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FIXED POINT METHODS FOR COMPUTING A Z-EIGENPAIR OF GENERAL SQUARE TENSORS

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Abstract: The tensor eigenvalue problem has many important applications, such as in the best rank-one approximation, higher order Markov chains, and hypergraphs. In this paper, we propose a fixed point algorithm for computing a Z-eigenvalue of tensors and show that the proposed algorithm is convergent for general square tensors. We also show that the shifted power method for computing a Z-eigenpair of tensor is convergent for general square tensors under some assumptions. Numerical results are reported to show the effectiveness of the proposed algorithm.

Key words: *Z-eigenvalue, general tensors, fixed point algorithm, convergence.*

Mathematics Subject Classification: *90C33, 90C30, 65H10.*

1 Introduction

Recently, the eigenvalue problem of high-order tensors has attracted significant attention since it plays a fundamental role in the best rank-one approximation. The best rank-one approximation of higher order tensors has numerous applications in engineering and higher order statistics, such as Statistical Data Analysis [6, 11, 23]. It is also discovered that the tensor eigenvalue problem has some connections to the transition probability tensors of higher order Markov chains in [14–16]. In [9, 13], Qi et al. extended the notions and results regarding the algebraic connectivity of a graph in spectral graph theory to k-uniform hypergraphs by using Z-eigenvalues. In [10], it is shown that Z-eigenvalues have a new exciting application in quantum entanglement problem.

Various kinds of methods have been developed to compute a Z-eigenvalue of a tensor. In particular, Kofidis [11] extended the power method for computing the largest eigenvalue of a matrix to a Z-eigenpair of a symmetric tensor, which is known as the symmetric higher-order power method(SHOPM). Kolda and Mayo [12] imposed a shifted term into the above SHOPM to form the shifted symmetric higher-order power method (SSHOPM) and established the convergence of this method. In [8], Han reformulated the Z-eigenpair problem for even order symmetric tensors to an unconstrained optimization problem, which can be solved by using some powerful optimization algorithms, such as the BFGS method. Chen et al. [3] proposed a maximum block improvement method to find a singular-pair of a symmetric tensor and then adjusted the singular vectors to get a Z-eigenpair. In lower dimensional

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case, Qi et al. [19] proposed a method to compute the largest Z-eigenvalue directly. Cui et al. [5] proposed a Jacobian semidefinite relaxation approach to compute all real eigenvalues for symmetric tensors.

Those existing methods, including algorithms and their convergence analysis, are mostly for computing the eigenpair of symmetric tensors. However, there are very few methods for computing the eigenpair of asymmetric tensors. The asymmetric tensors also have widely applications in science and engineering, such as the velocity gradient tensor and the deformation gradient tensors. In [4], the eigenvalue regions of the velocity gradient tensor denote the place where the rotational component dominates the shear component of deformation. In [24], the eigenvalue visualization of the deformation gradient tensors enables us to examine the relative strengths of fluid expansion (contraction), rotations, and the rate of shear strain in one single plot. While it is possible to convert the asymmetric tensor field into a symmetric one by multiplying with its transpose or simply removing the antisymmetric component, such an approach can cause information loss. So direct analysis and visualization of asymmetric tensor fields are highly desirable [1].

In this paper, we shall focus on the methods of computing a Z-eigenpair for any square tensors, symmetric or not. First, we show the shifted power method [12] is convergent for any square tensors under a weak assumption. Then, we reformulate the Z-eigenvalue problem into a projection equation which is related to a projection operator on the unit ball. Finally, we use a fixed point method to solve this equation and prove the linear convergence of the proposed method for any square tensors.

The paper is organized as follows. In Section 2, we list some preliminaries. In Section 3, we prove that the shifted power method [12] is convergent for general square tensors under a weak assumption. In Section 4, we propose a fixed point method and prove the linear convergence of the proposed algorithm. Some preliminary numerical results are also reported in Section 5.

2 Preliminaries

Let $\mathcal{R}$ be the real field, we consider an m-order n-dimensional tensor $\mathcal{A}$ consisting of n^m entries in $\mathcal{R}$:

$$\mathcal{A} = (a_{i_1 i_2 \dots i_m}), a_{i_1 i_2 \dots i_m} \in \mathcal{R}, 1 \leq i_1, i_2, \dots, i_m \leq n \tag{2.1}$$

If $a_{i_{p(1)} i_{p(2)} \dots i_{p(m)}} = a_{i_1 \dots i_m}$ for all $i_1, i_2, \dots i_m \in \{1, 2, \dots, n\}$ and p belongs to the set of all permutations of $\{1, \dots, m\}$, then tensor $\mathcal{A}$ is called to be symmetric.

