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AN OPTIMAL TIMETABLE FOR THE TWO TRAIN SEPARATION PROBLEM ON LEVEL TRACK*

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Abstract: When two trains travel along the same track in the same direction it is a common safety requirement that they should always be separated by at least one signal. If the signals are located at fixed positions that divide the track into different segments this means that the two trains cannot occupy the same segment of track at the same time. Nevertheless it is desirable to find driving strategies for each train that minimize the total energy consumption and allow the trains to reach their final destinations at the designated times while maintaining adequate separation. For each increasing sequence of intermediate signal times—where each intermediate time specifies the latest allowed time for the leading train to leave a given segment and the earliest allowed time for the following train to enter that same segment—it has been recently proposed that the total energy consumption is minimized by two characteristic optimal driving strategies. The leading train must use an initial phase of maximum power followed by alternate phases of speedhold and coast before a final phase of maximum brake. The following train must use an initial phase of maximum power followed by alternate phases of speedhold and maximum power before final phases of speedhold, coast and maximum brake. Key formulæ were also proposed that relate the train speeds at every signal point to the optimal holding speeds on the adjoining track segments. We fully justify the proposed formulæ and extend these ideas to find necessary conditions on an analytic solution for the optimal sequence of prescribed intermediate signal times—a sequence that defines the combined driving strategy, minimizes the total energy consumption and ensures safe separation of the trains.

Key words: *optimal train control, safe separation of trains, energy-efficient driving strategies.*

Mathematics Subject Classification: *49K15, 90B06, 90B35.*

1 Introduction

Consider a level rail track $[0, X]$ with signals placed at points $0 = x_0 < \cdots < x_n = X$. It is a common safety requirement for trains travelling from $x_0 = 0$ to $x_n = X$ that no two trains are allowed on the same segment (x_s, x_{s+1}) at the same time. We wish to find position and speed profiles $(x_{[\ell]}, v_{[\ell]}) = (x_{[\ell]}(t), v_{[\ell]}(t))$ for a leading train starting at time $t = 0$ and finishing at time $t = T_\ell$ and position and speed profiles $(x_{[f]}, v_{[f]}) = (x_{[f]}(t), v_{[f]}(t))$ for a following train starting at time $t = \Delta T$ and finishing at time $t = \Delta T + T_f$ such that:

- for each $s = 0, \ldots, n-1$, the time at which the following train enters the segment $[x_s, x_{s+1}]$ is always later than the time at which the leading train leaves the segment—thus we require $x_{[f]}^{-1}(x_s) \geq x_{[\ell]}^{-1}(x_{s+1})$; and

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- the total energy consumption is minimized.

We refer to this problem as the *two train separation problem on level track* and we solve it in two stages. For the first stage we assume a given set of times $0 = h_0 < \cdots < h_{n+1}$ where $h_1 \leq \Delta T$, $h_n = T_\ell$ and $h_{n+1} = \Delta T + T_f$. At this stage we wish to solve two problems—*the leading train problem* and *the following train problem*. That is we want to find $(x_{[\ell]}, v_{[\ell]})$ for the leading train so that $x_{[\ell]}(0) = 0$, $x_{[\ell]}(h_s) \geq x_s$ for each $s = 1, \ldots, n-1$ and $x_{[\ell]}(T_\ell) = X$ in such a way that energy consumption is minimized. We also want to find $(x_{[f]}, v_{[f]})$ for the following train so that $x_{[f]}(\Delta T) = 0$, $x_{[f]}(h_{s+1}) \leq x_s$ for $s = 1, \ldots, n-1$ and $x_{[f]}(\Delta T + T_f) = X$ in such a way that energy consumption is minimized. For the second stage we consider the set of all feasible prescribed times $\{h_s\}_{s=0}^{n+1}$ and find the set that minimizes the total energy consumed.

1.1 Some General Remarks about the Model and the Objectives

We assume a point-mass model for each train and note that Howlett and Pudney [21] have shown that any train control problem for a train with distributed mass can be replaced by an equivalent problem for a point-mass train. We do not model internal forces and we do not calculate the energy dissipated by these forces. Hence, in our analysis, an optimal strategy is one that minimizes the energy required to move the train along the track from one station to the next. For each train and each specified control function there is a uniquely-determined speed profile. We do not take into account the lengths of the trains and hence do not calculate the time taken for the trains to pass any particular signal location. Nor do we allow any additional time for the signal system to record the train movements. These considerations do not change the basic theoretical arguments.

1.2 Optimal Train Control

The modern theory is described in [2, 5, 7, 8, 11, 19, 20, 23, 24, 27] and references therein. The fundamental problem is to minimize the energy required to drive a train from one station to the next within a given time. The optimal strategy is generally a *maximum power–speedhold–coast–maximum brake* strategy except that the *singular* speedhold phase with a uniquely-determined optimal driving speed and positive power must be interrupted by phases of *maximum power* to negotiate steep uphill sections and *coast* to negotiate steep downhill sections. Thus the optimal strategy becomes an optimal switching strategy. If regenerative braking is available there will be additional speedhold phases at a higher speed using negative power for steep downhill grades [7, 8]. A similar theory of optimal control applies to solar-powered racing cars [22]. By considering the necessary conditions for optimal switching it has been shown [8, 23] that optimal switching points can be determined for each steep section by minimising an associated local energy functional. In [5, 8] the local energy functional was used to find a new constructive proof that the optimal strategy is unique. In practice the optimal strategy is also constrained by speed limits. As a general rule in such cases it is best to follow an unconstrained optimal speed profile except where a speed limit is violated in which case the speed limit is followed [7].

The use of optimal speed profiles allows a significant reduction in fuel consumption. As a journey evolves and circumstances change it is necessary to continually recalculate the optimal profile. Specialized numerical algorithms developed by the Scheduling and Control Group (SCG) at the University of South Australia for Sydney-based company TTG Transportation Technology are an essential component of the Energymiser® system. This system provides on-board advice to train drivers about energy-efficient driving strategies. It is used by major rail operators in Australia and the United Kingdom and is currently being implemented by a large European railway as a smartphone application for all drivers of high-speed passenger trains. The algorithms can compute an optimal speed profile for a journey of several hundred kilometres in a few seconds on a standard laptop or smartphone. More information about Energymiser® can be obtained from the TTG website‡. General methods of computational control are not suited to real-time calculation of optimal train controls [23].

1.3 Scheduling and Control of Trains

There are many other studies related to the scheduling and control of trains. These include integration of track maintenance and scheduling [1, 3], integration of optimal control and scheduling [32], development of robust schedules [4], the study of in-train forces [12, 33, 35, 36] and route mapping and train position estimation using GPS technology [9, 10]. There is extensive literature related to the planning and implementation of train schedules on complex rail networks. The survey paper by Cordeau *et al.* [13] describes much of the early work. Kraay *et al.* [25] and Kraay and Harker [26] used mathematical programming to find strategies for optimal pacing of trains and real-time scheduling. Dorfman and Medanic [14] used a discrete event model of railway traffic to define network schedules. The single-line scheduling problem to determine the order in which trains cross at designated crossing loops and the associated crossing times is an NP-hard problem. Higgins *et al.* [18] used integer programming and heuristic search techniques while Liu and Kozan [28, 29] used a job-shop scheduling approach. Some of the most successful methods for solution of realistic rail scheduling problems use probabilistic search techniques. In particular we mention the Problem Space Search (PSS) technique pioneered by Storer [34] for job-shop scheduling but more recently applied to scheduling on single-line long-haul corridors [1, 3, 30, 31].

1.4 Train Separation

It is common practice for railways to define different track segments using fixed signals. A normal safety requirement for two trains travelling in the same direction on the same line is that the following train must not enter any particular segment of track until the leading train has left it. Although the structure of the uniquely-defined optimal driving strategy for a single train is well established [2, 5, 7, 8, 11, 19, 20, 23, 24, 27] there is no corresponding comprehensive theory to determine optimal driving strategies for the two train separation problem on level track. Our purpose here is to justify and extend a solution proposed in [6].

2 The Single Train Control Problem on Level Track

We will begin by discussing the solution to the single train control problem on level track. That is, we wish to find a strategy that drives a train from $x = 0$ to $x = X$ on level track within time T in such a way that energy consumption is minimized. The solution is well known [20, 24, 27] but many of the algebraic intricacies are not easily found in the published literature. If resistance due to track curvature is ignored then the equations of motion can

‡See www.ttgtransportationtechnology.com

be written as

$$\dot{x} = v \tag{2.1}$$

$$\dot{v} = p/v - q - r(v) \tag{2.2}$$

where $x = x(t) \in [0, X]$ is the distance travelled, $v = v(t) \in [0, \infty)$ is the speed and t is the elapsed time, $p = p(t) \in [0, P]$ is the power per unit mass, $q = q(t) \in [0, Q]$ is the braking force per unit mass and $r = r(v)$ is the resistive force per unit mass. We have used the notation $\dot{x} = dx/dt$ and $\dot{v} = dv/dt$ to denote the respective derivatives. Define auxiliary functions $\varphi(v) = vr(v)$ and $\psi(v) = v^2 r'(v)$ for all $v \geq 0$. As in [23] we assume that $\varphi(v)$ is non-negative, increasing and strictly convex in v and that $\psi(v)$ is strictly increasing in v. The cost per unit mass of the strategy is

$$J = \int_0^T p\, dt. \tag{2.3}$$

We assume that the minimum-time journey [23] using a *power-brake* strategy is feasible. More general models[§] can be used but the basic method of analysis remains much the same and similar results are obtained.

2.1 Pontryagin Analysis

For a full discussion of the Pontryagin principle we refer to [15–17]. Define the Hamiltonian function

$$\mathcal{H} = (-1)p + \mu_1 v + \mu_2(p/v - q - r(v)) \tag{2.4}$$

and an associated Lagrangian function

$$\mathcal{L} = \mathcal{H} + \nu_1(P - p) + \nu_2 p + \nu_3(Q - q) + \nu_4 q \tag{2.5}$$

where $p = p(t) \in [0, P]$ and $q = q(t) \in [0, Q]$ are the controls, $(\mu_1, \mu_2) = (\mu_1(t), \mu_2(t))$ are absolutely continuous adjoint variables satisfying the differential equations

$$\dot{\mu_1} = 0 \tag{2.6}$$

$$\dot{\mu_2} = \mu_2 p/v^2 + \mu_2 r'(v) - \mu_1 \tag{2.7}$$

and $\nu_1 \geq 0$, $\nu_2 \geq 0$, $\nu_3 \geq 0$ and $\nu_4 \geq 0$ are Lagrange multipliers. To find necessary conditions on the optimal controls (p, q) we must maximize the Hamiltonian subject to the control constraints and the state space constraints. The Karush-Kuhn-Tucker (KKT) conditions require $\partial\mathcal{L}/\partial p = 0$ and $\partial\mathcal{L}/\partial q = 0$ which gives

$$-1 + \mu_2/v - \nu_1 + \nu_2 = 0 \tag{2.8}$$

$$-\mu_2 - \nu_3 + \nu_4 = 0. \tag{2.9}$$

It is also necessary to impose the complementary slackness conditions $\nu_1(P-p) = 0$, $\nu_2 p = 0$, $\nu_3(Q - q) = 0$ and $\nu_4 q = 0$. Since $\mu_1(t)$ is absolutely continuous it follows from (2.6) that $\mu_1(t) = c \in \mathbb{R}$ for all $t \in [0, T]$. We identify the following optimal control modes.

