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LOWER SEMICONTINUITY OF APPROXIMATE SOLUTION MAPPING TO PARAMETRIC SET-VALUED WEAK VECTOR EQUILIBRIUM PROBLEMS*

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Abstract: In this paper, some stability results for parametric set-valued weak vector equilibrium problems are obtained. Under suitable assumptions, we establish the lower semicontinuity of the approximate solution mapping to a class of parametric set-valued weak vector equilibrium problem by using the scalarization method. These results extend and improve the corresponding ones in the literature. Some examples are given to illustrate the conclusions.

Key words: *lower semicontinuity, scalarization, parametric set-valued weak vector equilibrium problem, approximate solution mapping.*

Mathematics Subject Classification: *49K40, 90C29, 90C31.*

1 Introduction

We all know that the vector equilibrium problem is a unified model of several problems, such as vector optimization problems, vector variational inequalities, Nash economic equilibrium problems, variational inclusion problems, complementarity problems, and so on (see [11,14]). Existence results for various types of (generalized) vector equilibrium problems have been investigated intensively, e.g., see [13–15, 21, 23, 27] and the references therein.

The stability analysis of solution mappings for vector equilibrium problems and vector variational inequalities is another important topic in optimization theory and applications. Gong [16] extended the scalarization results from vector optimization problems to vector equilibrium problems and discussed efficiency and Henig efficiency for vector equilibrium problems. Cheng and Zhu [10] obtained a lower semicontinuity result of the solution mapping for a weak vector variational inequality in finite dimensional spaces by using a scalarization method. Huang et al. [20] obtained the upper semicontinuity and lower semicontinuity of

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the solution mapping for parametric implicit vector equilibrium problems. By using the idea of Cheng and Zhu [10], Gong [17] established the continuity of the solution mapping to parametric weak vector equilibrium problems with vector-valued mappings. Based on a scalarization representation of the solution mapping and a property involving the union of a family of lower semicontinuous set-valued mappings, Chen et al. [9] discussed the lower semicontinuity and continuity of the solution mapping to a parametric generalized vector equilibrium problem involving set-valued mappings. Hou et al. [21] obtained results on the existence and semicontinuity of solutions for generalized Ky Fan inequality problems with trifunctions. Chen and Li [8] studied the continuity of various efficient solution mappings for a parametric generalized vector equilibrium problem without the uniform compactness assumption and improved the results of [17, 18]. Under new assumptions, Peng et al. [30] studied the lower semicontinuity of solution mappings to two classes of parametric weak generalized Ky Fan inequalities with set-valued mappings in Hausdorff topological vector spaces. Recently, Chen and Huang [6] established the continuity of solution mappings to the two kinds of parametric generalized vector equilibrium problems under suitable assumptions. Very recently, Wang et al. [31, 32] obtained the lower semicontinuity of the solution mapping to a parametric generalized vector equilibrium problem and the continuity of the solution mapping to a parametric generalized strong vector equilibrium problem, respectively.

On the other hand, in recent years, many reseachers have been interested in approximate solutions of optimization (or equilibrium) problems. There are several important reasons for considering this kind of solutions. One of them is that exact solutions of the problems may not exist in many practical problems, but approximate solutions of problems can be computed by using iterative algorithms or heuristic methods. Khanh and Luu [22] established the semicontinuity of solution mappings and approximate solution mappings of parametric multivalued quasivariational inequalities in topological vector spaces. Anh and Khanh [1] considered two kinds of approximate solutions and approximate solution mappings to multivalued quasiequilibrium problems, and established the sufficient conditions for their Hausdorff semicontinuity (semicontinuity). By using a scalarization method, Li and Li [26] discussed the continuity of a approximate solution mapping for a parametric vector equilibrium problem, which is different from the corresponding ones in [1, 22]. Chen et al. [7] proved the connectedness results of ϵ-weak efficient and ϵ-efficient solutions mappings for vector equilibrium problems under some suitable conditions. Recently, Qiu and Yang [29] obtained some scalar characterization of approximate weakly efficient solutions and approximate Henig efficient solutions for vector equilibrium problems.