A tensor is a natural generalization of a matrix. A matrix is simply a two-order tensor. We denote the set of all m-order n-dimensional tensors by $\mathcal{R}^{[m,n]}$ in the rest of this paper.

Let $\mathcal{A} \in \mathcal{R}^{[m,n]}$ and $x = (x_1, x_2, \dots, x_n)^T \in \mathcal{R}^n$. Let

$$\mathcal{A}x^m := \sum_{i_1, \dots, i_m = 1}^{n} a_{i_1 i_2 \dots i_m} x_{i_1} \dots x_{i_m}, \tag{2.2}$$

$$\mathcal{A}x^{m-1} := \Big(\sum_{i_2, \dots, i_m = 1}^{n} a_{i i_2 \dots i_m} x_{i_2} \dots x_{i_m} \Big)_{1 \leq i \leq n}, \tag{2.3}$$

and

$$\mathcal{A}x^{m-2} := \Big(\sum_{i_3, \dots, i_m = 1}^{n} a_{i j i_3 \dots i_m} x_{i_3} \dots x_{i_m} \Big)_{ij}, \qquad 1 \leq i \leq n, 1 \leq j \leq n. \tag{2.4}$$

It is easy to see that $\mathcal{A}x^m = x^T(\mathcal{A}x^{m-1}) = x^T(\mathcal{A}x^{m-2})x$.

Various definitions of real eigenpairs for tensors have been introduced in the literature, including H-eigenvalues [20], Z-eigenvalues [20], D-eigenvalues [21], and so on. In this paper, we devote ourselves to find a Z-eigenpair for any square tensors.

Definition 2.1. Let $\mathcal{A} \in \mathcal{R}^{[m,n]}$. $(\lambda, x) \in \mathcal{C} \times \mathcal{C}^n \setminus \{0\}$ is called an E-eigenpair of $\mathcal{A}$ if

$$\begin{cases} \mathcal{A}x^{m-1} = \lambda x \\ x^T x = 1. \end{cases} \tag{2.5}$$

We call (λ, x) a Z-eigenpair if both x and λ are real.

$\boxed{2.1}$ Properties of $f(x) = \mathcal{A}x^m$ and $F(x) = \mathcal{A}x^{m-1}$

Lemma 2.2. *Let $\mathcal{A} \in \mathcal{R}^{[m,n]}$, the gradient of $f(x) = \mathcal{A}x^m$ is*

$$\nabla_j f(x) = \sum_{i_1,\dots,i_m=1}^{n} \sum_{q=1}^{m} a_{i_1 i_2 \dots i_m} x_{i_1} \dots x_{i_{q-1}} \delta_{i_q,j} x_{i_{q+1}} \dots x_{i_m}, 1 \le j \le n. \tag{2.6}$$

If $\mathcal{A}$ is symmetric, then

$$\nabla_j f(x) = m(\mathcal{A}x^{m-1})_j, \tag{2.7}$$

hence, $\nabla f(x) = m\mathcal{A}x^{m-1}$. Moreover, for any square tensors, symmetric or not, we have

$$mf(x) = x^T \nabla f(x). \tag{2.8}$$

Proof. From the proof of Lemma 3.1 in [12], we have (2.6) and (2.7). Then,

$$\begin{aligned} x^T \nabla f(x) &= \sum_{j=1}^{n} x_j \nabla_j f(x) \\ &= \sum_{j=1}^{n} \sum_{i_1,\dots,i_m=1}^{n} \sum_{q=1}^{m} a_{i_1 \dots i_m} x_j \cdot x_{i_1} \dots x_{i_{q-1}} \delta_{i_q,j} x_{i_{q+1}} \dots x_{i_m} \\ &= \sum_{q=1}^{m} \sum_{j=1}^{n} \sum_{\{i_1,\dots i_m\}/i_q=1}^{n} a_{i_1 \dots i_{q-1} j i_{q+1} \dots i_m} (x_j \cdot x_{i_1} \dots x_{i_{q-1}} x_{i_{q+1}} \dots x_{i_m}) \\ &= mAx^m. \end{aligned}$$

□

Similarly, we can obtain the Jacobian matrix of $F(x)$ in the following.