[§]The equation of motion (2.2) on level track can be written more generally in the form $\dot{v} = u - r(v)$ where the acceleration $u = u(t)$ satisfies bounds of the form $U_-(v) \leq u \leq U_+(v)$. If regenerative braking is allowed then the cost per unit mass is given by

$$J = \int_0^T [(u + |u|)v/2 + \rho(u - |u|)v/2]\, dt$$

where $0 \leq \rho < 1$ is an efficiency constant. Our model corresponds to the case where $U_+(v) = P/v$, $U_-(v) = -Q$ and $\rho = 0$. For an extended discussion of the modelling process see [7].

Mode 1: $\mu_2 > v$. If this condition is maintained on a nontrivial time interval then (2.8) gives $\mu_2/v - 1 = \nu_1 - \nu_2 > 0$. Since $\nu_1, \nu_2 \geq 0$ it follows that $\nu_1 = \mu_2/v - 1 > 0$ and $\nu_2 = 0$. Hence $p = P$. Now (2.9) gives $\mu_2 = \nu_4 - \nu_3 > 0$ and since $\nu_3, \nu_4 \geq 0$ we know that $\nu_4 = \mu_2 > 0$ and $\nu_3 = 0$. Therefore $q = 0$. This is a regular phase of *maximum power* with $(p, q) = (P, 0)$.

Mode 2: $\mu_2 = v$. If this condition is maintained on a nontrivial time interval then (2.8) gives $\nu_1 - \nu_2 = 0$. Since $\nu_1, \nu_2 \geq 0$ it follows that $\nu_1 = \nu_2 = 0$. At the same time (2.9) gives $v = \nu_4 - \nu_3 > 0$ and since $\nu_3, \nu_4 \geq 0$ we deduce that $\nu_3 = 0$ and $\nu_4 = v$. Hence $q = 0$. To maintain this condition we must have $\dot{\mu}_2 = \dot{v}$ and so we have $p/v + vr'(v) - c = p/v - r(v) \Rightarrow \varphi'(v) = c$. Since $\varphi'(v)$ is nonnegative and strictly increasing there is a uniquely-defined speed $v = V$ that solves this equation. Thus $c = \varphi'(V)$. Hence this is a singular phase of *speedhold* with $v = V$ and partial power $(p, q) = (\varphi(V), 0)$. We call V the optimal driving speed.

Mode 3: $v > \mu_2 > 0$. If this condition is maintained on a nontrivial time interval then (2.8) gives $1 - \mu_2/v = \nu_2 - \nu_1 > 0$. Since $\nu_1, \nu_2 \geq 0$ it follows that $\nu_1 = 0$ and $\nu_2 = 1 - \mu_2/v > 0$. Hence $p = 0$. Now (2.9) gives $\mu_2 = \nu_4 - \nu_3 > 0$ and since $\nu_3, \nu_4 \geq 0$ we deduce that $\nu_4 = \mu_2 > 0$ and $\nu_3 = 0$. Therefore $q = 0$. This is a regular *coast* phase with $(p, q) = (0, 0)$.

Mode 4: $\mu_2 = 0$. If this condition is maintained on a nontrivial time interval then (2.8) gives $1 = \nu_2 - \nu_1 > 0$. Since $\nu_1, \nu_2 \geq 0$ it follows that $\nu_1 = 0$ and $\nu_2 = 1 > 0$. Hence $p = 0$. At the same time (2.9) gives $\nu_4 - \nu_3 = 0$ and since $\nu_3, \nu_4 \geq 0$ we deduce that $\nu_3 = \nu_4 = 0$. To maintain the condition $\mu_2 = 0$ we must have $\dot{\mu}_2 = 0$ and so it follows that $\mu_1 = c = 0$. This is a *partial brake* phase with $(p, q) \in (0, [0, Q])$. Since the important *speedhold* phase with partial power requires $c > 0$ we may conclude that *partial brake* does not occur.

Mode 5: $\mu_2 < 0$. If this condition is maintained on a nontrivial time interval then (2.8) gives $1 - \mu_2/v = \nu_2 - \nu_1 > 0$. Since $\nu_1, \nu_2 \geq 0$ it follows that $\nu_1 = 0$ and $\nu_2 = 1 - \mu_2/v > 0$. Hence $p = 0$. At the same time (2.9) gives $\nu_3 - \nu_4 = -\mu_2 > 0$ and since $\nu_3, \nu_4 \geq 0$ we deduce that $\nu_3 = -\mu_2 > 0$ and $\nu_4 = 0$. Therefore $q = Q$. Hence this is a regular phase of *maximum brake* with $(p, q) = (0, Q)$.

Thus an optimal strategy on level track involves only four possible modes—a regular mode of *maximum power*, a singular *speedhold* mode where the optimal driving speed $v = V$ is maintained, a regular *coast* mode, and a regular mode of *maximum brake*. A strategy that satisfies the necessary conditions for optimality will be called a *strategy of optimal type*.

2.2 Evolution of the Optimal Strategy During Regular Control

We define a modified adjoint variable $\eta = \eta(t)$ by the formula $\eta = \mu_2/v - 1$ and consider evolution of η during a phase of regular control in a strategy of optimal type on level track.

For a phase of *maximum power* with $(p, q) = (P, 0)$ and $\eta > 0$ we have

$$\dot{\eta} = \dot{\mu}_2/v - (\mu_2/v^2)\dot{v} = (\varphi'(v)/v)\eta + [\varphi'(v) - \varphi'(V)]/v \tag{2.10}$$

and hence using (2.2) we find that

$$d\eta/dv - [\varphi'(v)/(P - \varphi(v))]\,\eta = [\varphi'(v) - \varphi'(V)]/(P - \varphi(v)) \tag{2.11}$$

from which it follows that

$$(P - \varphi(v))\eta = \varphi(v) - \varphi'(V)v + C \tag{2.12}$$

where C is a constant of integration. The adjoint variable is absolutely continuous and we know that $\eta \to 0$ when $v \to V$. Therefore $C = -\varphi(V) + \varphi'(V)V$. Hence, finally, we have

$$\eta = [\varphi(v) - L_V(v)]/[P - \varphi(v)] \tag{2.13}$$

for $v \neq V$ where we have written $L_V(v) = \varphi(V) + \varphi'(V)(v - V)$ for the tangential approximation to $\varphi(v)$ at $v = V$. The strict convexity of $\varphi(v)$ means that $\varphi(v) > L_V(v) \iff \eta > 1$ for all $v \neq V$. Since (2.13) shows that η depends only on v we will now write $\eta = \eta[v] = \eta[v(t)]$ if we wish to highlight the dependence on v. On level track the speed always increases during a phase of *maximum power*. Consequently if $v_0 = v(t_0) > V$ at some time during an optimal phase of *maximum power* then the phase must continue indefinitely with $v(t) > v_0 > V$ and $\eta[v] > \eta[v_0] > 0$ for all $v > v_0$.

Remark 2.1. On an interval $t \in [a, b]$ where $v(a) < v(b)$ there exist strategies of optimal type with optimal driving speed $V \in (v(a), v(b))$ in the form *maximum power–speedhold–maximum power*. If so, the speed increases during the initial *maximum power* phase from $v = v(a)$ to $v = V$, then remains constant with $v = V$ during the *speedhold* phase, and then increases again during the final *maximum power* phase from $v = V$ to $v = v(b)$. The modified adjoint variable decreases during the initial *maximum power* phase from $\eta = \eta(a) > 0$ to $\eta = 0$ during the *speedhold* phase and then increases during the final *maximum power* phase from $\eta = 0$ to $\eta = \eta(b) > 0$. We shall see later that on each segment a *maximum power–speedhold–maximum power* strategy is typical for a following train.

For a phase of *coast* with $(p, q) = (0, 0)$ and $-1 < \eta < 0$ we have

$$\dot{\eta} = \dot{\mu}_2/v - (\mu_2/v^2)\dot{v} = (\varphi'(v)/v)\eta + [\varphi'(v) - \varphi'(V)]/v \tag{2.14}$$

and hence using (2.2) we can establish that

$$d\eta/dv + [\varphi'(v)/\varphi(v)]\,\eta = [\varphi'(V) - \varphi'(v)]/\varphi(v) \tag{2.15}$$

from which it follows that

$$\varphi(v)\eta = \varphi'(V)v - \varphi(v) + D \tag{2.16}$$

where D is a constant of integration. The adjoint variable is absolutely continuous and we know that $\eta \to 0$ when $v \to V$. Therefore $D = \varphi(V) - \varphi'(V)V$. Hence it follows that

$$\eta = (-1)[\varphi(v) - L_V(v)]/\varphi(v) = L_V(v)/\varphi(v) - 1 \tag{2.17}$$

provided $v \neq 0$. Once again (2.17) shows that $\eta = \eta[v] = \eta[v(t)]$ depends only on v. The strict convexity of $\varphi(v)$ means that $L_V(v) < \varphi(v) \iff \eta < 0$ for all $v \neq V$. We must have $\eta > -1$ during *coast* and so $L_V(v) > 0 \iff v > V - \varphi(V)/\varphi'(V) = \psi(V)/\varphi'(V) = U > 0$. Since $\eta \to -1$ as $v \to U$ it follows that $v = U$ must be the speed at which braking begins.

Remark 2.2. On an interval $t \in [a, b]$ where $v(a) > v(b)$ there exist strategies of optimal type in the form *coast–speedhold–coast* with optimal driving speed $V \in (v(b), v(a))$. In such strategies the speed decreases during the initial *coast* phase from $v = v(a)$ to $v = V$, remains constant with $v = V$ during the *speedhold* phase and then decreases during the final *coast* phase from $v = V$ to $v = v(b)$. The adjoint variable increases during the initial *coast* phase from $\eta = \eta(a) < 0$ to $\eta = 0$ during the *speedhold* phase and then decreases during the final *coast* phase from $\eta = 0$ to $\eta = \eta(b) < 0$. We shall see later that on each segment a *coast–speedhold–coast* strategy is typical for a leading train.

For a phase of *maximum brake* with $(p,q) = (0,Q)$ and $\eta < -1$ similar arguments can be used to show that

$$\eta = (-1)[Qv + \varphi(v) - L_V(v)]/[Qv + \varphi(v)] = L_V(v)/[Qv + \varphi(v)] - 1 \tag{2.18}$$

where we again use $L_V(v) = \varphi(V) + \varphi'(V)(v - V)$ for the tangential approximation to $\varphi(v)$ at $v = V$. Once again $\eta = \eta[v] = \eta[v(t)]$ depends only on v. Note that $L_V(v) < 0 \iff \eta < -1$ for all $v < U$. On level track the speed will always decrease during a phase of *maximum brake*. Consequently if an optimal phase of *maximum brake* begins at time $t = a$ with $v(a) = U$ then $v(t) < U$ for all $t > a$ and hence $\eta[v] < -1$ for all $v < U$. Therefore the phase must continue until the end of the journey.

Remark 2.3. If a strategy of optimal type on level track contains a phase of *maximum brake* then it is always the final phase.

2.3 Convenient Notation and Key Formulæ

We will introduce some convenient notation and some key formulæ.

For a *power* phase with $(p,q) = (P,0)$ from $v = 0$ to $v = V$ the distance travelled and elapsed time are respectively

$$\Delta_p x(0,V) = \int_0^V v^2 dv/(P - \varphi(v)) \tag{2.19}$$

and

$$\Delta_p t(0,V) = \int_0^V v dv/(P - \varphi(v)) \tag{2.20}$$

and the cost is given by $\Delta_p J(0,V) = P\Delta_p t(0,V)$.