Motivated and inspired by the research works mentioned above, in this paper, some stability results for parametric set-valued weak vector equilibrium problems are discussed. Under suitable assumptions, which do not contain any information about solution mappings, we establish the lower semicontinuity of the approximate solution mapping to a class of parametric set-valued weak vector equilibrium problem by using a scalarization technique. These results extend and improve the corresponding ones in the literature ([6, 17, 30–32]). Some examples are given to illustrate the conclusions.

2 Preliminaries

Throughout this paper, unless specified otherwise, let X be a real Hausdorff topological vector space, Y be a real locally convex Hausdorff topological vector space and Y^* be the topological dual space of Y, Z be a topological space. Let C be a closed, convex and pointed cone in Y with nonempty interior int$C$.

Let

$$C^* := \{f \in Y^* : f(y) \geq 0,\ \forall y \in C\}$$

be the dual cone of C. Denote the quasi-interior of C^* by $C^\sharp$, i.e.,

$$C^\sharp := \{f \in Y^* : f(y) > 0,\ \forall y \in C \setminus \{0\}\}.$$

It is easy to see that $C^\sharp \neq \emptyset$ if and only if C has a base.

Let $e \in \mathrm{int}C$ be fixed and let

$$B_e^* = \{f \in C^* \setminus \{0\} : f(e) = 1\},$$

then B_e^* is a weak* compact base of C^*.

Let the set A be a nonempty subset of X, and $F : A \times A \to 2^Y$ be a set-valued mapping. When the mapping F and A are perturbed by a parameter μ which varies over a subset Λ of Z, we can consider the following parametric set-valued weak vector equilibrium problem of finding $x \in A(\mu)$ such that

$$\text{(PSWVEP)} \qquad F(x, y, \mu) \cap (-\mathrm{int}\, C) = \emptyset, \quad \forall y \in A(\mu),$$

where $A : \Lambda \to 2^X$ is a set-valued mapping with nonempty values, $F : B \times B \times \Lambda \subset X \times X \times Z \to 2^Y$ is a set-valued mapping with $A(\Lambda) = \bigcup_{\mu \in \Lambda} A(\mu) \subset B$.

For each $\mu \in \Lambda, \epsilon \geq 0$, let $V^W(\mu), V_\epsilon^W(\mu)$ denote the solution set and the approximate solution set of (PSWVEP), respectively, i.e.,

$$V^W(\mu) = \{x \in A(\mu) : F(x, y, \mu) \cap (-\mathrm{int}\, C) = \emptyset, \forall y \in A(\mu)\}$$

and

$$V_\epsilon^W(\mu) = \{x \in A(\mu) : (F(x, y, \mu) + \epsilon e) \cap (-\mathrm{int}\, C) = \emptyset, \forall y \in A(\mu)\},$$

where $e \in \mathrm{int}C$ be a fixed element.

For each $f \in C^* \setminus \{0\}$ and for each $\mu \in \Lambda$, the f-solution set of (PSWVEP) is defined by

$$V^f(\mu) := \{x \in A(\mu) : \inf_{z \in F(x,y,\mu)} f(z) \geq 0, \forall y \in A(\mu)\}.$$

For each $f \in B_e^*$ and for each $\mu \in \Lambda$, the approximate f-solution set of (PSWVEP) is defined by

$$V_\epsilon^f(\mu) := \{x \in A(\mu) : \inf_{z \in F(x,y,\mu)} f(z) + \epsilon \geq 0, \forall y \in A(\mu)\}.$$

Throughout this paper, we always assume $V^f(\mu) \neq \emptyset$ for each $f \in C^* \setminus \{0\}$, $\mu \in \Lambda$ and $V_\epsilon^f(\mu) \neq \emptyset$ for each $f \in B_e^*$, $\mu \in \Lambda$. In this paper, we will investigate the lower semicontinuity of the approximate solution mapping $V_\epsilon^W(\mu)$ to (PSWVEP). Now we recall some basic definitions and their properties which are needed in the following sections.