Lemma 2.3. *Let $\mathcal{A} \in \mathcal{R}^{[m,n]}$, the Jacobian of $F(x) = \mathcal{A}x^{m-1}$ is*

$$[JF(x)]_{ij} = \sum_{i_2,\dots,i_m=1}^{n} \sum_{q=2}^{m} a_{i i_2 \dots i_m} x_{i_2} \dots x_{i_{q-1}} \delta_{i_q,j} x_{i_{q+1}} \dots x_{i_m}, 1 \le i,j \le n. \tag{2.9}$$

If $\mathcal{A}$ is symmetric, then

$$[JF(x)]_{ij} = (m-1)(\mathcal{A}x^{m-2})_{ij}, 1 \le i,j \le n, \tag{2.10}$$

hence, $JF(x) = (m-1)\mathcal{A}x^{m-2}$. Moreover, for any square tensors, symmetric or not, we have

$$(m-1)f(x) = x^T JF(x)x. \tag{2.11}$$

Proof. For any tensor $\mathcal{A}$, the mapping $F : \mathcal{R}^n \to \mathcal{R}^n$ is defined by

$$F(x) = \mathcal{A}x^{m-1} = \left(\sum_{i_2,\dots,i_m=1}^{n} a_{ii_2\dots i_m} x_{i_2} \dots x_{i_m} \right)_{1\le i\le n}. \tag{2.12}$$

Then the Jacobian matrix of the mapping F is as follows:

$$\begin{aligned}
[JF(x)]_{ij} &= \frac{\partial(\sum_{i_2,\dots,i_m=1}^{n} a_{ii_2\dots i_m} x_{i_2} \dots x_{i_m})}{\partial x_j} \\
&= \sum_{i_2,\dots,i_m=1}^{n} \frac{\partial(a_{ii_2\dots i_m} x_{i_2} \dots x_{i_m})}{\partial x_j} \\
&= \sum_{i_2,\dots,i_m=1}^{n} \sum_{q=2}^{m} a_{ii_2\dots i_m} x_{i_2} \dots x_{i_{q-1}} \delta_{i_q,j} x_{i_{q+1}} \dots x_{i_m}
\end{aligned}$$

When $\mathcal{A}$ is symmetric, its entries $a_{i_1 i_2 \dots i_m}$ are invariant under any permutation of their indices $\{i_1, i_2, \dots, i_m\}$, so it is easy to compute that

$$\begin{aligned}
[JF(x)]_{ij} &= \sum_{q=2}^{m} \sum_{i_2,\dots,i_m=1}^{n} a_{ii_2\dots i_m} x_{i_2} \dots x_{i_{q-1}} \delta_{i_q,j} x_{i_{q+1}} \dots x_{i_m} \\
&= \sum_{q=2}^{m} \sum_{\{i_2,\dots i_m\}/i_q=1}^{n} a_{ii_2\dots j\dots i_m} x_{i_2} \dots x_{i_{q-1}} x_{i_{q+1}} \dots x_{i_m} \\
&= \sum_{q=2}^{m} \sum_{i_3,\dots,i_m=1}^{n} a_{iji_3\dots i_m} x_{i_3} \dots x_{i_m} \\
&= (m-1)\mathcal{A}x^{m-2}.
\end{aligned}$$

Similar to the proof of (2.8), we get (2.11) by changing the order of the finite sum over j, q, and i_m. $\square$

For the symmetric tensor $\mathcal{A}$, Lim [17] observes the following result based on the result $\nabla \mathcal{A}x^m = m\mathcal{A}x^{m-1}$.

Lemma 2.4. *Any eigenpair (λ, x) of symmetric tensor $\mathcal{A}$ is a Karush-Kuhn-Tucker (KKT) point of the nonlinear optimization problem*

$$\max_{x^T x=1, x\in\mathcal{R}^n} \mathcal{A}x^m. \tag{2.13}$$

When the tensor is not symmetric, we do not have $\nabla \mathcal{A}x^m = m\mathcal{A}x^{m-1}$. So the Z-eigenvalue problem for asymmetric tensor cannot be solved by the nonlinear optimization problem. The above result can be extended to the weak symmetric tensor. A tensor $\mathcal{A}$ is called to be a *weak symmetric tensor* if $\nabla \mathcal{A}x^m = m\mathcal{A}x^{m-1}$ is satisfied.

