(x_k) - (g(x_k) + u_k)\| = \|u_k\|$ and $\|h(x_k)\| = \|v_k\|$, thus

$$[\|g_+(x_k)\| + \|h(x_k)\| + \sum_{j=1, j\neq i}^{p} |(f_j(x_k) - f_j(\bar{x}))_+|]^\sigma \leq \|(u_k, v_k, y_k)\|^\sigma,$$

in view of (3.6), we obtain

$$\begin{aligned} & f_i(x_k) + k[\|g_+(x_k)\| + \|h(x_k)\| + \sum_{j=1,j\neq i}^{p} |(f_j(x_k) - f_j(\bar{x}))_+|]^\sigma \\ = \; & f_i(x_k) + k\|(u_k, v_k, y^{i(k)})\|^\sigma + k[(\|g_+(x_k)\| + \|h(x_k)\| \\ & + \sum_{j=1,j\neq i}^{p} |(f_j(x_k) - f_j(\bar{x}))_+|)^\sigma - \|(u_k, v_k, y^{i(k)})\|^\sigma] \\ \leq \; & f_i(x_k) + k\|(u_k, v_k, y^{i(k)})\|^\sigma < f_i(\bar{x}), \end{aligned}$$

which implies that the penalty problem with order σ does not admit a local exact penalization at $\bar{x}$ since $\{x_k\} \in K$ and $x_k \to \bar{x}$. □

4 Strong Karush-Kuhn-Tucker Conditions for (MOP)

In general nonlinear programming, we know that a calmness condition with order 1 can lead to the KKT condition. Now we can obtain strong KKT condition for the multiobjective optimization problem under the (MOP)-calmness condition with order 1.

Theorem 4.1 (Strong KKT conditions)**.** *Let $\bar{x} \in X$ be a local weak efficient solution for* (MOP) *and* (MOP) *is* (MOP)*-calm with order* 1 *at $\bar{x}$, then there exist $\lambda_j > 0$ $(j = 1, 2, \ldots, p), \beta_i \geq 0$ $(i = 1, 2, \ldots, m), \gamma_l \in \mathbb{R}$ $(l = 1, 2, \ldots, s)$ such that*

$$\begin{aligned} & 0 \in \sum_{j=1}^{p} \lambda_j \partial_C f_j(\bar{x}) + \sum_{i=1}^{m} \beta_i \partial_C g_i(\bar{x}) + \sum_{l=1}^{s} \gamma_l \partial_C h_l(\bar{x}) + N(K, \bar{x}), \\ & \beta_i g_i(\bar{x}) = 0, \quad i = 1, 2, \ldots, m. \end{aligned}$$

Proof. Since $\bar{x} \in X$ is a local weak efficient solution for (MOP) and (MOP) is (MOP)-calm with order 1 at $\bar{x}$, together with (i) and (iii) of Theorem 3.8 , it follows that there exists some $\hat{\mu} > 0$ such that for any $\mu \geq \hat{\mu}, i \in \{1, 2, \ldots, p\}$, $\bar{x}$ is a local weak efficient solution for (II) with order 1. For simplicity, let the real-valued function $\Gamma : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\Gamma(x) = \|g_+(x)\| + \|h(x)\| + \sum_{j=1,j\neq i}^{p} |(f_i(x) - f_i(\bar{x}))_+|, \quad \forall x \in \mathbb{R}^n.$$

Note that $f_i(i = 1, 2, \ldots, p), g, h$ are locally Lipschitz, so Γ is locally Lipschitz and the penalty function $f_i(\cdot) + \hat{\mu}\Gamma(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is also locally Lipschitz. By (ii) of Proposition 2.3, we have

$$0 \in \partial_C(f_i(\cdot) + \hat{\mu}\Gamma(\cdot))(\bar{x}) + N(K, \bar{x}). \tag{4.1}$$

In view of (i) of Proposition 2.3 and (i) of Proposition 2.4, we have

$$\partial_C(f_i(\cdot) + \hat{\mu}\Gamma(\cdot))(\bar{x}) \subset \partial_C f_i(\bar{x}) + \hat{\mu}\partial_C\Gamma(\bar{x}) \tag{4.2}$$

and

$$\partial_C\Gamma(\bar{x}) \subset \partial_C\|g_+(\cdot)\|(\bar{x}) + \partial_C\|h(\cdot)\|(\bar{x}) + \Sigma_{j=1,j\neq i}^{p}\partial_C|(f_j(x) - f_j(\cdot))_+|(\bar{x}). \tag{4.3}$$