For a *speedhold* phase at speed $v = V$ with $(p,q) = (\varphi(V),0)$ from $t = a$ to $t = b$ the distance travelled and elapsed time are

$$\Delta_{sh} x(a,b,V) = V(b-a) \quad \text{and} \quad \Delta_{sh} t(a,b) = b - a \tag{2.21}$$

and the cost is given by $\Delta_{sh} J(a,b,V) = \varphi(V)(b-a)$.

For a *coast* phase with $(p,q) = (0,0)$ from $v = V$ to $v = U$ the distance travelled and elapsed time are

$$\Delta_c x(V,U) = \int_U^V v dv/r(v) \quad \text{and} \quad \Delta_c t(V,U) = \int_U^V dv/r(v) \tag{2.22}$$

and the cost is $\Delta_c J(V,U) = 0$.

For a *brake* phase with $(p,q) = (0,Q)$ from $v = U$ to $v = 0$ the distance travelled and elapsed time are

$$\Delta_b x(U,0) = \int_0^U v dv/(Q + r(v)) \quad \text{and} \quad \Delta_b t(U,0) = \int_0^U dv/(Q + r(v)) \tag{2.23}$$

and the cost is $\Delta_b J(U,0) = 0$.

3 The Leading Train Problem

We are now ready to begin our detailed discussion of the leading train problem. Recall that for given intermediate times $\{h_s\}_{s=1}^{n-1}$ we wish to find position and speed profiles $(x_{[\ell]}, v_{[\ell]})$ for the leading train so that $x_{[\ell]}(0) = 0$, $x_{[\ell]}(h_s) \geq x_s$ for each $s = 1, \ldots, n-1$ and $x_{[\ell]}(T_\ell) = X$ in such a way that energy consumption is minimized. For the optimal strategy we must understand that only some of the intermediate position constraints are active. Thus we define indices $0 = \ell(0) < \ell(1) < \cdots < \ell(k) = n$ such that $x_{[\ell]}(h_{\ell(i)}) = x_{\ell(i)}$ for $i = 0, \ldots, k$ and $x_{[\ell]}(h_s) > x_s$ for $s \neq \ell(i)$ for all $0 \leq i \leq k$. In such cases we will say that the optimal strategy is *restricted*. We have the following elementary but important result.

Theorem 3.1. *Let $(x_{[\ell]}, v_{[\ell]})$ be an optimal solution to the restricted leading train problem with $0 = \ell(0) < \ell(1) < \cdots < \ell(k) = n$ such that $x_{[\ell]}(h_{\ell(i)}) = x_{\ell(i)}$ for $i = 0, \ldots, k$ and $x_{[\ell]}(h_s) > x_s$ if $s \neq \ell(i)$ for all $i = 0, \ldots, k$. Then for each segment $(x_{\ell(i)}, x_{\ell(i+1)})$ the profile $(x_{[\ell]}, v_{[\ell]})$ is an optimal unrestricted single train strategy subject to the initial and final speed constraints $v(h_{\ell(i)}) = v_{[\ell]}(h_{\ell(i)})$ and $v(h_{\ell(i+1)}) = v_{[\ell]}(h_{\ell(i+1)})$ for $i = 0, \ldots, k-1$.* □

Thus we may solve the *restricted* leading train problem by solving a series of *unrestricted* single train optimal control problems.

3.1 An Inductive Solution Process

Suppose the position constraints $x(h_s) \geq x_s$ for $s = 1, \ldots, n-1$ are not all satisfied for the optimal *unrestricted* strategy. We assume the minimum-time *power-brake* strategy is feasible and so the first violation cannot occur during the initial *power* phase. Let us assume—for the sake of argument—that the first violation occurs at time $t = h_{\ell(1)}$ during the *speedhold* phase with $v = V$. Thus we have

$$\Delta_p x(0, V) + V\left(h_{\ell(1)} - \Delta_p t(0, V)\right) < x_{\ell(1)}.$$

We wish to find an optimal *power-speedhold-coast* strategy on the interval $(0, h_{\ell(1)})$ with hold speed $V_0 > V$ and final speed $U_1 = v(h_{\ell(1)})$ which satisfies the position constraint

$$\Delta_p x(0, V_0) + \Delta_c x(V_0, U_1) + V_0\left(h_{\ell(1)} - \Delta_p t(0, V_0) - \Delta_c t(V_0, U_1)\right) = x_{\ell(1)}$$

and an optimal *coast-speedhold-coast-brake* strategy on the interval $(h_{\ell(1)}, T_\ell)$ with initial speed U_1, hold speed $V_1 < V$ and brake speed $U < V_1$ such that the position constraint

$$\Delta_c x(U_1, U) + \Delta_b x(U, 0) + V_1\left(T_\ell - h_{\ell(1)} - \Delta_c t(U_1, U) - \Delta_b t(U, 0)\right) = X - x_{\ell(1)}$$

is satisfied and so that the cost of the overall restricted strategy

$$\begin{aligned} J(V_0, U_1, V_1, U) &= P\Delta_p J(0, V_0) + \varphi(V_0)\left[h_{\ell(1)} - \Delta_p t(0, V_0) - \Delta_c t(V_0, U_1)\right] \\ &\quad + \varphi(V_1)\left[T_\ell - h_{\ell(1)} - \Delta_c t(U_1, U) - \Delta_b t(U, 0)\right] \end{aligned}$$

is minimized. Once the optimal values for V_0, V_1, U_1, U are determined then we must check to see if any subsequent constraints are violated. Let us assume, for the sake of argument, that the next violated constraint occurs at time $t = h_{\ell(2)}$ when $v = U_2$ during the *coast* phase from $v = V_1$ to $v = U$. Thus we have

$$\Delta_c x(U_1, U_2) + V_1\left(h_{\ell(2)} - h_{\ell(1)} - \Delta_c t(U_1, U_2)\right) < x_{\ell(2)} - x_{\ell(1)}.$$

Now we seek an optimal *power-speedhold-coast* strategy on $(0, h_{\ell(1)})$ with hold speed V_0 and final speed $v(h_{\ell(1)}) = U_1$ satisfying the position constraint

$$\Delta_p x(0, V_0) + \Delta_c x(V_0, U_1) + V_0 \left(h_{\ell(1)} - \Delta_p t(0, V_0) - \Delta_c t(V_0, U_1)\right) = x_{\ell(1)}$$

followed by an optimal *coast-speedhold-coast* strategy on $(h_{\ell(1)}, h_{\ell(2)})$ with initial speed U_1, hold speed V_1 and final speed $v(h_{\ell(2)}) = U_2$ satisfying the position constraint

$$\Delta_c x(U_1, U_2) + V_1(h_{\ell(2)} - h_{\ell(1)} - \Delta_c t(U_1, U_2)) = x_{\ell(2)} - x_{\ell(1)}$$

and an optimal *coast-speedhold-coast-brake* strategy with initial speed U_2, hold speed V_2, speed $v = U$ at which braking begins and final speed $v(T_\ell) = 0$ on the interval $(h_{\ell(2)}, T_\ell)$ such that the position constraint

$$\Delta_c x(U_2, U) + V(T_\ell - h_{\ell(2)} - \Delta_c t(U_2, U) - \Delta_b t(U, 0)) + \Delta_b x(U, 0) = X - x_{\ell(2)}$$

is satisfied. Subject to any given speeds U_1 at $x_{\ell(1)}$ and U_2 at $x_{\ell(2)}$ we can adjust the hold speeds $V_0 > V_1 > V_2$ and the speed U at which braking begins so that the single train strategy on each segment is optimal. More generally we could also adjust the speeds $U_1 > U_2$ to minimize the total cost of the overall restricted strategy¶ subject to the position constraints $x(h_{\ell(1)}) = x_{\ell(1)}$, $x(h_{\ell(2)}) = x_{\ell(2)}$ and $x(T_\ell) = X$. Now we check the interval $(h_{\ell(2)}, T_\ell)$ to see if there are any subsequent violations of intermediate position constraints. The process continues until there are no more violations.

3.2 The Leading Train Problem on Level Track

On the basis of the previous discussion we can formulate and solve the leading train problem on level track in the following way.

Problem 3.2. We wish to find optimal position and speed profiles $(x_{[\ell]}(t), v_{[\ell]}(t))$ for a leading train to travel from $(x_{[\ell]}(0), v_{[\ell]}(0)) = (0, 0)$ to $(x_{[\ell]}(T_\ell), v_{[\ell]}(T_\ell)) = (X, 0)$ on level track subject to additional intermediate position constraints $x_{[\ell]}(h_s) \geq x_s$ for $s = 0, \ldots, n-1$ where the signal positions $\{x_s\}_{s=0}^n$ with $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = X$ and corresponding times $\{h_s\}_{s=0}^n$ with $0 = h_0 < h_1 < \cdots < h_{n-1} < h_n = T_\ell$ are given. For the optimal strategy we assume that only some of the intermediate position constraints are active—that is we assume indices $0 = \ell(0) < \ell(1) < \cdots < \ell(k) = n$ such that $x_{[\ell]}(h_{\ell(i)}) = x_{\ell(i)}$ for $i = 0, \ldots, k$ and $x_{[\ell]}(h_s) > x_s$ for $s \neq \ell(i)$. We propose a strategy with (1) a *power* phase from $(x, v, t) = (0, 0, 0)$ to speed V_0 followed by a *speedhold* phase at speed V_0 and a *coast* phase to $(x, v, t) = (x_{\ell(1)}, U_1, h_{\ell(1)})$, (2) a *coast* phase from $(x, v, t) = (x_{\ell(i)}, U_i, h_{\ell(i)})$ to speed V_i followed by a *speedhold* phase at speed V_i and a *coast* phase to $(x, v, t) = (x_{\ell(i+1)}, U_{i+1}, h_{\ell(i+1)})$ for each $i = 1, \ldots, k-2$ and (3) a *coast* phase from $(x, v, t) = (x_{\ell(k-1)}, U_{k-1}, h_{\ell(k-1)})$ to speed V_{k-1}, a *speedhold* phase at speed V_{k-1}, a *coast* phase to speed $U = U_k$ and a final *brake* phase to $(x, v, t) = (x_{\ell(k)}, 0, h_{\ell(k)}) = (X, 0, T_\ell)$. The cost of the strategy per unit mass is given by $J_\ell = J_\ell(\boldsymbol{U}, \boldsymbol{V})$ where $\boldsymbol{U} = (U_1, \ldots, U_k)$ and $\boldsymbol{V} = (V_0, \ldots, V_{k-1})$ and where

$$\begin{aligned} J_\ell &= P\Delta_p t(0, V_0) + \varphi(V_0)\left[h_{\ell(1)} - \Delta_p t(0, V_0) - \Delta_c t(V_0, U_1)\right] \\ &\quad + \sum_{i=1}^{k-1} \varphi(V_i)\left[h_{\ell(i+1)} - h_{\ell(i)} - \Delta_c t(U_i, U_{i+1})\right]. \end{aligned} \tag{3.1}$$

¶For an optimal strategy we shall see later that the speed U_i is uniquely determined by the speeds V_i and V_{i+1}.