In [2], D. Cartwright and B. Sturmfels studied the number of the E-eigenvalues of both general square tensors and symmetric square tensors.

Lemma 2.5. *The set of E-eigenvalues of a tensor is either finite or it consists of all complex numbers in the complement of a finite set.*

Lemma 2.6. *Every symmetric tensor $\mathcal{A}$ has at most $((m-1)n-1)/(m-2)$ distinct eigenvalues.*

2.2 A Shifted Term

The idea of adding a "shift" term has been proposed in the context of independent component analysis (ICA) by Regalia and Kofidis [22], and Erdogan [7]. It has also been used in [12] to assure the convex or the concave of the underlying function. In [18], Liu et al. also used this technique to assure that the underlying tensor is primitive. In this paper, we will add a "shift" term to guarantee the positive definition of matrix $JF(x)$ when $\|x\| = 1$. That means that our method works with a suitably modified function

$$\hat{F}(x) = F(x) + \alpha x. \tag{2.14}$$

If (λ, x) is a Z-eigenpair of $\hat{F}(x)$, then we have

$$\begin{cases} \hat{F}(x) = F(x) + \alpha x = \lambda x \\ x^T x = 1. \end{cases} \tag{2.15}$$

This means that $(\lambda - \alpha, x)$ is a Z-eigenpair of $F(x)$.

For a matrix, we have the following property.

Lemma 2.7. *For any matrix $M \in \mathcal{R}^{n\times n}$, there exists real numbers $a > 0$ and $b > 0$ such that $aA + I$ and $A + bI$ are positive definite.*

From the above lemma, we can choose a suitable α such that $J\hat{F}(x) = JF(x) + \alpha I$ is positive definite for all x satifying $\|x\| = 1$.

3 The Shifted Power Method(SPM) and its convergence

Based on Lemmas 2.4 and 2.6, Kolda and Mayo [12] proposed a shifted symmetric higher-order power method(SSHOPM) for computing a Z-eigenpair of symmetric tensors, and proved that it is convergent for symmetric tensors. In this section, we will use this method for computing a Z-eigenpair of general square tensors, and we call it as SPM.

Algorithm 3.1. Shifted Power Method (SPM)

Step 0. Choose $x^0 \in \mathcal{R}^n$ with $\|x^0\| = 1$. Let $\lambda_0 = \mathcal{A}(x^0)^m$. Set $k := 0$.

Step 1. If $\alpha \geq 0$, $\bar{x}^{k+1} = F(x^k) + \alpha x^k$; else $\bar{x}^{k+1} = -F(x^k) - \alpha x^k$.

Step 2. $x^{k+1} = \bar{x}^{k+1}/\|\bar{x}^{k+1}\|$, $\lambda^{k+1} = \mathcal{A}(x^{k+1})^m$ and $k := k + 1$; go to Step 1.

Lemma 3.2 ([12])**.** *Let $\mathcal{A} \in \mathcal{R}^{[m,n]}$ be symmetric. For $\alpha > \beta(\mathcal{A})$, where $\beta(\mathcal{A}) := (m - 1)\max_{x\in\Sigma} \rho(\mathcal{A}x^{m-2})$, the iterates $\{\lambda_k, x^k\}$ produced by Algorithm 3.1 satisfy the following properties.*

(a) *The sequence $\{\lambda_k\}$ is nondecreasing, and there exists λ_* such that $\lambda_k \to \lambda_*$.*

(b) *The sequence $\{x^k\}$ has an accumulation point.*

(c) *For every such accumulation point x^*, the pair (λ_*, x^*) is an eigenpair of $\mathcal{A}$.*

(d) *If $\mathcal{A}$ has finitely many real eigenvectors, then there exists x^* such that $x^k \to x^*$.*

It is easy to see that Algorithm 3.1 can be used to compute a Z-eigenvalue of general square tensors, including asymmetric tensors. However, the eigenvalue problem cannot be reformulated as a nonlinear convex optimization problem when tensor $\mathcal{A}$ is not symmetric. So we cannot obtain the convergence of Algorithm 3.1. It is nature to ask whether or not we can show the convergence of Algorithm 3.1 for general square tensors. In the following, we give a positive answer under an assumption that $\|x^{k+1} - x^k\| \to 0$.