In view of (ii) of Proposition 2.4, for all $i \in \{1, 2, \ldots, m\}$, we have

$$\partial_C \max\{0, g_i(\cdot)\}(\bar{x}) = \begin{cases} 0 & \text{if} \quad g_i(\bar{x}) < 0 \\ [0,1]\partial_C g_i(\bar{x}) & \text{if} \quad g_i(\bar{x}) = 0, \end{cases} \tag{4.4}$$

and for all $l \in \{1, 2, \ldots, s\}$ and $k \in \{1, 2, \ldots, p\}$, we have

$$\partial_C |h_l(\cdot)|(\bar{x}) = [-1, 1]\partial_C h_l(\bar{x}), \tag{4.5}$$

$$\partial_C |(f_k(\cdot) - f_k(\bar{x}))_+|(\bar{x}) = [0, 1]\partial_C f_k(\bar{x}). \tag{4.6}$$

Then we conclude from (4.3-4.6) that

$$\partial_C \Gamma(\bar{x}) \subset \sum_{i\in\mathfrak{J}(\bar{x})} [0, 1]\partial_C g_i(\bar{x}) + \sum_{l=1}^{s} [-1, 1]\partial_C h_l(\bar{x}) + \sum_{k=1,k\neq i}^{p} [0, 1]\partial_C f_k(\bar{x}).$$

Together with (4.1) and (4.2), there exist $\bar{\beta_j}^{(i)} \geq 0$ with $j \in \mathfrak{J}(\bar{x})$, $\bar{\gamma_l}^{(i)} \in \mathbb{R}$ and $\bar{t_k}^{(i)} \geq 0$ such that

$$\begin{aligned} 0 \in & \partial_C f_i(\bar{x}) + \hat{\mu}\Big(\sum_{j\in\mathfrak{J}(\bar{x})} \bar{\beta_j}^{(i)} \partial_C g_j(\bar{x}) + \sum_{l=1}^{s} \bar{\gamma_l}^{(i)} \partial_C h_l(\bar{x}) + \sum_{k=1,k\neq i}^{p} \bar{t_k}^{(i)} \partial_C f_k(\bar{x})\Big) + N(K, \bar{x}) \\ = & \partial_C f_i(\bar{x}) + \hat{\mu} \sum_{j\in\mathfrak{J}(\bar{x})} \bar{\beta_j}^{(i)} \partial_C g_j(\bar{x}) + \hat{\mu} \sum_{l=1}^{s} \bar{\gamma_l}^{(i)} \partial_C h_l(\bar{x}) + \hat{\mu} \sum_{k=1,k\neq i}^{p} \bar{t_k}^{(i)} \partial_C f_k(\bar{x}) + N(K, \bar{x}). \end{aligned}$$

Taking $\beta_j^{(i)} \geq 0$ with $\beta_j^{(i)} = \hat{\mu}\bar{\beta_j}^{(i)}, j \in \mathfrak{J}(\bar{x})$ and $\beta_j^{(i)} = 0, j \in \{1, 2, \ldots, m\}\backslash\mathfrak{J}(\bar{x})$, $\gamma_l^{(i)} \in \mathbb{R}$ with $\gamma_l^{(i)} = \hat{\mu}\bar{\gamma_l}^{(i)}$, $t_k^{(i)} \geq 0$ with $t_k^{(i)} = \hat{\mu}\bar{t_k}^{(i)}$, then we have

$$0 \in \partial_C f_i(\bar{x}) + \sum_{j=1}^{m} \beta_j^{(i)} \partial_C g_j(\bar{x}) + \sum_{l=1}^{s} \gamma_l^{(i)} \partial_C h_l(\bar{x}) + \sum_{k=1,k\neq i}^{p} t_k^{(i)} \partial_C f_k(\bar{x}) + N(K, \bar{x}), \tag{4.7}$$