The distance travelled during the time interval $(h_{\ell(i)}, h_{\ell(i+1)})$ is given by

$$\delta_0 = \Delta_p x(0, V_0) + V_0[h_{\ell(1)} - \Delta_p t(0, V_0) - \Delta_c t(V_0, U_1)] + \Delta_c x(V_0, U_1) \tag{3.2}$$

for the first interval with $i = 0$, by

$$\delta_i = \Delta_c x(U_i, U_{i+1}) + V_i[h_{\ell(i+1)} - h_{\ell(i)} - \Delta_c t(U_i, U_{i+1})] \tag{3.3}$$

for each intermediate interval with $i = 1, \ldots, k-2$ and by

$$\begin{aligned}\delta_{k-1} \quad = \quad & \Delta_c x(U_{k-1}, U_k) + \Delta_b x(U_k, 0) \\ & +V_{k-1}[T_\ell - h_{\ell(k-1)} - \Delta_c t(U_{k-1}, U_k) - \Delta_b t(U_k, 0)]\end{aligned} \tag{3.4}$$

for the final interval with $i = k - 1$. We wish to minimize J_ℓ subject to the constraints $x_{\ell(i+1)} - x_{\ell(i)} \le \delta_i$ for each $i = 0, \ldots, k-1$. □

Define a Lagrangian function

$$\mathcal{J}_\ell = J_\ell + \sum_{i=0}^{k-1} \mu_i (x_{\ell(i+1)} - x_{\ell(i)} - \delta_i) \tag{3.5}$$

and solve the equations $\partial \mathcal{J}_\ell / \partial V_i = 0$ and $\partial \mathcal{J}_\ell / \partial U_{i+1} = 0$ for each $i = 0, \ldots, k-1$. The condition $\partial \mathcal{J}_\ell / \partial V_i = 0$ implies

$$(\varphi'(V_0) - \mu_0)\,[h_{\ell(1)} - \Delta_p t(0, V_0) - \Delta_c t(V_0, U_1)] = 0 \tag{3.6}$$

when $i = 0$,

$$(\varphi'(V_i) - \mu_i)\,[h_{\ell(i+1)} - h_{\ell(i)} - \Delta_c t(U_i, U_{i+1})] = 0 \tag{3.7}$$

for $i = 1, \ldots, k-2$, and

$$(\varphi'(V_{k-1}) - \mu_{k-1})\,[T_\ell - h_{\ell(k-1)} - \Delta_c t(U_{k-1}, U_k) - \Delta_b t(0, U_k)] = 0 \tag{3.8}$$

when $i = k-1$. The condition $\partial \mathcal{J}_\ell / \partial U_{i+1} = 0$ implies

$$\varphi(V_0) - \varphi(V_1) - \mu_0(V_0 - U_1) - \mu_1(U_1 - V_1) = 0 \tag{3.9}$$

when $i = 0$,

$$[\varphi(V_i) - \varphi(V_{i+1}) + (\mu_i - \mu_{i+1})U_{i+1} + \mu_{i+1}V_{i+1} - \mu_i V_i]/\varphi(U_{i+1}) = 0 \tag{3.10}$$

for $i = 1, \ldots, k-2$ and

$$[\varphi(V_{k-1}) - \varphi(V_k) + (\mu_{k-1} - \mu_k)U_k + \mu_k V_k - \mu_{k-1}V_{k-1}]/\varphi(U_k) = 0 \tag{3.11}$$

when $i = k-1$. From (3.6), (3.7) and (3.8) it can be seen that

$$\mu_i = \varphi'(V_i) \tag{3.12}$$

for each $i = 0, \ldots, k-1$. Hence it follows from (3.9), (3.10) and (3.11) that

$$U_{i+1} = [\psi(V_i) - \psi(V_{i+1})]/[\varphi'(V_i) - \varphi'(V_{i+1})] \tag{3.13}$$

for each $i = 0, 1, \ldots, k-2$. Finally we have

$$U_k = V_{k-1} - \varphi(V_{k-1})/\varphi'(V_{k-1}) = \psi(V_{k-1})/\varphi'(V_{k-1}) \tag{3.14}$$

when $i = k-1$. The convexity of $\varphi(v)$ means $V_i \ge U_{i+1} \ge V_{i+1}$ for each $i = 0, 1, \ldots, k-2$. From (3.14) it is clear that $U_k < V_{k-1}$.

4 The Following Train Problem

We can formulate and solve the following train problem on level track by using an analogous inductive argument to that for the leading train. However, rather than present the details we will simply state the corresponding structural result. At this stage the solution of the following train problem is not directly related to the solution of the leading train problem. The only connection is that each problem is specified in terms of a common set of prescribed times. For this reason—and because the resulting formulæ for the optimal speeds at the signal positions are precisely the same—it is convenient to use the same notation for each problem. In Section 8 when we discuss optimization of the prescribed times by considering both trains simultaneously it will be necessary to use a more discerning notation.

Theorem 4.1. *Let $(x_{[f]}, v_{[f]})$ be an optimal solution to the restricted following train problem with $0 = f(0) < f(1) < \cdots < f(m) = n$ such that $x_{[f]}(\Delta T) = 0$, $x_{[f]}(h_{f(j)+1}) = x_{f(j)}$ for $j = 1, \ldots, m-1$, $x_{[f]}(\Delta T + T_f) = X$ and $x_{[f]}(h_{s+1}) < x_s$ if $s \neq f(j)$ for all $j = 1, \ldots, m$. Then for each segment $(x_{f(j)}, x_{f(j+1)})$ the profile $(x_{[f]}, v_{[f]})$ is an optimal unrestricted single train strategy subject to the initial and final speed constraints $v(\Delta T) = v_{[f]}(\Delta T) = 0$ and $v(h_{f(1)+1}) = v_{[f]}(h_{f(1)+1})$ on the first interval, $v(h_{f(j)+1}) = v_{[f]}(h_{f(j)+1})$ and $v(h_{f(j+1)+1}) = v_{[f]}(h_{f(j+1)+1})$ on the intermediate intervals for $j = 1, \ldots, m-2$ and $v(h_{f(m-1)+1}) = v_{[f]}(h_{f(m-1)+1})$ and $v(\Delta T + T_f) = v_{[f]}(\Delta T + T_f) = 0$ on the final interval.*

Problem 4.2. We wish to find optimal position and speed profiles $(x_{[f]}(t), v_{[f]}(t))$ for a following train to travel from $(x_{[f]}(\Delta T), v_{[f]}(\Delta T)) = (0, 0)$ to $(x_{[f]}(\Delta T+T_f), v_{[f]}(\Delta T+T_f)) = (X, 0)$ on level track subject to additional intermediate position constraints $x_{[f]}(h_{s+1}) \leq x_s$ where the signal positions $\{x_s\}_{s=0}^{n}$ with $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = X$ and corresponding times $\{h_{s+1}\}_{s=0}^{n}$ with $0 < h_1 < \cdots < h_{n-1} < h_n < h_{n+1}$ where $h_1 \leq \Delta T$ and $h_{n+1} = \Delta T + T_f$ are given. For the optimal strategy we assume that only some of the intermediate position constraints are active—that is we assume $0 = f(0) < f(1) < \cdots < f(m) = n$ such that $x_{[f]}(h_{f(j)+1}) = x_{f(j)}$ for each $j = 1, \ldots, m$ and $x_{[f]}(h_{s+1}) < x_s$ for $s \neq f(j)$. We propose a strategy with (1) a *power* phase from $(x, v, t) = (0, 0, \Delta T)$ to speed V_0 followed by a *speedhold* phase at speed V_0 and a *power* phase to $(x, v, t) = (x_{f(1)}, U_1, h_{f(1)+1})$, (2) a *power* phase from $(x, v, t) = (x_{f(j)}, U_j, h_{f(j)+1})$ to speed V_j followed by a *speedhold* phase at speed V_j and a *power* phase to $(x, v, t) = (x_{f(j+1)}, U_{j+1}, h_{f(j+1)+1})$ for each $j = 1, \ldots, m-2$ and (3) a *power* phase from $(x, v, t) = (x_{f(m-1)}, U_{m-1}, h_{f(m-1)+1})$ to speed V_{m-1}, a *speedhold* phase at speed V_{m-1}, a *coast* phase to speed U_m and a final *brake* phase to $(x, v, t) = (x_{f(m)}, 0, h_{f(m)+1}) = (X, 0, \Delta T + T_f)$. The cost of the strategy per unit mass is given by $J_f = J_f(\boldsymbol{U}, \boldsymbol{V})$ where $\boldsymbol{U} = (U_1, \ldots, U_m)$ and $\boldsymbol{V} = (V_0, \ldots, V_{m-1})$ and where

$$\begin{aligned} J_f \quad = \quad & P\Delta_p t(0, U_1) + \varphi(V_0)\left[h_{f(1)+1} - \Delta T - \Delta_p t(0, U_1)\right] \\ & + \sum_{j=1}^{m-2} P\Delta_p t(U_j, U_{j+1}) \\ & + \sum_{j=1}^{m-2} \varphi(V_j)\left[h_{f(j+1)+1} - h_{f(j)+1} - \Delta_p t(U_j, U_{j+1})\right] \\ & + P\Delta_p t(U_{m-1}, V_{m-1}) \\ & + \varphi(V_{m-1})\left[\Delta T + T_f - h_{f(m-1)+1} - \Delta t_{m-1}\right] \end{aligned} \tag{4.1}$$

where we have written

$$\Delta t_{m-1} = \Delta_p t(U_{m-1}, V_{m-1}) + \Delta_c t(V_{m-1}, U_m) + \Delta_b t(U_m, 0)$$

for convenience. The distance travelled during $(\Delta T, h_{f(1)+1})$ is

$$\delta_0 = \Delta_p x(0, U_1) + V_0[h_{f(1)+1} - \Delta T - \Delta_p t(0, U_1)] \tag{4.2}$$

on the first interval, the distance travelled during $(h_{f(j)+1}, h_{f(j+1)+1})$ is

$$\delta_j = \Delta_p x(U_j, U_{j+1}) + V_j[h_{f(j+1)+1} - h_{f(j)+1} - \Delta_p t(U_j, U_{j+1})] \tag{4.3}$$

on each intermediate interval with $j = 1, \ldots, m-2$ and the distance travelled during $(h_{f(m-1)+1}, h_{f(m)+1})$ is

$$\begin{aligned} \delta_{m-1} &= \Delta_p x(U_{m-1}, V_{m-1}) + V_{m-1}[\Delta T + T_f - h_{f(m-1)+1} - \Delta t_{m-1}] \\ &\qquad + \Delta_c x(V_{m-1}, U_m) + \Delta_b x(U_m, 0) \end{aligned} \tag{4.4}$$

on the final interval, where we have again used the notation Δt_{m-1} for convenience. We wish to minimize J_f subject to the constraints $\delta_j \le x_{f(j+1)} - x_{f(j)}$ for each $j = 0, \ldots, m-1$ and the overall position constraint $x_n \le \delta_0 + \delta_1 + \cdots + \delta_{m-1}$. □