Theorem 3.3. *Let $\mathcal{A} \in \mathcal{R}^{[m,n]}$ be a general square tensor. For any $\alpha \in \mathcal{R}$, if the iterates $\{\lambda_k, x^k\}$ produced by Algorithm 3.1 satisfy $\|x^{k+1} - x^k\| \to 0$, then: (1) the sequence $\{x^k\}$ has an accumulation point x^*; (2) for each accumulation point x^*, if we denote $\mathcal{A}(x^*)^m$ as λ_*, then the pair (λ_*, x^*) is a Z-eigenpair of $\mathcal{A}$; (3) If $\mathcal{A}$ has finitely many real eigenvectors, then there exists x^* such that $x^k \to x^*$.*

Proof. (1). Suppose that $\{x^k\}$ is an infinite sequence on a compact set $\Sigma = \{x \in \mathcal{R}^n | \|x\| = 1\}$. So there is an accumulation point $x^* \in \Sigma$ by the Bolzano-Weierstrass theorem.

(2). From Step 2 in Algorithm 3.1, we have

$$x^{k+1} = \hat{F}(x^k)/\|\hat{F}(x^k)\|.$$

By the continuity of F, every accumulation point x^* satisfies

$$x^* = \hat{F}(x^*)/\|\hat{F}(x^*)\|,$$

that is

$$\hat{F}(x^*) = F(x^*) + \alpha x^* = \|\hat{F}(x^*)\| x^* \quad \text{and} \quad \|x^*\| = 1.$$

So $(\|\hat{F}(x^*)\| - \alpha, x^*)$ is a Z-eigenpair of $\mathcal{A}$. According to the definition of λ_*, we have

$$\begin{aligned}
\lambda_* &:= \mathcal{A}(x^*)^m = \langle x^*, F(x^*)\rangle \\
&= \left\langle \frac{\hat{F}(x^*)}{\|\hat{F}(x^*)\|}, \hat{F}(x^*) - \alpha x^* \right\rangle \\
&= \|\hat{F}(x^*)\| - \alpha \frac{\langle \hat{F}(x^*), x^*\rangle}{\|\hat{F}(x^*)\|} \\
&= \|\hat{F}(x^*)\| - \alpha.
\end{aligned}$$

So (λ_*, x^*) is an eigenpair of $\mathcal{A}$.

(3). By the result of (1), there exists a subsequence $\{x^{k_i}\} \subset \{x^k\}$ such that $\|x^{k_i} - x^*\| \to 0$. Associated with the condition $\|x^{k+1} - x^k\| \to 0$ and the assumption that $\mathcal{A}$ has finitely many real eigenvectors, we know that the whole sequence $\{x^k\}$ is convergent to x^*. The proof is similar to the proof of Theorem 4.4(d) in [12]. □

The above theorem stated that the SSHOPM is convergent for general square tensors under the assumption of $\|x^{k+1} - x^k\| \to 0$. In next section, we will propose a fixed point algorithm and show that it is convergent for general square tensors without this assumption.

4 A Fixed Point Algorithm and its Convergence

First, we reformulate the Z-eigenvalue problem as a projection equation. Then we use a fixed point algorithm to solve this projection equation.

Lemma 4.1. *For any $\tau > 0$, suppose x^* is a solution of $x = \Pi_{\mathcal{B}}(x - \tau F(x))$, where $\mathcal{B} = \{s \in \mathcal{R}^n : \|s\| \leq 1\}$ is an unit ball and $\Pi_{\mathcal{B}}(s)$ is the projection operator onto $\mathcal{B}$ defined as:*

$$\Pi_{\mathcal{B}}(s) = \begin{cases} \frac{s}{\|s\|} & \text{if} \quad \|s\| > 1, \\ s & \text{if} \quad \|s\| \leq 1. \end{cases} \tag{4.1}$$

(i) *if $\|x^* - \tau F(x^*)\| > 1$, then (λ, x^*) is a Z-eigenpair of tensor $\mathcal{A}$, where $\lambda = (1 - \|x^* - \tau F(x^*)\|)/\tau$.*

(ii) *if $\|x^* - \tau F(x^*)\| \leq 1$ and $x^* \neq 0$, then $(0, \frac{x^*}{\|x^*\|})$ is a Z-eigenpair of tensor $\mathcal{A}$.*

Proof. (i) Suppose that $\|x^* - \tau F(x^*)\| > 1$. From the definition of $\Pi_{\mathcal{B}}$, we have $x^* = \frac{x^* - \tau F(x^*)}{\|x^* - \tau F(x^*)\|}$. Consequently,

$$\|x^*\| = 1$$

and

$$x^* \|x^* - \tau F(x^*)\| = x^* - \tau F(x^*).$$

According to Definition 2.1, we know that (λ, x^*) is a Z-eigenpair of tensor $\mathcal{A}$.