$$\beta_j^{(i)} g_j(\bar{x}) = 0, \quad j = 1, 2, \ldots, m. \tag{4.8}$$

Summing (4.7) from $i = 1$ to p, and together with the convexness of Clarke subdifferential, we obtain

$$\begin{aligned} 0 \in \; & (1 + t_1^{(2)} + t_1^{(3)} + \cdots + t_1^{(p)})\partial_C f_1(\bar{x}) + (1 + t_2^{(1)} + t_2^{(3)} + \cdots + t_2^{(p)})\partial_C f_2(\bar{x}) \\ & + (1 + t_3^{(1)} + t_3^{(2)} + \cdots + t_3^{(p)})\partial_C f_3(\bar{x}) + \cdots + (1 + t_p^{(1)} + t_p^{(2)} + \cdots + t_p^{(p-1)})\partial_C f_p(\bar{x}) \\ & + \sum_{j=1}^{m} \beta_j \partial_C g_j(\bar{x}) + \sum_{l=1}^{s} \gamma_l \partial_C h_l(\bar{x}) + N(K, \bar{x}), \end{aligned}$$

where

$$\sum_{i=1}^{p} \beta_j^{(i)} = \beta_j, \qquad \sum_{i=1}^{p} \gamma_l^{(i)} = \gamma_l.$$

As $\beta_j^{(i)} \geq 0, j \in \mathfrak{J}(\bar{x})$ and $\beta_j^{(i)} = 0, j \in \{1, 2, \ldots, m\}\backslash\mathfrak{J}(\bar{x})$ and $t_k^{(i)} \geq 0$, then for each $i \in \{1, 2, \ldots, p\}$, $\lambda_i = 1 + \sum_{k=1,k\neq i}^{p} t_k^{(i)} > 0$, $\beta_j \geq 0, j \in \mathfrak{J}(\bar{x})$ and $\beta_j = 0, j \in \{1, 2, \ldots, m\}\backslash\mathfrak{J}(\bar{x})$. From above, we conclude that there exist $\lambda_i > 0$ $(i = 1, 2, \ldots, p), \beta_j \geq 0$ $(j = 1, 2, \ldots, m)$ and $\gamma_l \in \mathbb{R}$ $(l = 1, 2, \ldots, s)$ such that

$$\begin{aligned} & 0 \in \sum_{i=1}^{p} \lambda_i \partial_C f_i(\bar{x}) + \sum_{j=1}^{m} \beta_j \partial_C g_j(\bar{x}) + \sum_{l=1}^{s} \gamma_l \partial_C h_l(\bar{x}) + N(K, \bar{x}), \\ & \beta_j g_j(\bar{x}) = 0, \quad j = 1, 2, \ldots, m. \end{aligned}$$

This completes the proof. □

Remark 4.2. The (MOP)-calmness condition in [22] was defined as $f(x)+M\|(u,v,y^i)\|^\sigma e \notin f(\bar{x})-\text{int}\mathbb{R}^p$, where $e=\{1,1,\ldots,1\}\in\text{int}\mathbb{R}^p$. Obviously, it is weaker than that in Definition 3.1. However, under the calmness condition in [22], the KKT conditions obtained are just weak KKT conditions, in which some multipliers corresponding to the objective functions may equal to zero.

We now present an example to verify the strong KKT conditions.

Example 4.3. Consider the multiobjective programming in Example 3.6.

Similarly to the analysis in Example 3.6, this problem is (MOP)-calm with order 1 at $\bar{x}=(0,0)$. And by the formulas of f_1, f_2, g, h and K, we obtain

$$\partial_C g(\bar{x}) = \{(c,1) \mid -1 \le c \le 1\}, \partial_C h(\bar{x}) = \{(1,c) \mid -2 \le c \le 2\}$$

and

$$\partial_C f_1(\bar{x}) = (0,0), \partial_C f_2(\bar{x}) = (0,0), (0,0) \in N(K,\bar{x}).$$

For $(0,0)\in N(K,\bar{x}), (-1,1)\in\partial_C g(\bar{x}), (1,-1)\in\partial_C h(\bar{x})$ and $\forall\lambda_1>0, \lambda_2>0$, there exist $\beta=1, \gamma=1$ such that

$$\begin{aligned}&0_{\mathbb{R}^2} = \lambda_1(0,0)+\lambda_2(0,0)+\beta(-1,1)+\gamma(1,-1)+(0,0),\\ &\beta g(\bar{x})=0.\end{aligned}$$

Thus the strong KKT condition holds at $\bar{x}$.