Define a Lagrangian function

$$\begin{aligned} \mathcal{J}_f &= J_f + \sum_{j=0}^{m-1} \mu_j(\delta_j - x_{f(j+1)} + x_{f(j)}) + \mu \left(x_n - \sum_{j=0}^{m-1} \delta_j \right) \\ &= J_f + \sum_{j=0}^{m-1} (\mu_j - \mu)(\delta_j - x_{f(j+1)} + x_{f(j)}) \end{aligned} \tag{4.5}$$

and solve the equations $\partial \mathcal{J}_f / \partial V_j = 0$ and $\partial \mathcal{J}_f / \partial U_{j+1} = 0$ for each $j = 0, 1, \ldots, m-1$. For $j = 0$ we have $\partial \mathcal{J}_f / \partial V_0$ implies

$$(\varphi'(V_0) - (\mu - \mu_0))\, [h_{f(1)+1} - \Delta T - \Delta_p t(U_0, U_1)] = 0 \tag{4.6}$$

and $\partial \mathcal{J}_f / \partial U_1 = 0$ implies

$$\varphi(V_1) - \varphi(V_0) - (\mu - \mu_0)(U_1 - V_0) + (\mu - \mu_1)(U_1 - V_1) = 0. \tag{4.7}$$

For $j = 1, \ldots, m-2$ we have $\partial \mathcal{J}_f / \partial V_j = 0$ implies

$$(\varphi'(V_j) - (\mu - \mu_j))\, [h_{f(j+1)+1} - h_{f(j)+1} - \Delta_p t(U_j, U_{j+1})] = 0 \tag{4.8}$$

and $\partial \mathcal{J}_f / \partial U_{j+1} = 0$ implies

$$\varphi(V_{j+1}) - \varphi(V_j) - (\mu - \mu_j)(U_{j+1} - V_j) + (\mu - \mu_{j+1})(U_{j+1} - V_{j+1}) = 0. \tag{4.9}$$

For $j = m-1$ we can see that $\partial \mathcal{J}_f / \partial V_{m-1} = 0$ implies

$$(\varphi'(V_{m-1}) - (\mu - \mu_{m-1})) \Big[\Delta T + T_f - h_{f(m-1)+1} - \Delta t_{m-1} \Big] = 0, \tag{4.10}$$

where we have once again written

$$\Delta t_{m-1} = \Delta_p(U_{m-1}, V_{m-1}) - \Delta_c t(V_{m-1}, U_m) - \Delta_b t(U_m, 0)$$

for convenience, and that $\partial \mathcal{J}_f / \partial U_m = 0$ implies

$$\varphi(V_{m-1}) + (\mu_{m-1} - \mu)(V_{m-1} - U_m) = 0. \tag{4.11}$$

We can solve these equations in the same way we solved the corresponding equations for the leading train to give

$$U_{j+1} = [\psi(V_{j+1}) - \psi(V_j)]/[\varphi'(V_{j+1}) - \varphi'(V_j)] \tag{4.12}$$

for each $j = 0, \dots, m-2$. It is easy to show that the convexity of $\varphi(v)$ means that $V_j \leq U_{j+1} \leq V_{j+1}$. Lastly we find that the optimal value of U_m, the speed at which braking begins, is given by

$$U_m = \psi(V_{m-1})/\varphi'(V_{m-1}). \tag{4.13}$$

5 An Important Note About the Optimal Speeds

For both the leading train and the following train we have assumed that only a subset of the prescribed times involve active constraints. The active set was denoted by $\{\ell(i)\}_{i=0}^{k}$ for the leading train and by $\{f(j)\}_{j=0}^{m}$ for the following train. If the time constraint at a particular signal point is active then for both the leading train and the following train the optimal speed at this point is given by

$$U(V, W) = [\psi(W) - \psi(V)]/[\varphi'(W) - \varphi'(V)] \tag{5.1}$$

for $W \neq V$ where $(V, W) = (V_\ell, W_\ell)$ and $(V, W) = (V_f, W_f)$ are the respective optimal hold speeds for the leading train and the following train before and after the signal point. Now we note that

$$\lim_{W \to V} U(V, W) = \lim_{W \to V} \psi'(W)/\varphi''(W) = \lim_{W \to V} W = V. \tag{5.2}$$

If the time constraint at a particular signal point is not active then for both the leading train and the following train we know that the holding speed does not change and so $U = W = V$. Thus we may interpret (5.1) more generally so that henceforth

$$U(V, W) = \begin{cases} [\psi(W) - \psi(V)]/[\varphi'(W) - \varphi'(V)] & \text{for } W \neq V \\ V & \text{for } W = V. \end{cases} \tag{5.3}$$

This is important later in the paper where it will be convenient to allow degenerate phases. By adopting the convention described in (5.3) we can use the formulæ derived earlier, even if it turns out that a particular phase becomes degenerate in the search for optimal prescribed intermediate times. For instance it is possible that a *coast—speedhold—coast* sequence on a particular interval will degenerate into a single *speedhold* phase as the prescribed times are changed. Nevertheless all relevant formulæ remain valid.

6 The Two Train Separation Problem—an Example with Prescribed Times

We consider two identical trains‖ with identical allowed journey times $T = T_\ell = T_f$. The maximum power per unit mass is $P = 3\,\mathrm{m}^2\mathrm{s}^{-3}$ and the maximum braking force per unit mass is $Q = 0.3\,\mathrm{ms}^{-2}$. The resistive force per unit mass is given by the formula

$$r(v) = 6.75 \times 10^{-3} + 5 \times 10^{-5} v^2\,\mathrm{ms}^{-2}.$$

We take $X = 144 \times 10^3$ m with signal locations given by

$$\boldsymbol{x} = (0, 20, 84, 132, 144) \times 10^3\,\mathrm{m}.$$

We let $T = 72 \times 10^2$ s and $\Delta T = 12 \times 10^2$ s. In the first instance we suppose the specified times are

$$\boldsymbol{h} = (0, 12, 36, 60, 72) \times 10^2\,\mathrm{s}.$$

The leading train must pass the point $x_1 = 20000$ by time $h_1 = 1200$, pass $x_2 = 84000$ by $h_2 = 3600$, pass $x_3 = 132000$ by $h_3 = 6000$ and reach $x_4 = 144000$ at time $T = 7200$. We used MATLAB to calculate the optimal values

$$\boldsymbol{V} = (23.56, 23.56, 20.14, 10.90)\,\mathrm{ms}^{-1}, \quad \boldsymbol{U} = (23.56, 21.90, 15.98, 5.27)\,\mathrm{ms}^{-1}$$

for the key parameters. The cost of the optimal strategy is $J_\ell = 4334\ \mathrm{m}^2\mathrm{s}^{-2}$.

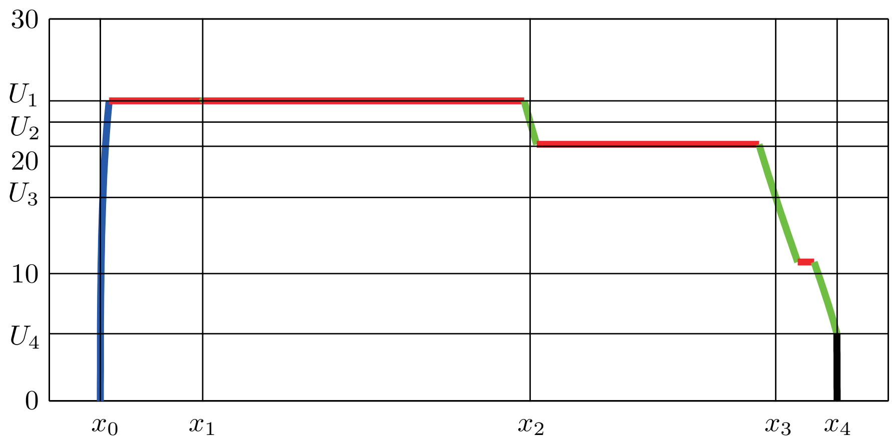

Figure 1: Optimal speed profile for a leading train on level track with initial prescribed times.

The following train must not leave the point $x_0 = 0$ until time $\Delta T = h_1 = 1200$, must not reach $x_1 = 20000$ until $h_2 = 3600$, $x_2 = 84000$ until $h_3 = 6000$, $x_3 = 132000$ until $h_4 = 7200$ but must reach the final point $x_4 = 144000$ by time $\Delta T + T = 8400$. We used MATLAB to calculate the optimal values

$$\boldsymbol{V} = (8.21, 26.48, 26.48, 26.48)\,\mathrm{ms}^{-1}, \quad \boldsymbol{U} = (18.95, 26.48, 26.48, 16.59)\,\mathrm{ms}^{-1}$$

‖In order to correctly compare costs we assume the trains have the same total mass.

for the key parameters. The cost of the optimal strategy is $J_f = 5414$ m^{2}s^{-2}. If possible we would like to reduce the total cost $J = J_\ell + J_f = 9748$ m^{2}s^{-2} by changing the intermediate times.

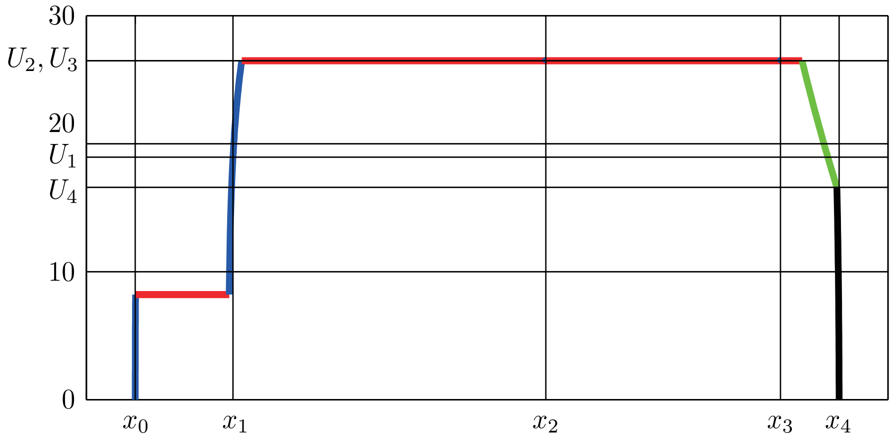

Figure 2: Optimal speed profile for a following train on level track with initial prescribed times.

We note that the maximum holding speed of the leading train 23.56 ms^{-1} for the initial prescribed times is significantly less than the maximum holding speed of the following train 26.48 ms^{-1}. Since high speeds mean high resistance and high cost it would seem sensible—if possible—to adjust the prescribed times in such a way that the maximum holding speed of the leading train is increased and the maximum holding speed of the following train is decreased.

7 The Two Train Separation Problem—an Example with Modified Prescribed Times

We use the same train parameters, the same journey times and the same signal locations as in the previous example. This time we suppose the specified times are

$$\boldsymbol{h} = (0, 12, 34, 58, 72) \times 10^2\,\text{s}.$$

The leading train must pass the point $x_1 = 20000$ by time $h_1 = 1200$, pass $x_2 = 84000$ by $h_2 = 3400$, $x_3 = 132000$ by $h_3 = 5800$ and must reach $x_4 = 144000$ at $T = 7200$. We used Matlab to calculate the optimal values

$$\boldsymbol{V} = (25.01, 25.01, 20.20, 8.77)\,\text{ms}^{-1}, \quad \boldsymbol{U} = (25.01, 22.69, 15.24, 3.69)\,\text{ms}^{-1}$$

for the key parameters. The cost of the optimal strategy is $J_\ell = 4602$ m^{2}s^{-2}.