(ii) If $\|x^* - \tau F(x^*)\| \leq 1$, then $\Pi_{\mathcal{B}}(x^* - \tau F(x^*)) = x^* - \tau F(x^*)$. So we have $x^* = x^* - \tau F(x^*)$. Consequently, $F(x^*) = 0$. Associated with $x^* \neq 0$, it is easy to see that $(0, \frac{x^*}{\|x^*\|})$ is a Z-eigenpair of tensor $\mathcal{A}$. □

From Lemma 4.1, we present the following fixed point method to find a Z-eigenpair of tensor $\mathcal{A}$.

Algorithm 4.2. Fixed Point Method (FPM)

Step 0. Choose $x^0 \in \mathcal{B}$. Set $k := 0$.

Step 1. If $x^k = \Pi_{\mathcal{B}}(x^k - \tau F(x^k))$, stop.

Step 2. $x^{k+1} = \Pi_{\mathcal{B}}(x^k - \tau F(x^k))$ and $k := k + 1$; go to Step 1.

The following result gives sufficient conditions on the mapping F to ensure the convergence of the above algorithm.

Theorem 4.3. *Let $F : \mathcal{B} \to \mathcal{R}^n$, suppose L and μ are such that for any x and y in $\mathcal{B}$,*

$$(F(x) - F(y))^T (x - y) \geq \mu \|x - y\|_2^2 \tag{4.2}$$

and

$$\|F(x) - F(y)\|_2 \leq L \|x - y\|_2. \tag{4.3}$$

If

$$\tau \in \begin{cases} (0, \frac{\mu - \sqrt{\mu^2 - L^2}}{L^2}) \cup (\frac{\mu + \sqrt{\mu^2 - L^2}}{L^2}, \frac{2\mu}{L^2}), & \mu \geq L, \\ (0, \frac{2\mu}{L^2}), & \mu < L. \end{cases} \tag{4.4}$$

Then the mapping $\Pi_{\mathcal{B}}(x - \tau F(x))$ is a contraction from $\mathcal{B}$ to $\mathcal{B}$.

Proof. For any x and y in $\mathcal{B}$, since $\mathcal{B}$ is a closed convex set, we have

$$\begin{aligned}
& \|\Pi_{\mathcal{B}}(x-\tau F(x)) - \Pi_{\mathcal{B}}(y-\tau F(y))\|^2 \\
\leq\ & \|x-\tau F(x) - y + \tau F(y)\|^2 \\
\leq\ & \|x-y\|^2 + \tau^2\|F(x)-F(y)\|^2 - 2\tau(F(x)-F(y))^T(x-y) \\
\leq\ & (1+\tau^2L^2-2\tau\mu)\|x-y\|^2.
\end{aligned}$$

It is easy to see that $1+\tau^2L^2-2\tau\mu \in (0,1)$ when τ satisfies (4.4), so the mapping $\Pi_{\mathcal{B}}(x-\tau F(x))$ is contract. □

Next, we analyse that the strong monotonicity (4.2) and the Lipschitz continuity (4.3) can be satisfied for general tensors by adding a shifted term αx into the mapping $F(x)$.

Lemma 4.4. *For a general square tensor $\mathcal{A}$ and $\hat{F}(x) = \mathcal{A}x^{m-1} + \alpha x$, if $\alpha > \beta(\mathcal{A})$, where $\beta(\mathcal{A})$ is defined by*

$$\beta(\mathcal{A}) = (m-1)\sum_{i_1,i_2,\ldots,i_m=1}^{n} |a_{i_1i_2\ldots i_m}|. \tag{4.5}$$

Then $\hat{F}$ is strong monotone, i.e.

$$(\hat{F}(x)-\hat{F}(y))^T(x-y) \geq \mu\|x-y\|_2^2,$$

where $\mu = \alpha - \beta(\mathcal{A})$.