Definition 2.1. Let D be a nonempty convex subset of X, $G : D \to 2^Y$ be a set-valued mapping.

(i) G is called C-convex on D, if for any $x_1, x_2 \in D$ and $t \in [0, 1]$, $tG(x_1) + (1-t)G(x_2) \subset G(tx_1 + (1-t)x_2) + C$.

(ii) G is called C-convexlike on D, if for any $x_1, x_2 \in D$ and $t \in [0, 1]$, there exists $x_3 \in D$ such that $tG(x_1) + (1-t)G(x_2) \subset G(x_3) + C$.

Definition 2.2. (i) The mapping $G : D \times D \to 2^Y$ is said to be C-monotone on $D \times D$ if, for all $x, y \in D$, $G(x, y) + G(y, x) \subset -C$.

(ii) The mapping $G : D \times D \to 2^Y$ is said to be C-strictly monotone on $D \times D$, if G is C-monotone on $D \times D$, and for any $x, y \in D$ with $x \neq y$, one has $G(x, y) + G(y, x) \subset -\text{int}C$.

Definition 2.3 ([3])**.** Let X and Y be topological spaces, $G : X \to 2^Y$ be a set-valued mapping.

(i) G is said to be upper semicontinuous (u.s.c, for short) at $x_0 \in X$, if for any open set V with $G(x_0) \subset V$, there exists a neighborhood U of x_0 in X such that $G(x) \subset V$ for all $x \in U$.

(ii) G is said to be lower semicontinuous (l.s.c, for short) at $x_0 \in X$, if for any open set V with $G(x_0) \cap V \neq \emptyset$, there exists a neighborhood U of x_0 in X such that $G(x) \cap V \neq \emptyset$ for all $x \in U$.

(iii) G is said to be continuous at $x_0 \in X$, if it is both l.s.c and u.s.c at $x_0 \in X$. G is said to be l.s.c (resp. u.s.c) on X, iff it is l.s.c (resp. u.s.c) at each $x \in X$.

Definition 2.4 ([19])**.** Let X and Y be topological vector spaces, $G : X \to 2^Y$ be a set-valued mapping.

(i) G is said to be C-upper semicontinuous at x_0, if for any neighborhood U of 0 in Y, there exists a neighborhood $U(x_0)$ of x_0 such that

$$G(x) \subset G(x_0) + U + C, \ \forall x \in U(x_0) \cap X.$$

(ii) G is said to be C-lower semicontinuous at x_0, if for each $z \in G(x_0)$, and any neighborhood U of 0 in Y, there exists a neighborhood $U(x_0)$ of x_0 such that

$$G(x) \cap (z + U - C) \neq \emptyset, \ \forall x \in U(x_0) \cap X.$$

The following example is given to show that G is C-lower semicontinuous, while it is not necessarily upper semicontinuous.

Example 2.5. Let $X = \mathbb{R}, C = \mathbb{R}_+$. $G : \mathbb{R} \to 2^{\mathbb{R}}$ is defined by

$$G(x) = \begin{cases} \{0\}, & \text{if} \quad x = 0, \\ [-1, 1], & \text{if} \quad x \neq 0. \end{cases}$$

Then, G is C-lower semicontinuous at $x = 0$, but G is not upper semicontinuous at $x = 0$.

Definition 2.6 ([33])**.** A set-valued mapping G is said to be nearly C-subconvexlike on A, if cl cone$(G(A) + C)$ is a convex set.

Remark 2.7. It is clear that $G(A) + C$ is a convex set implies that cl cone$(G(A) + C)$ is a convex set, but it follows from Example 3.1 of [33] that the converse is not true.

From [3, 12], we have the following properties for Definition 2.3.