In view of Theorem 3.5, Proposition 3.7 and Theorem 4.1, we immediately obtain the following result.

Corollary 4.4. *Let $\bar{x}\in X$ be a local efficient solution for* (MOP)*. Suppose that the extension constraint system of* (MOP) *has a local error bound with order* 1 *at $\bar{x}$, or equivalently, the set-valued map $Q_i:\mathbb{R}^{m+s+p-1}\rightrightarrows\mathbb{R}^n$, defined in* (3.1)*, is calm with order* 1 *at $(0_{\mathbb{R}^{m+s+p-1}},\bar{x})$, or equivalently,* (iii) *of Proposition* 3.7 *is satisfied for $\bar{x}$. Then there exist $\lambda_j>0$ $(j=1,2,\ldots,p), \beta_i\ge 0$ $(i=1,2,\ldots,m)$ and $\gamma_l\in\mathbb{R}$ $(l=1,2,\ldots,s)$ such that*

$$\begin{aligned}&0\in\sum_{j=1}^{p}\lambda_j\partial_C f_j(\bar{x})+\sum_{i=1}^{m}\beta_i\partial_C g_i(\bar{x})+\sum_{l=1}^{s}\gamma_l\partial_C h_l(\bar{x})+N(K,\bar{x}),\\ &\beta_i g_i(\bar{x})=0,\quad i=1,2,\ldots,m.\end{aligned}$$

As we all know, the no nonzero abnormal multiplier constraint qualification, which is presented in [18], is very useful in optimization. In Fritz John conditions, it ensures the multiplier of the objective function is positive for nonlinear programming. The following notion is an extension of no nonzero abnormal multiplier constraint qualification. In the following part, we assume that all emerging functions in (MOP) are smooth.

Definition 4.5. The generalized no nonzero abnormal multiplier constraint qualification (GNNAMCQ) holds at $\bar{x}\in X$ if for every $i\in\{1,2,\ldots,p\}$, there is no nonzero multiplier $y=(\beta,\gamma,\alpha)\in\mathbb{R}^{m+s+p-1}$, where $\beta=(\beta_1,\beta_2,\ldots,\beta_m), \gamma=(\gamma_1,\gamma_2,\ldots,\gamma_s), \alpha=$

$(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_p)$ such that

$$\begin{aligned}
&0 \in \sum_{i=1}^{m} \beta_i \nabla g_i(\bar{x}) + \sum_{j=1}^{s} \gamma_j \nabla h_j(\bar{x}) + \sum_{k=1, k\neq i}^{p} \alpha_k \nabla f_k(\bar{x}) + N(K, \bar{x}),\\
&\beta_i \geq 0, \quad i = 1, 2, \ldots, m,\\
&\gamma_j \in \mathbb{R}, \quad j = 1, 2, \ldots, s,\\
&\alpha_k \geq 0, \quad k = 1, 2, \ldots, p, k \neq i,\\
&\beta_i g_i(\bar{x}) = 0, \quad i = 1, 2, \ldots, m.
\end{aligned}$$

Now we are in the position to compare GNNAMCQ and the error bound condition of (MOP), next lemma states that the former implies the latter.

Lemma 4.6. *Suppose that all emerging functions in* (MOP) *are smooth and* GNNAMCQ *holds at $\bar{x} \in X$. Then there exist $\delta > 0$ and $\kappa > 0$ such that*

$$d(x, Q_i) < \kappa d((g(x), h(x), p_i(x), x), \mathbb{R}^m_- \times \{0\}_{\mathbb{R}^s} \times \mathbb{R}^{p-1}_- \times K),$$

where

$$p_i(x) = (f_1(x) - f_1(\bar{x}), f_2(x) - f_2(\bar{x}), \ldots, f_{i-1}(x) - f_{i-1}(\bar{x}), f_{i+1}(x) - f_{i+1}(\bar{x}), \ldots, f_p(x) - f_p(\bar{x})).$$