Proof. For any $x, y \in \mathcal{B}$, we have

$$\begin{aligned}
(\hat{F}(x)-\hat{F}(y))^T(x-y) &= (F(x)-F(y))^T(x-y) + \alpha\|x-y\|^2 \\
&\geq -\|F(x)-F(y)\|\cdot\|x-y\| + \alpha\|x-y\|^2 \\
&= -\|JF(\xi)(x-y)\|\cdot\|x-y\| + \alpha\|x-y\|^2 \\
&\geq -\|JF(\xi)\|\cdot\|x-y\|^2 + \alpha\|x-y\|^2 \\
&\geq (\alpha-\beta(\mathcal{A}))\|x-y\|^2,
\end{aligned}$$

where the first inequality comes from the Hölder inequality, $\xi = tx+(1-t)y \in \mathcal{B}$, $0<t<1$, and the last inequality is because of the definition of JF and $\beta(\mathcal{A})$. □

Lemma 4.5. *For a general square tensor $\mathcal{A}$ and $F(x) = \mathcal{A}x^{m-1}$, F is Lipschitz continuous, i.e.*

$$\|F(x)-F(y)\|_2 \leq L\|x-y\|_2.$$

where $L = \beta(\mathcal{A})$.

Proof. For any $x, y \in \mathcal{B}$, we have

$$\|F(x)-F(y)\|_2 = \|JF(\xi)(x-y)\|_2 \leq \beta(\mathcal{A})\|x-y\|_2,$$

where $\xi = tx+(1-t)y \in \mathcal{B}$, $0<t<1$, and the last inequality is because of the definition of JF and $\beta(\mathcal{A})$. □

From Lemma 4.5, we can easily obtain that $\hat{F}(x) = F(x) + \alpha x$ is Lipschitz continuous with the Lipschitz constant $L = \alpha + \beta(\mathcal{A})$. Then we find that the Lipschitz constant L is bigger than the strong monotonicity constant μ. Associated with Theorem 4.3, Lemma 4.4, and Lemma 4.5, we have the following convergence result for any square tensor.

Theorem 4.6. *For any square tensor $\mathcal{A}$, if $\alpha > \beta(\mathcal{A})$ and $\tau \in (0, \frac{2\mu}{L^2})$, then Algorithm 4.2 imposed on $\hat{F}(x) = Ax^{m-1} + \alpha x$ either finds a Z-eigenvector of tensor $\mathcal{A}$ or generates a sequence $\{x^k\}$. Moreover, the whole sequence $\{x^k\}$ is convergent to x^*. Suppose that $x^* \neq 0$, then x^* is a Z-eigenvector of $\mathcal{A}$.*

Proof. If the Algorithm 4.2 stopped at step 1, then there exists a $k_0 \in Z^+$ such that

$$x_{k_0} = \Pi_{\mathcal{B}}(x_{k_0} - \tau \hat{F}(x^{k_0})).$$

We consider two cases.

Case 1: $x^{k_0} \neq 0$. A Z-eigenpair of $\mathcal{A}$ is obtained from Lemma 4.1.

Case 2: $x^{k_0} = 0$. This means that $x^{k_0-1} \neq 0$, $x^{k_0-1} \in \mathcal{B}$ and

$$\Pi_{\mathcal{B}}(x^{k_0-1} - \tau \hat{F}(x^{k_0-1})) = 0.$$

From the definition of $\Pi_{\mathcal{B}}$, we have

$$x^{k_0-1} - \tau \hat{F}(x^{k_0-1}) = x^{k_0-1} - \tau(F(x^{k_0-1}) + \alpha x^{k_0-1}) = 0,$$

then it implies

$$F\Big(\frac{x^{k_0-1}}{\|x^{k_0-1}\|}\Big) = \|x^{k_0-1}\|^{1-m} F(x^{k_0-1}) = \|x^{k_0-1}\|^{2-m}\Big(\frac{1}{\tau} - \alpha\Big)\Big(\frac{x^{k_0-1}}{\|x^{k_0-1}\|}\Big).$$

So $(\|x^{k_0-1}\|^{2-m}(\frac{1}{\tau} - \alpha), \frac{x^{k_0-1}}{\|x^{k_0-1}\|})$ is a Z-eigenpair of $\mathcal{A}$.