Proposition 2.8. *Let X and Y be topological vector spaces, $G : X \to 2^Y$ be a set-valued mapping.*

(i) *G is l.s.c at $x_0 \in X$ if and only if for any net $\{x_\alpha\} \subset X$ with $x_\alpha \to x_0$ and any $y_0 \in G(x_0)$, there exists $y_\alpha \in G(x_\alpha)$ such that $y_\alpha \to y_0$.*

(ii) *If G has compact values (i.e., $G(x)$ is a compact set for each $x \in X$), then G is u.s.c at x_0 if and only if for any net $\{x_\alpha\} \subset X$ with $x_\alpha \to x_0$ and for any $y_\alpha \in G(x_\alpha)$, there exist $y_0 \in G(x_0)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y_0$.*

Lemma 2.9 ([4])**.** *The union $\Gamma = \bigcup_{i\in I} \Gamma_i$ of a family of l.s.c set-valued mappings Γ_i from a topological space X into a topological space Y is also a l.s.c set-valued mapping from X into Y, where I is an index set.*

Lemma 2.10 ([29])**.** *Let $C \subset Y$ be a convex cone with $intC \neq \emptyset$. Then*

$$intC = \{y \in Y : f(y) > 0, \forall f \in C^* \setminus \{0\}\}.$$

3 Lower Semicontinuity of Approximate Solution Mapping for (PSWVEP)

In this section, we mainly discuss the lower semicontinuity of the approximate solution mapping to (PSWVEP). To obtain the lower semicontinuity of the approximate solution mapping V_ϵ^W, we introduce the following assumption (S) :

Let $\epsilon \geq 0, f \in B_e^*$ and let $F : A(\Lambda) \times A(\Lambda) \times \Lambda \to 2^Y$ be a set-valued mapping. $\inf_{z\in F(x,y,\mu)} f(z) + \epsilon = 0$ implies that $x = y$.

The following example is given to illustrate the assumption (S).

Example 3.1. Let $X = Z = \mathbb{R}, Y = \mathbb{R}^2, C = \mathbb{R}^2_+, \Lambda = [1, 2], \epsilon = \frac{1}{12}, e = (1, 1), f = (1, 0)$. Let $A : \Lambda \to 2^X$ be a set-valued mapping defined by $A(\mu) = [\frac{1}{2}, 3]$ and let $F : X \times X \times \Lambda \to 2^Y$ defined by

$$F(x, y, \mu) = \left[x^2 - y^2 - x + y - \frac{1}{12}, 30\right] \times [(\mu^2 + 1)(x^2 + y^2 - 2), +\infty).$$

Then, $\inf_{z\in F(x,y,\mu)} f(z) + \epsilon = 0$ implies that $x = y$. So the assumption (S) holds.

Lemma 3.2. *Let $f \in B_e^*, \epsilon \geq 0$. Suppose that the following conditions are satisfied:*

(i) *$A(\cdot)$ is continuous with compact values on Λ;*

(ii) *$F(\cdot, \cdot, \cdot)$ is lower semicontinuous on $B \times B \times \Lambda$;*

(iii) *For any given $\mu \in \Lambda$,*

$$(\inf_{z\in F(x,y,\mu)} f(z) + \epsilon)(\inf_{z\in F(y,x,\mu)} f(z) + \epsilon) \leq 0, \forall x, y \in A(\mu);$$

(iv) *The assumption (S) holds for f.*

Then, $V_\cdot^f(\cdot)$ is l.s.c on $\epsilon \times \Lambda$.

Proof. Suppose that there exists $\mu_0 \in \Lambda$ such that $V_\cdot^f(\cdot)$ is not l.s.c at (ϵ, μ_0). Then, there exist a net $\{\mu_\alpha\}$ with $\mu_\alpha \to \mu_0, \epsilon_\alpha \to \epsilon$ with $\epsilon_\alpha \in [0, +\infty)$ and $x_0 \in V_\epsilon^f(\mu_0)$ such that for any $x_\alpha \in V_{\epsilon_\alpha}^f(\mu_\alpha), x_\alpha \not\to x_0$.