The following train must not leave the point $x_0 = 0$ until time $\Delta T = h_1 = 1200$, must not reach $x_1 = 20000$ until $h_2 = 3400$, $x_2 = 84000$ until $h_3 = 5800$, $x_3 = 132000$ until $h_4 = 7200$ but must reach the final point $x_4 = 144000$ by time $\Delta T + T = 8400$. We used Matlab to calculate the optimal values

$$\boldsymbol{V} = (8.99, 25.40, 25.40, 25.40)\,\text{ms}^{-1}, \quad \boldsymbol{U} = (18.50, 25.40, 25.40, 15.83)$$

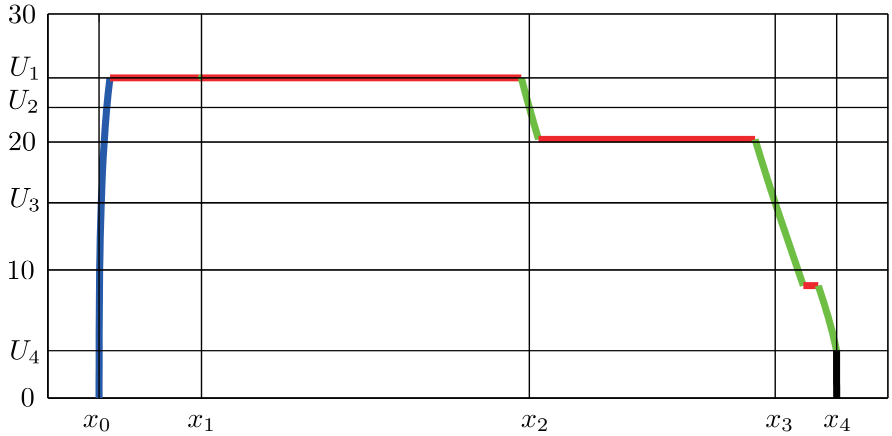

Figure 3: Optimal speed profile for a leading train on level track with modified prescribed times.

for the key parameters. The cost of the optimal strategy is $J_f = 5078$ m^{2}s^{-2}. Hence the total cost $J = J_\ell + J_f = 9680$ m^{2}s^{-2} with the modified prescribed times is less than the total cost with the original prescribed times.

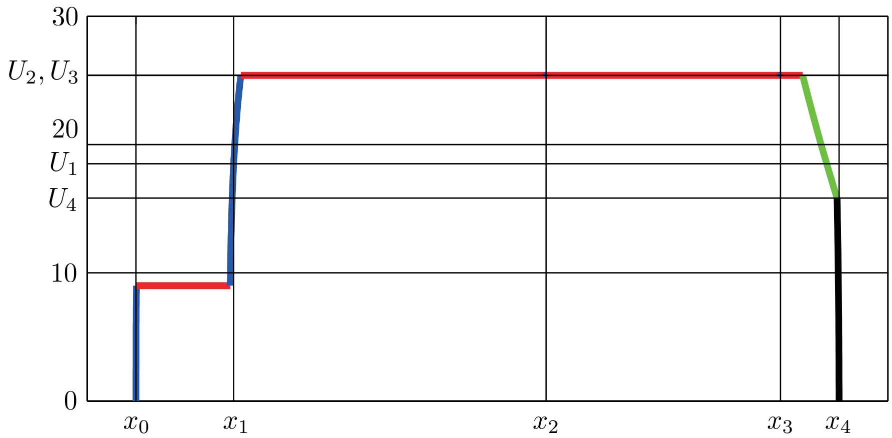

Figure 4: Optimal speed profile for a following train on level track with modified prescribed times.

We can now see that the modified prescribed times have indeed forced an increase in the maximum holding speed of the leading train to 25.01 ms^{-1} and have allowed a decrease in the maximum holding speed of the following train to 25.40 ms^{-1}. Although these modified times have resulted in decreased energy usage they are still not optimal** and more work is needed to find an efficient algorithm that will systematically determine the optimal set of prescribed times.

**Subsequent *ad hoc* calculations after acceptance of the original manuscript suggest that the times $\boldsymbol{h} \approx$

8 An Analytic Solution for the Optimal Prescribed Intermediate Times

We consider the case where the track consists of only four segments[††] and derive necessary conditions on an analytic solution for the optimal prescribed times. Thus we have $0 = x_0 < \cdots < x_4 = X$. We assume that the trains are identical, that the total journey time allowed for each train is the same and that the relative starting times are fixed. Thus we assume $T_\ell = T_f = T$ and that the journeys are completed during $t \in [0, T]$ for the leading train and $t \in [\Delta T, \Delta T + T]$ for the following train where $\Delta T > 0$ is given.

The leading train uses a strategy of *power–hold–coast* on (x_0, x_1) followed by *coast–hold–coast* on (x_s, x_{s+1}) for $s = 1, 2$ and *coast–hold–coast–brake* on (x_3, x_4). The hold speeds are V_s on (x_s, x_{s+1}) for $s = 0, \ldots, 3$. The speed at x_{s+1} is $U_{s+1} = U(V_s, V_{s+1})$ given by (5.3) for $s = 0, 1, 2$ and the speed at which braking begins is $U_4 = U_4(V_3) = \psi(V_3)/\varphi'(V_3)$.

The following train uses a strategy of *power–hold–power* on (x_s, x_{s+1}) for $s = 0, 1, 2$ followed by *power–hold–coast–brake* on (x_3, x_4). The hold speeds are Y_s on (x_s, x_{s+1}) for $s = 0, \ldots, 3$. The speed at x_{s+1} is $Z_{s+1} = Z_{s+1}(Y_s, Y_{s+1}) = U(Y_s, Y_{s+1})$ given by (5.3) for $s = 0, 1, 2$ and the speed at which braking begins is $Z_4 = Z_4(Y_3) = \psi(Y_3)/\varphi'(Y_3)$.

Remark 8.1. In this formulation we note that degenerate phases are allowed. See the earlier remarks in Section 5.

Our first task is to define the main constraints. For the leading train we calculate the time taken to traverse the various segments as

$$\begin{aligned} f_0(V_0, V_1) &= \Delta_p t(0, V_0) + \Delta_c t(V_0, U_1) \\ &\quad + (1/V_0)\left[(x_1 - x_0) - \Delta_p x(0, V_0) - \Delta_c x(V_0, U_1)\right] \end{aligned} \tag{8.1}$$

for the segment (x_0, x_1),

$$\begin{aligned} f_s(V_{s-1}, V_s, V_{s+1}) &= \Delta_c t(U_s, U_{s+1}) \\ &\quad + (1/V_s)\left[(x_{s+1} - x_s) - \Delta_c x(U_s, U_{s+1})\right] \end{aligned} \tag{8.2}$$

for the segments (x_s, x_{s+1}) when $s = 1, 2$ and

$$\begin{aligned} f_3(V_2, V_3) &= \Delta_c t(U_3, U_4) + \Delta_b t(U_4, 0) \\ &\quad + (1/V_3)\left[(x_4 - x_3) - \Delta_c x(U_3, U_4) - \Delta_b x(U_4, 0)\right] \end{aligned} \tag{8.3}$$

for the segment (x_3, x_4).

For the following train the times taken to traverse the various segments are given by

$$g_0(Y_0, Y_1) = \Delta_p t(0, Z_1) + (1/Y_0)\left[(x_1 - x_0) - \Delta_p x(0, Z_1)\right] \tag{8.4}$$

for the segment (x_0, x_1),

$$\begin{aligned} g_s(Y_{s-1}, Y_s, Y_{s+1}) &= \Delta_p t(Z_s, Z_{s+1}) \\ &\quad + (1/Y_s)\left[(x_{s+1} - x_s) - \Delta_p x(Z_s, Z_{s+1})\right], \end{aligned} \tag{8.5}$$

$(0, 12, 32.65, 59.76, 72) \times 10^2$ s are close to optimal. The corresponding optimal values for the key speeds are

$$\boldsymbol{V} \approx (26.134, 26.134, 17.660, 10.985)\ \mathrm{ms}^{-1}, \quad \boldsymbol{U} \approx (26.134, 22.170, 14.582, 5.334)\ \mathrm{ms}^{-1}$$

for the leading train and

$$\boldsymbol{V} \approx (9.624, 23.653, 25.920, 25.920)\ \mathrm{ms}^{-1}, \quad \boldsymbol{U} \approx (17.624, 24.804, 25.920, 16.196)\ \mathrm{ms}^{-1}$$

for the following train. The total cost is $J = J_\ell + J_f \approx 4633.83 + 4900.66 = 9534.49\ \mathrm{m}^2\mathrm{s}^{-2}$.

[††]A general argument could be expected to proceed along similar lines.

for the segments (x_s, x_{s+1}) when $s = 1, 2$ and

$$\begin{aligned} g_3(Y_2, Y_3) &= \Delta_p t(Z_3, Y_3) + \Delta_c t(Y_3, Z_4) + \Delta_b t(Z_4, 0) \\ &\quad + (1/Y_3)\left[(x_4 - x_3) - \Delta_p x(Z_3, Y_3) - \Delta_c x(Y_3, Z_4) - \Delta_b x(Z_4, 0)\right] \end{aligned} \tag{8.6}$$

for the segment (x_3, x_4).

The following train must start at time ΔT and so the time taken for the leading train to traverse the segment (x_0, x_1) is at most ΔT. Therefore

$$f_0 \le \Delta T. \tag{8.7}$$

The time taken for the leading train to traverse the combined segment (x_0, x_{s+1}) is at most equal to the initial delay for the following train plus the time taken for the following train to traverse the combined segment (x_0, x_s). Thus we have

$$f_0 + f_1 + \cdots + f_{s+1} \le \Delta T + g_0 + \cdots + g_s \tag{8.8}$$

for each $s = 0, 1, 2$. However, for both trains, the maximum allowed journey time is T. Therefore

$$f_0 + \cdots + f_3 \le T \tag{8.9}$$

and

$$g_0 + \cdots + g_3 \le T. \tag{8.10}$$

We will also impose the speed constraints

$$0 \le V_3 \le V_2 \le V_1 \le V_0 \tag{8.11}$$

and

$$0 \le Y_0 \le Y_1 \le Y_2 \le Y_3. \tag{8.12}$$

Our next task is to define the cost for each train. For the leading train the cost $J_\ell = J_\ell(\boldsymbol{V})$ is given by

$$\begin{aligned} J_\ell &= P\Delta_p t(0, V_0) + r(V_0)\left[(x_1 - x_0) - \Delta_p x(0, V_0) - \Delta_c x(V_0, U_1)\right] \\ &\quad + \sum_{s=1}^{2} r(V_s)\left[(x_{s+1} - x_s) - \Delta_c x(U_s, U_{s+1})\right] \\ &\quad + r(V_3)\left[(x_4 - x_3) - \Delta_c x(U_3, U_4) - \Delta_b x(U_4, 0)\right] \end{aligned} \tag{8.13}$$

and for the following train the cost $J_f = J_f(\boldsymbol{Y})$ is given by

$$\begin{aligned} J_f &= P\Delta_p t(0, Y_3) + r(Y_0)\left[(x_1 - x_0) - \Delta_p x(0, Z_1)\right] \\ &\quad + \sum_{s=1}^{2} r(Y_s)\left[(x_{s+1} - x_s) - \Delta_p x(Z_s, Z_{s+1})\right] \\ &\quad + r(Y_3)\left[(x_4 - x_3) - \Delta_p x(Z_3, Y_3) - \Delta_c x(Y_3, Z_4) - \Delta_b x(Z_4, 0)\right]. \end{aligned} \tag{8.14}$$

To find the minimum total cost subject to the required constraints we form a Lagrangian function

$$\begin{aligned} \mathcal{J} &= J_\ell + J_f + \kappa_0[f_0 - \Delta T] + \kappa_1[f_0 + f_1 - (\Delta T + g_0)] \\ &\quad + \kappa_2[f_0 + f_1 + f_2 - (\Delta T + g_0 + g_1)] \\ &\quad + \kappa_3[f_0 + f_1 + f_2 + f_3 - (\Delta T + g_0 + g_1 + g_2)] \\ &\quad + \lambda_\ell(f_0 + \cdots + f_3 - T) + \lambda_f(g_0 + \cdots + g_3 - T) \\ &\quad + \mu_1(V_1 - V_0) + \mu_2(V_2 - V_1) + \mu_3(V_3 - V_2) - \mu_4 V_3 \\ &\quad - \nu_1 Y_0 + \nu_2(Y_0 - Y_1) + \nu_3(Y_1 - Y_2) + \nu_4(Y_2 - Y_3) \end{aligned} \tag{8.15}$$

and apply the usual Karush–Kuhn–Tucker (KKT) equations and complementary slackness conditions.