If the Algorithm 4.2 did not stop at step 1, then an infinite sequence $\{x^k\}$ can be generated. From Theorem 4.3, Lemma 4.4, Lemma 4.5 and the Brouwer's Fixed-point theorem, we know that the whole sequence $\{x^k\}$ is convergent to x^*. By Lemma 4.1, x^* is a Z-eigenvector of tensor $\mathcal{A}$ when $x^* \neq 0$. □

5 Numerical Results

In this section, we are going to test the performance of Algorithm 4.2. All experiments are conducted using a PC with 2.33GHz CPU and 0.99G memory. The operating system is Windows XP and the implementation is done by using MATLAB 7.0.1.

The parameters used in this test were as follows:

$$MaxIter = 1000, \quad Tolx = 10^{-6}.$$

The stop criterion is

$$\|x^{k+1} - x^k\| < Tolx.$$

We use some randomly generated 3^{th} order n-dimensional tensors to test Algorithm 4.2 and SPM [12]. The nonnegative tensor $\mathcal{A}$ is generated by $\mathcal{A} = \text{rand}(n, n, n)$. The general tensor $\mathcal{A}$ is generated by $\mathcal{B} = \text{rand}(n, n, n)$ and $\mathcal{C} = 1/2*\ \text{ones}(n, n, n)$; then $\mathcal{A} = \mathcal{B} + \mathcal{C}$.

Two groups of experiments were done by us: First, comparing the SPM with Algorithm 4.2 for randomly generated nonnegative tensors from a randomly generated start point. Second, testing the successful times of Algorithm 4.2 for one randomly generated tensor from 100 randomly generated start point.

The results of the first experiment are shown in Tables 1-2, where **CPU Time** and **Accuracy** denote the averages of 100 trials of CPU times in seconds and accuracy. Accuracy is the final value of $\|F(x)-\lambda x\|$. The results of the second experiment are shown in Table 3, where **kk** denotes the number of tests to find a Z-eigenpair of $\mathcal{A}$ and **ko** denotes the number of tests in which the convergent point is $x^*=0$.

The parameters used in the first experiment were $\alpha = 1, \tau = 2$ and in the second experiment were $\alpha = 3, \tau = 1$.

Table 1: Numerical results of Algorithm 4.2 and SPM for small size problems

Problem		Algorithm 2		SPM	
n	m	CPU Time	Accuracy	CPU Time	Accuracy
10	3	1.54e-002	4.69e-007	1.92e-002	5.46e-007
15	3	1.50e-002	6.18e-007	1.93e-002	5.70e-007
20	3	1.86e-002	4.37e-007	2.26e-002	2.65e-007
25	3	2.11e-002	2.08e-007	2.26e-002	7.30e-007
30	3	2.26e-002	1.23e-007	2.40e-002	6.26e-007
35	3	2.06e-002	3.30e-007	1.97e-002	3.62e-007
40	3	2.34e-002	8.68e-007	3.09e-002	1.90e-007
50	3	3.39e-002	7.44e-007	3.82e-002	9.27e-008

Table 2: Numerical results of Algorithm 4.2 and SPM for big size problems

Problem		Algorithm 2		SPM	
n	m	CPU Time	Accuracy	CPU Time	Accuracy
60	3	4.59e-002	5.52e-007	5.29e-002	8.74e-008
80	3	9.89e-002	2.61e-007	9.83e-002	8.74e-007
100	3	1.84e-001	1.54e-007	1.81e-001	5.92e-007
120	3	2.97e-001	9.90e-008	2.97e-001	3.66e-007
140	3	4.61e-001	6.80e-008	4.62e-001	2.32e-007
160	3	1.14e+000	5.03e-008	1.14e+000	1.67e-007
180	3	9.65e-001	3.88e-008	9.70e-001	1.24e-007
200	3	1.29e+000	3.37e-008	1.29e+000	9.09e-008

Table 3: Numerical results of Algorithm 4.2

Problem		Algorithm 2	
n	m	kk	ko
3	3	99	0
5	3	99	0
10	3	83	12

From the numerical results given in Tables 1-3, we see that the efficiency of Algorithm 4.2 is almost the same as that of SPM, including the CPU time and accuracy. However, we know that the convergence result of Algorithm 4.2 can be obtained under a weaker condition than that of SPM; See Theorem 3.1 and Theorem 4.3.

It is worth to note that although the reformulation of Z-eigenpair problem, used in Algorithm 4.2, enlarge the feasible region of the original problem, numerical results for nonnegative asymmetric tensors show that we can obtain the solution of the original problem. For the general square asymmetric tensors, Algorithm 4.2 can find a Z-eigenpair of $\mathcal{A}$ in most cases.

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