Proof. Let $p(x) := (g(x), h(x), p_i(x), x)$, define $S(x) := p(x) - \mathbb{R}^m_- \times \{0\}_{\mathbb{R}^s} \times \mathbb{R}^{p-1}_- \times K$, then its inverse be $S^{-1}(u) = \{x \mid u \in S(x)\}$. Obviously, $S^{-1}(0) = \{x \mid p(x) \in \mathbb{R}^m_- \times \{0\}_{\mathbb{R}^s} \times \mathbb{R}^{p-1}_- \times K\} = Q_i$. Since the GNNAMCQ holds at $\bar{x} \in X$, it follows from [18] (or the Mordukhovich criterion) that there exist $\delta_1 > 0, \ell > 0$ such that

$$d(x, S^{-1}(u)) \leq \ell d(u, S(x)), \quad \forall x \in \mathbb{B}(\bar{x}, \delta_1), \forall u \in \mathbb{B}(0, \delta_1).$$

Therefore, there exists $0 < \kappa < \ell$ such that

$$d(x, S^{-1}(u)) < \kappa d(u, S(x)), \quad \forall x \in \mathbb{B}(\bar{x}, \delta_1), \forall u \in \mathbb{B}(0, \delta_1),$$

and the desired result is obtained by setting $u = 0$. □

Remark 4.7. Suppose $\bar{x} \in X$ is a local efficient solution of (MOP), in view of Lemma 4.6, Proposition 3.7 and Theorem 3.5, the following implication holds true:

GNNAMCQ holds at $\bar{x} \Rightarrow$ (MOP) is (MOP)-calm at $\bar{x}$ with order 1.

Now we say that the generalized Mangasarian-Fromovitz constraint qualification for (MOP), which is presented in [8] for nonsmooth multiobjective programming problems, holds at $\bar{x}$ if the following statements hold:

(i) $0 \in \sum_{k=1}^{m} \nu_k \nabla h_k(\bar{x}) + N(K, \bar{x}) \Rightarrow \nu = 0$,

(ii) $(F^i)^s \cap G^s \cap H^- \cap T_C(K, \bar{x}) \neq \emptyset$, for $i = 1, 2, \ldots, p$,

where $F^i := \cup_{j\in\{1,2,\ldots,p\}, j\neq i} \nabla f_j(\bar{x}), G := \cup_{j\in\mathfrak{J}(\bar{x})} \nabla g_j(\bar{x}), H := (\cup_{k\in\{1,2,\ldots,s\}} \nabla h_k(\bar{x})) \bigcup (\cup_{k\in\{1,2,\ldots,s\}} \nabla(-h_k)(\bar{x}))$ and for a set $C \subset \mathbb{R}^n$, the C^- and C^s are the negative polar and strictly negative polar of C, defined respectively by

$$C^- := \{\xi \in \mathbb{R}^n | \langle \xi, v \rangle \leq 0, \forall v \in C\},$$

$$C^s := \{\xi \in \mathbb{R}^n | \langle \xi, v \rangle < 0, \forall v \in C\}.$$

In [8], Golestani and Nobakhtian present the strong KKT condition for (MOP) under the generalized Mangasarian-Fromovitz constraint qualification. Now we are in the position to compare the generalized Mangasarian-Fromovitz constraint qualification and the calmness condition of (MOP) when all emerging functions in (MOP) are smooth.

Theorem 4.8. *Suppose $\bar{x} \in X$ and the generalized Mangasarian-Fromovitz constraint qualification is satisfied. Then* GNNAMCQ *is hold at $\bar{x}$.*

Proof. Suppose to the contrary that the GNNAMCQ does not hold at $\bar{x}$, this yields that there exists nonzero multiplier $(\mu, \nu, \lambda) \in \mathbb{R}^{m+s+p-1}$, where $\mu = (\mu_1, \mu_2, \ldots, \mu_m), \nu = (\nu_1, \nu_2, \ldots, \nu_s)$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_p)$, such that

$$\begin{aligned}
&0 \in \sum_{l=1}^{m} \mu_l \nabla g_l(\bar{x}) + \sum_{k=1}^{s} \nu_k \nabla h_k(\bar{x}) + \sum_{j=1, j\neq i}^{p} \lambda_j \nabla f_j(\bar{x}) + N(K, \bar{x}),\\
&\mu_l \geq 0, \quad l = 1, 2, \ldots, m,\\
&\nu_k \in \mathbb{R}, \quad k = 1, 2, \ldots, s,\\
&\lambda_j \geq 0, \quad j = 1, 2, \ldots, p, j \neq i,\\
&\mu_l g_l(\bar{x}) = 0, \quad l = 1, 2, \ldots, m.
\end{aligned}$$