First we differentiate with respect to V_0 to give

$$\frac{\partial J_\ell}{\partial V_0} = r'(V_0)\,[(x_1 - x_0) - \Delta_p x(0, V_0) - \Delta_c x(V_0, U_1)] + [r(V_0) - r(V_1)]\frac{U_1}{r(U_1)}\frac{\partial U_1}{\partial V_0}, \tag{8.16}$$

$$\frac{\partial f_0}{\partial V_0} = -\frac{1}{V_0^2}\,[(x_1 - x_0) - \Delta_p x(0, V_0) - \Delta_c x(V_0, U_1)] + \left[\frac{1}{V_0} - \frac{1}{U_1}\right]\frac{U_1}{r(U_1)}\frac{\partial U_1}{\partial V_0}, \tag{8.17}$$

$$\frac{\partial f_1}{\partial V_0} = \left[\frac{1}{U_1} - \frac{1}{V_1}\right]\frac{U_1}{r(U_1)}\frac{\partial U_1}{\partial V_0}. \tag{8.18}$$

Second we differentiate with respect to V_1 to give

$$\frac{\partial J_\ell}{\partial V_1} = r'(V_1)\,[(x_2 - x_1) - \Delta_c x(U_1, U_2)] + \sum_{s=0}^{1}[r(V_s) - r(V_{s+1})]\frac{U_{s+1}}{r(U_{s+1})}\frac{\partial U_{s+1}}{\partial V_1}, \tag{8.19}$$

$$\frac{\partial f_0}{\partial V_1} = \left[\frac{1}{V_0} - \frac{1}{U_1}\right]\frac{U_1}{r(U_1)}\frac{\partial U_1}{\partial V_1}, \tag{8.20}$$

$$\frac{\partial f_1}{\partial V_1} = -\frac{1}{V_1^2}\,[(x_2 - x_1) - \Delta_c x(U_1, U_2)] + \left[\frac{1}{U_1} - \frac{1}{V_1}\right]\frac{U_1}{r(U_1)}\frac{\partial U_1}{\partial V_1} + \left[\frac{1}{V_1} - \frac{1}{U_2}\right]\frac{U_2}{r(U_2)}\frac{\partial U_2}{\partial V_1}, \tag{8.21}$$

$$\frac{\partial f_2}{\partial V_1} = \left[\frac{1}{U_2} - \frac{1}{V_2}\right]\frac{U_2}{r(U_2)}\frac{\partial U_2}{\partial V_1}. \tag{8.22}$$

Third we differentiate with respect to V_2 to give

$$\frac{\partial J_\ell}{\partial V_2} = r'(V_2)\,[(x_3 - x_2) - \Delta_c x(U_2, U_3)] + \sum_{s=1}^{2}[r(V_s) - r(V_{s+1})]\frac{U_{s+1}}{r(U_{s+1})}\frac{\partial U_{s+1}}{\partial V_2}, \tag{8.23}$$

$$\frac{\partial f_1}{\partial V_2} = \left[\frac{1}{V_1} - \frac{1}{U_2}\right]\frac{U_2}{r(U_2)}\frac{\partial U_2}{\partial V_2}, \tag{8.24}$$

$$\frac{\partial f_2}{\partial V_2} = -\frac{1}{V_2^2}\left[(x_3 - x_2) - \Delta_c x(U_2, U_3)\right] + \left[\frac{1}{U_2} - \frac{1}{V_2}\right]\frac{U_2}{r(U_2)}\frac{\partial U_2}{\partial V_2} + \left[\frac{1}{V_2} - \frac{1}{U_3}\right]\frac{U_3}{r(U_3)}\frac{\partial U_3}{\partial V_2}, \tag{8.25}$$

$$\frac{\partial f_3}{\partial V_2} = \left[\frac{1}{U_3} - \frac{1}{V_3}\right]\frac{U_3}{r(U_3)}\frac{\partial U_3}{\partial V_2}. \tag{8.26}$$

Fourth we differentiate with respect to V_3 to give

$$\frac{\partial J_\ell}{\partial V_3} = r'(V_3)\left[(x_4 - x_3) - \Delta_c x(U_3, U_4) - \Delta_b x(0, U_4)\right] + [r(V_2) - r(V_3)]\frac{U_3}{r(U_3)}\frac{\partial U_3}{\partial V_3} + r(V_3)\left[\frac{U_4}{r(U_4)} - \frac{U_4}{Q + r(U_4)}\right]\frac{\partial U_4}{\partial V_3}, \tag{8.27}$$

$$\frac{\partial f_2}{\partial V_3} = \left[\frac{1}{V_2} - \frac{1}{U_3}\right]\frac{U_3}{r(U_3)}\frac{\partial U_3}{\partial V_3}, \tag{8.28}$$

$$\frac{\partial f_3}{\partial V_3} = -\frac{1}{V_3^2}\left[(x_4 - x_3) - \Delta_c x(U_3, U_4) - \Delta_b x(0, U_4)\right] + \left[\frac{1}{U_3} - \frac{1}{V_3}\right]\frac{U_3}{r(U_3)}\frac{\partial U_3}{\partial V_3} + \left[\frac{1}{V_3} - \frac{1}{U_4}\right]\left[\frac{U_4}{r(U_4)} - \frac{U_4}{Q + r(U_4)}\right]\frac{\partial U_4}{\partial V_3}. \tag{8.29}$$

Fifth we differentiate with respect to Y_0 to give

$$\frac{\partial J_f}{\partial Y_0} = r'(Y_0)\left[(x_1 - x_0) - \Delta_p x(0, Z_1)\right] + [r(Y_1) - r(Y_0)]\frac{Z_1^2}{P - \varphi(Z_1)}\frac{\partial Z_1}{\partial Y_0}, \tag{8.30}$$

$$\frac{\partial g_0}{\partial Y_0} = -\frac{1}{Y_0^2}\left[(x_1 - x_0) - \Delta_p x(0, Z_1)\right] + \left[\frac{1}{Z_1} - \frac{1}{Y_0}\right]\frac{Z_1^2}{P - \varphi(Z_1)}\frac{\partial Z_1}{\partial Y_0}, \tag{8.31}$$

$$\frac{\partial g_1}{\partial Y_0} = \left[\frac{1}{Y_1} - \frac{1}{Z_1}\right]\frac{Z_1^2}{P - \varphi(Z_1)}\frac{\partial Z_1}{\partial Y_0}. \tag{8.32}$$

Sixth we differentiate with respect to Y_1 to give

$$\frac{\partial J_f}{\partial Y_1} = r'(Y_1)\left[(x_2 - x_1) - \Delta_p x(Z_1, Z_2)\right] + \sum_{s=0}^{1}[r(Y_{s+1}) - r(Y_s)]\frac{Z_{s+1}^2}{P - \varphi(Z_{s+1})}\frac{\partial Z_{s+1}}{\partial Y_1}, \tag{8.33}$$

$$\frac{\partial g_0}{\partial Y_1} = \left[\frac{1}{Z_1} - \frac{1}{Y_0}\right] \frac{Z_1^2}{P - \varphi(Z_1)} \frac{\partial Z_1}{\partial Y_1}, \tag{8.34}$$

$$\begin{aligned}\frac{\partial g_1}{\partial Y_1} = & -\frac{1}{Y_1^2}\left[(x_2 - x_1) - \Delta_p x(Z_1, Z_2)\right] \\ & + \left[\frac{1}{Y_1} - \frac{1}{Z_1}\right] \frac{Z_1^2}{P - \varphi(Z_1)} \frac{\partial Z_1}{\partial Y_1} \\ & + \left[\frac{1}{Z_2} - \frac{1}{Y_1}\right] \frac{Z_2^2}{P - \varphi(Z_2)} \frac{\partial Z_2}{\partial Y_1},\end{aligned} \tag{8.35}$$

$$\frac{\partial g_2}{\partial Y_1} = \left[\frac{1}{Y_2} - \frac{1}{Z_2}\right] \frac{Z_2^2}{P - \varphi(Z_2)} \frac{\partial Z_2}{\partial Y_1}. \tag{8.36}$$

Seventh we differentiate with respect to Y_2 to give

$$\begin{aligned}\frac{\partial J_f}{\partial Y_2} = & \, r'(Y_2)\left[(x_3 - x_2) - \Delta_p x(Z_2, Z_3)\right] \\ & + \sum_{s=1}^{2} [r(Y_{s+1}) - r(Y_s)] \frac{Z_{s+1}^2}{P - \varphi(Z_{s+1})} \frac{\partial Z_{s+1}}{\partial Y_2},\end{aligned} \tag{8.37}$$

$$\frac{\partial g_1}{\partial Y_2} = \left[\frac{1}{Z_2} - \frac{1}{Y_1}\right] \frac{Z_2^2}{P - \varphi(Z_2)} \frac{\partial Z_2}{\partial Y_2}, \tag{8.38}$$

$$\begin{aligned}\frac{\partial g_2}{\partial Y_2} = & -\frac{1}{Y_2^2}\left[(x_3 - x_2) - \Delta_p x(Z_2, Z_3)\right] \\ & + \left[\frac{1}{Y_2} - \frac{1}{Z_2}\right] \frac{Z_2^2}{P - \varphi(Z_2)} \frac{\partial Z_2}{\partial Y_2} \\ & + \left[\frac{1}{Z_3} - \frac{1}{Y_2}\right] \frac{Z_3^2}{P - \varphi(Z_3)} \frac{\partial Z_3}{\partial Y_2},\end{aligned} \tag{8.39}$$

$$\frac{\partial g_3}{\partial Y_2} = \left[\frac{1}{Y_3} - \frac{1}{Z_3}\right] \frac{Z_3^2}{P - \varphi(Z_3)} \frac{\partial Z_3}{\partial Y_2}. \tag{8.40}$$