By (ii) of the generalized Mangasarian-Fromovitz constraint qualification, there exists $\eta \in T_C(K, \bar{x})$ such that

$$\begin{aligned}
&\langle \eta, \nabla f_j(\bar{x}) \rangle < 0, \quad j \in \{1, 2, \ldots, p\}, j \neq i,\\
&\langle \eta, \nabla g_j(\bar{x}) \rangle < 0, \quad j \in \mathfrak{J}(\bar{x}),\\
&\langle \eta, \nabla h_k(\bar{x}) \rangle = 0, \quad k \in \{1, 2, \ldots m\},\\
&\langle \eta, d \rangle \leq 0, \quad d \in N(K, \bar{x}).
\end{aligned}$$

As $\mu_l g_l(\bar{x}) = 0, l = 1, 2, \ldots, m$, thus $\mu_l = 0, l \in \{1, 2, \ldots, m\} \backslash \mathfrak{J}(\bar{x})$. And we have that there exists $\bar{d} \in N(K, \bar{x})$ such that

$$\langle \eta, \sum_{l=1}^{m} \mu_l \nabla g_l(\bar{x}) + \sum_{k=1}^{s} \nu_k \nabla h_k(\bar{x}) + \sum_{j=1, j\neq i}^{p} \lambda_j \nabla f_j(\bar{x}) + \bar{d} \rangle = 0. \tag{4.9}$$

From the above, we obtain $\lambda_j = 0, j = 1, \ldots, p, j \neq i, \mu_l = 0, l = 1, \ldots, m$. Thus we have

$$0 \in \sum_{k=1}^{s} \nu_k \nabla h_k(\bar{x}) + N(K, \bar{x}),$$

in view of (i) of the generalized Mangasarian-Fromovitz constraint qualification, we get $\nu_k = 0, k = 1, \ldots, m$, thus we have a contradiction to $(\mu, \nu, \lambda) \neq 0_{\mathbb{R}^{m+s+p-1}}$, so GNNAMCQ holds at $\bar{x}$. □

Sometimes, for $\bar{x} \in X$, the GNNAMCQ is satisfied, but the generalized Mangasarian-Fromovitz constraint qualification does not hold at $\bar{x}$.

Example 4.9. For $n = 2, p = 2, m = s = 1, K = \mathbb{R}^2$,

$$\begin{aligned} \min \quad & f(x) = (f_1(x), f_2(x)) = (x_1^2 + x_1, x_2^2) \\ s.t. \quad & g(x) = x_2 \leq 0, \\ & h(x) = x_1 - x_2 = 0, \\ & x \in K. \end{aligned}$$

The set of all efficient solutions is given as $S = \{(x_1, x_2) \mid x_1 = x_2, -\frac{1}{2} \leq x_1 \leq 0\}$. Now we choose $\bar{x} = (-\frac{1}{4}, -\frac{1}{4})$. By the information of f_1, f_2, g, h, K, we have $\nabla f_1(\bar{x}) = (\frac{1}{2}, 0), \nabla f_2(\bar{x}) = (0, -\frac{1}{2}), \nabla g(\bar{x}) = (0, 1), \nabla h(\bar{x}) = (1, -1)$ and $N(K, \bar{x}) = (0, 0)$. It is easy to verify that the GNNAMCQ is satisfied.

As $(F^1)^s = \{(x_1, x_2) \mid x_1 < 0\}, G^s = \emptyset, H^- = \{(x_1, x_2) \mid x_1 - x_2 = 0\}$, then we have $(F^1)^s \cap G^s \cap H^- \cap T_C(K, \bar{x}) = \emptyset$, that is, the generalized Mangasarian-Fromovitz constraint qualification is not satisfied.