Eighth we differentiate with respect to Y_3 to give

$$\begin{aligned}\frac{\partial J_f}{\partial Y_3} = & \, r'(Y_3)\left[(x_4 - x_3) - \Delta_p x(Z_3, Y_3)\right. \\ & \left. - \Delta_c x(Y_3, Z_4) - \Delta_b x(Z_4, 0)\right] \\ & + [r(Y_3) - r(Y_2)] \frac{Z_3^2}{P - \varphi(Z_3)} \frac{\partial Z_3}{\partial Y_3} \\ & + r(Y_3)\left[\frac{Z_4}{r(Z_4)} - \frac{Z_4}{Q + r(Z_4)}\right] \frac{\partial Z_4}{\partial Y_3},\end{aligned} \tag{8.41}$$

$$\frac{\partial g_2}{\partial Y_3} = \left[\frac{1}{Z_3} - \frac{1}{Y_2}\right] \frac{Z_3^2}{P - \varphi(Z_3)} \frac{\partial Z_3}{\partial Y_3}, \tag{8.42}$$

$$\begin{aligned}\frac{\partial g_3}{\partial Y_3} &= -\frac{1}{Y_3^2}[(x_4-x_3)-\Delta_p x(Z_3,Y_3)\\ &\qquad -\Delta_c x(Y_3,Z_4)-\Delta_b x(Z_4,0)]\\ &\quad+\left[\frac{1}{Y_3}-\frac{1}{Z_3}\right]\frac{Z_3^2}{P-\varphi(Z_3)}\frac{\partial Z_3}{\partial Y_3}\\ &\quad+\left[\frac{1}{Y_3}-\frac{1}{Z_4}\right]\left[\frac{Z_4}{r(Z_4)}-\frac{Z_4}{Q+r(Z_4)}\right]\frac{\partial Z_4}{\partial Y_3}. \end{aligned}\tag{8.43}$$

We can write the KKT equations in vector form as $\partial\mathcal{J}/\partial\boldsymbol{V}=\boldsymbol{0}$ and $\partial\mathcal{J}/\partial\boldsymbol{Y}=\boldsymbol{0}$. For the first vector equation we have

$$\begin{aligned}\kappa_0\frac{\partial f_0}{\partial\boldsymbol{V}}&+\kappa_1\frac{\partial(f_0+f_1)}{\partial\boldsymbol{V}}+\kappa_2\frac{\partial(f_0+f_1+f_2)}{\partial\boldsymbol{V}}\\ &+(\kappa_3+\lambda_\ell)\frac{\partial(f_0+f_1+f_2+f_3)}{\partial\boldsymbol{V}}-\boldsymbol{\Delta\mu}=-\frac{\partial J_\ell}{\partial\boldsymbol{V}}\end{aligned}\tag{8.44}$$

where we have defined the vector $\boldsymbol{\Delta\mu}=\mu_1\boldsymbol{e}_1+(\mu_2-\mu_1)\boldsymbol{e}_2+(\mu_3-\mu_2)\boldsymbol{e}_3+(\mu_4-\mu_3)\boldsymbol{e}_4$ and $\boldsymbol{e}_1,\ldots,\boldsymbol{e}_4\in\mathbb{R}^4$ are the usual unit vectors. If we define $\sigma_q=\kappa_{q-1}+\cdots+\kappa_3+\lambda_\ell$ for each $q=1,\ldots,4$ then this becomes

$$\sigma_1\frac{\partial f_0}{\partial\boldsymbol{V}}+\sigma_2\frac{\partial f_1}{\partial\boldsymbol{V}}+\sigma_3\frac{\partial f_2}{\partial\boldsymbol{V}}+\sigma_4\frac{\partial f_3}{\partial\boldsymbol{V}}-\boldsymbol{\Delta\mu}=-\frac{\partial J_\ell}{\partial\boldsymbol{V}}.\tag{8.45}$$

Equivalently we may write

$$\left[\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{V}}\right]\boldsymbol{\sigma}=-\left(\frac{\partial J_\ell}{\partial\boldsymbol{V}}-\boldsymbol{\Delta\mu}\right)\tag{8.46}$$

where we note that the coefficient matrix in (8.46) is the Jacobian matrix of the transformation $\boldsymbol{V}\in\mathcal{V}\mapsto\boldsymbol{f}\in\boldsymbol{f}(\mathcal{V})=\mathcal{F}$ where $\mathcal{V}=\{\boldsymbol{V}\mid V_0\geq V_1\geq V_2\geq V_3\geq 0\}$ is the feasible set. Since there is a one–to–one correspondence between the traversal times $\boldsymbol{f}\in\mathcal{F}$ and the holding speeds $\boldsymbol{V}\in\mathcal{V}$ for the leading train it follows that the Jacobian matrix is non-singular on the interior of the feasible set $\mathcal{V}^\circ=\{\boldsymbol{V}\mid V_0>V_1>V_2>V_3>0\}$. At the same time we observe that the complementary slackness conditions from the minimization of (8.15) show that $\boldsymbol{\mu}=\boldsymbol{0}$ and hence also that $\boldsymbol{\Delta\mu}=\boldsymbol{0}$ for $\boldsymbol{V}\in\mathcal{V}^\circ$. Therefore, for each $\boldsymbol{V}\in\mathcal{V}^\circ$, the unique solution is defined by the equation

$$\left[\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{V}}\right]\boldsymbol{\sigma}=-\frac{\partial J_\ell}{\partial\boldsymbol{V}}.\tag{8.47}$$

We may extend our analysis to the entire set $\mathcal{V}$ by taking appropriate limits. Thus we may express the unique solution $\boldsymbol{\sigma}=\boldsymbol{\sigma}(\boldsymbol{V})$ in the form

$$\boldsymbol{\sigma}=-\left[\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{V}}\right]^\dagger\frac{\partial J_\ell}{\partial\boldsymbol{V}}\tag{8.48}$$

for all $\boldsymbol{V}\in\mathcal{V}$. The coefficient matrix on the right-hand side of (8.48) is written as a Moore–Penrose generalized inverse because it is possible that the Jacobian matrix may be singular on the boundary of $\mathcal{V}$.

For the second vector equation we have

$$\begin{aligned}\kappa_1\frac{\partial g_0}{\partial\boldsymbol{Y}}&+\kappa_2\frac{\partial(g_0+g_1)}{\partial\boldsymbol{Y}}+\kappa_3\frac{\partial(g_0+g_1+g_2)}{\partial\boldsymbol{Y}}\\ &-\lambda_f\frac{\partial(g_0+g_1+g_2+g_3)}{\partial\boldsymbol{Y}}-\boldsymbol{\Delta\nu}=\frac{\partial J_f}{\partial\boldsymbol{Y}}\end{aligned}\tag{8.49}$$

where we have defined $\Delta\boldsymbol{\nu} = (\nu_2 - \nu_1)\boldsymbol{e}_1 + (\nu_3 - \nu_2)\boldsymbol{e}_2 + (\nu_4 - \nu_3)\boldsymbol{e}_3 - \nu_4\boldsymbol{e}_4$. If we define $\tau_q = \kappa_q + \cdots + \kappa_3 - \lambda_f$ for each $q = 1, 2, 3$ and $\tau_4 = -\lambda_f$ then this becomes

$$\tau_1 \frac{\partial g_0}{\partial \boldsymbol{Y}} + \tau_2 \frac{\partial g_1}{\partial \boldsymbol{Y}} + \tau_3 \frac{\partial g_2}{\partial \boldsymbol{Y}} + \tau_4 \frac{\partial g_3}{\partial \boldsymbol{Y}} - \Delta\boldsymbol{\nu} = \frac{\partial J_f}{\partial \boldsymbol{Y}}. \tag{8.50}$$

Equivalently we may write

$$\left[\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{Y}}\right] \boldsymbol{\tau} = \frac{\partial J_f}{\partial \boldsymbol{Y}} + \Delta\boldsymbol{\nu} \tag{8.51}$$

where we note that the coefficient matrix in (8.51) is the Jacobian matrix of the transformation $\boldsymbol{Y} \in \mathcal{Y} \mapsto \boldsymbol{g} \in \boldsymbol{g}(\mathcal{Y}) = \mathcal{G}$ where $\mathcal{Y} = \{\boldsymbol{Y} \mid 0 \le Y_0 \le Y_1 \le Y_2 \le Y_3\}$ is the feasible set. Since there is a one–to–one correspondence between the traversal times $\boldsymbol{g} \in \mathcal{G}$ and the holding speeds $\boldsymbol{Y} \in \mathcal{Y}$ for the following train it follows that the Jacobian matrix is non-singular on the interior of the feasible set $\mathcal{Y}^\circ = \{\boldsymbol{Y} \mid 0 < Y_0 < Y_1 < Y_2 < Y_3\}$. At the same time we observe that the complementary slackness conditions from the minimization of (8.15) show that $\boldsymbol{\nu} = \boldsymbol{0}$ and hence also that $\Delta\boldsymbol{\nu} = \boldsymbol{0}$ for $\boldsymbol{Y} \in \mathcal{Y}^\circ$. Therefore, for each $\boldsymbol{Y} \in \mathcal{Y}^\circ$, the unique solution is defined by

$$\left[\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{Y}}\right] \boldsymbol{\tau} = \frac{\partial J_f}{\partial \boldsymbol{Y}}. \tag{8.52}$$

We may extend our analysis to the entire set $\mathcal{Y}$ by taking appropriate limits. Thus we may express the unique solution $\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{Y})$ in the form

$$\boldsymbol{\tau} = \left[\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{Y}}\right]^\dagger \frac{\partial J_f}{\partial \boldsymbol{Y}} \tag{8.53}$$

for each $\boldsymbol{Y} \in \mathcal{Y}$. Once again the Moore–Penrose generalized inverse is used because the Jacobian matrix may be singular on the boundary of $\mathcal{Y}$.

Since $\partial f_s/\partial V_t = 0$, $\partial g_s/\partial Y_t = 0$ for $t \ne s-1, s, s+1$ the coefficient matrices in (8.46) and (8.51) are tridiagonal. Hence for given $(\boldsymbol{V}, \boldsymbol{Y})$ numerical calculation of the generalized inverse matrices is straightforward. If we define the elementary permutation matrix $P = [\boldsymbol{e}_4, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3] \in \mathbb{R}^{4\times 4}$ then it follows from the definitions of $\boldsymbol{\sigma}, \boldsymbol{\tau}$ above that

$$-P\boldsymbol{\sigma} + \boldsymbol{\tau} + \kappa\boldsymbol{e}_4 + \lambda\boldsymbol{1} = \boldsymbol{0} \tag{8.54}$$

where we have written $\kappa = \kappa_0 + \cdots + \kappa_3$ and $\lambda = \lambda_\ell + \lambda_f$. Thus we obtain the necessary conditions for optimality (8.48), (8.53) and (8.54).

If intermediate times are prescribed it is relatively straightforward to set up simple iterations that allow us to determine $\boldsymbol{V}$ and $\boldsymbol{Y}$ as we did in the earlier examples. Hence we can easily use a package such as MATLAB to solve (8.47) and (8.52) and find $\boldsymbol{\sigma}(\boldsymbol{V})$ and $\boldsymbol{\tau}(\boldsymbol{Y})$. If (8.54) is satisfied then the prescribed times are optimal. If not we must try another set of prescribed times. Of course our ultimate aim is to find an algorithm which can be used to calculate the optimal values for $\boldsymbol{V}$ and $\boldsymbol{Y}$. In this regard we hope to exploit our knowledge of the marginal cost rates and the implicit relationship between $\boldsymbol{V}$ and $\boldsymbol{Y}$ embodied in (8.54).

9 Conclusions and Future Work

We have developed a theoretical methodology that could ultimately provide rail operators with a framework that will enable them to better understand the principles of energy-efficient train separation. Our particular contribution is to show how prescribed intermediate times can be used to find the most energy-efficient speed profiles for both a leading train and a following train while maintaining adequate train separation. Our next task is to develop an efficient numerical algorithm for optimization of the prescribed times. We would also like to extend the methods used in this paper to separation of a fleet of trains travelling in the same direction. In general we expect that trains in a fleet, other than the first and the last, will sometimes be constrained as leading trains and sometimes as following trains. We would also like to solve the corresponding problems on tracks with steep grades. There is much to be done!

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