Remark 4.10. Suppose $\bar{x} \in X$ is a local efficient solution of (MOP), in view of Remark 4.7 and Theorem 4.8, we have the following implications:

For $\bar{x} \in X$, the generalized Mangasarian-Fromovitz constraint qualification is satisfied
$\Rightarrow$ GNNAMCQ holds at $\bar{x}$
$\Rightarrow$ (MOP) is (MOP)-calm at $\bar{x}$ with order 1.

References

[1] Q.H. Ansari and J.-C. Yao (eds.), *Recent Developments in Vector Optimization*, Springer, Berlin, 2012.

[2] T.Q. Bao, P. Gupta and B.S. Mordukhovich, Necessary conditions in multiobjective optimization with equilibrium constraints, *J. Optim. Theory Appl.* 135 (2007) 179–203.

[3] G. Bigi, Optimality and Lagrangian regularity in vector optimization, PhD thesis, Universita di Pisa, 1999.

[4] R.S. Burachik and M.M. Rizvi, On weak and strong Kuhn-Tucker conditions for smooth multiobjective optimization, *J. Optim. Theory Appl.* 155 (2012) 477–491.

[5] F.H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.

[6] J. Dutta and S. Chandra, Convexifactors, generalized convexity and vector optimization, *Optimization* 53 (2004) 77–94.

[7] G. Giorgi, B. Jiménez and V. Novo, Strong Kuhn-Tucker conditions and constraint qualifications in locally Lipschitz multiobjective optimization problems, *Top.* 17 (2009) 288–304.

[8] M. Golestani and S. Nobakhtian, Nonsmooth multiobjective programming: strong Kuhn-Tucker conditions, *Positivity* 17 (2013) 711–732.

[9] M. Golestani and S. Nobakhtian, Nonsmooth multiobjective programming and constraint qualifications, *Optimization* 62 (2013) 783–795.

[10] M. Golestani and S. Nobakhtian, Convexificators and strong Kuhn-Tucker conditions, *Comp Math Appl.* 64 (2012) 550–557.

[11] X.X. Huang, K.L. Teo and X.Q. Yang, Calmness and exact penalization in vector optimization with cone constraints, *Comput. Optim. Appl.* 35 (2006) 47–67.

[12] X.F. Li and J.Z. Zhang, Stronger Kuhn-Tucker type conditions in nonsmooth multiobjective optimization: locally Lipschitz case, *J. Optim. Theory Appl.* 127 (2005) 367–388.

[13] X.J. Long and N.J. Huang, Optimality conditions for efficiency on nonsmooth multiobjective programming problems, *Taiwanese J. Math.* (2013), doi: 10.11650/tjm.18.2014.3730.

[14] D.V. Luu, Convexificators and necessary conditions for efficiency, *Optimization* 63 (2014) 321–335.

[15] D.V. Luu, Necessary and sufficient conditions for efficiency via convexificators, *J. Optim. Theory Appl.* 160 (2014) 510–526.

[16] T. Maeda, Constraint qualifications in multiobjective optimization problems: differentiable case, *J. Optim. Theory Appl.* 80 (1994) 483–500.

[17] V. Preda and I. Chitescu, On constraint qualification in multiobjective optimization problems: semidifferentiable case, *J. Optim. Theory Appl.* 100 (1999) 417–433.

[18] R.T. Rockafellar and R.J.-B. Wets, *Variational Analysis*, Springer, Berlin, 1998.

[19] S.M. Robinson, Stability theory for systems of inequalities, part II: Differentiable nonlinear systems, *SIAM J. Numer. Anal.* 13 (1976) 497–513.

[20] J.J. Ye, Necessary and sufficient optimality conditions for mathematical programs with equilibrium constraints, *J. Math. Anal. Appl.* 307 (2005) 350–369.

[21] J.J. Ye, Constraint qualifications and necessary optimality conditions for optimization problems with variational inequality constraints, *SIAM J. Optim.* 10 (2000) 943–962.

[22] S.K. Zhu and S.J. Li, Exact penalization and necessary optimality conditions for multiobjective optimization problems with equilibrium constraints, *Abstract and Applied Analysis* 2014 (2014) 1–13.

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