Since $x_0 \in A(\mu_0)$ and $A(\cdot)$ is l.s.c at μ_0, there exists $\{\bar{x}_\alpha\}$ with $\bar{x}_\alpha \in A(\mu_\alpha)$ such that

$$\bar{x}_\alpha \to x_0.$$

For any $y_\alpha \in V^f_{\epsilon_\alpha}(\mu_\alpha) \subset A(\mu_\alpha)$, since $A(\cdot)$ is u.s.c with compact values at μ_0, there exist $y_0 \in A(\mu_0)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that

$$y_\beta \to y_0.$$

It follows from $y_\beta \in V^f_{\epsilon_\beta}(\mu_\beta)$ and $\bar{x}_\beta \in A(\mu_\beta)$ that

$$\inf_{z\in F(y_\beta,\bar{x}_\beta,\mu_\beta)} f(z) + \epsilon_\beta \geq 0. \tag{3.1}$$

Since $F(\cdot,\cdot,\cdot)$ is lower semicontinuous at (y_0, x_0, μ_0), then for any given $z \in F(y_0, x_0, \mu_0)$, there exists $z_\beta \in F(y_\beta, \bar{x}_\beta, \mu_\beta)$ such that $z_\beta \to z$. Thus, by the continuity of f, we have

$$f(z_\beta) + \epsilon_\beta \to f(z) + \epsilon.$$

This together with (3.1) and the arbitrariness of $z \in F(y_0, x_0, \mu_0)$ yields

$$\inf_{z\in F(y_0,x_0,\mu_0)} f(z) + \epsilon \geq 0. \tag{3.2}$$

It follows from $x_0 \in V^f_\epsilon(\mu_0)$ and $y_0 \in A(\mu_0)$ that

$$\inf_{z\in F(x_0,y_0,\mu_0)} f(z) + \epsilon \geq 0. \tag{3.3}$$

If $\inf_{z\in F(x_0,y_0,\mu_0)} f(z) + \epsilon = 0$. Then, by assumption (iv), we have

$$x_0 = y_0.$$

If $\inf_{z\in F(x_0,y_0,\mu_0)} f(z) + \epsilon > 0$. Then, by assumption (iii), we have

$$\inf_{z\in F(y_0,x_0,\mu_0)} f(z) + \epsilon \leq 0. \tag{3.4}$$

From (3.2) and (3.4), one has

$$\inf_{z\in F(y_0,x_0,\mu_0)} f(z) + \epsilon = 0.$$

By assumption (iv) again , we can get $x_0 = y_0$. This is impossible by the contradiction assumption, and the proof is complete. □

Remark 3.3. In Lemma 3.2, we establish the lower semicontinuity of the approximate f-solution mapping for parametric set-valued weak vector equilibrium problem. Obviously, the result extends and improves Lemma 4.2 of [17] (the strict monotone is required) and Lemma 3.1 of [6] (the information of the solution mapping is required). The following example is given to illustrate this case.

Example 3.4. Let $X = Z = \mathbb{R}, Y = \mathbb{R}^2, C = \mathbb{R}^2_+, \Lambda = [1, 2], f = (0, 1), \epsilon = \frac{1}{2}, e = (1, 1)$. Let $A : \Lambda \to 2^X$ defined by $A(\mu) = [\frac{1}{2}, 2]$ and let $F : X \times X \times \Lambda \to 2^Y$ defined by

$$F(x, y, \mu) = [x(y - x)^2(x^2 - 2\mu - 5), 18] \times \Big[3x(y - x) - \frac{1}{2}, 20\Big].$$

Obviously, all conditions of Lemma 3.2 are satisfied. It follows from a direct computation that $V^f_\epsilon(\mu) = \{\frac{1}{2}\}, \forall \mu \in \Lambda$. Hence, $V^f_\cdot(\cdot)$ is l.s.c on $\epsilon \times \Lambda$.

However, $F(\cdot,\cdot,\mu)$ is not C-strictly monotone on $A(\mu)\times A(\mu)$ for any given $\mu\in\Lambda$. Thus, [9, Lemma 3.2] and [17, Lemma 4.2] are not applicable here. Moreover, the information of solution mapping is not needed, so Lemma 3.1 of [6] (Therem 3.3 of [30]) is not applicable here.

When $\epsilon=0$, we can get the following result.

Corollary 3.5. *Let $f\in C^*\setminus\{0\}$. Suppose that the following conditions are satisfied:*

(i) $A(\cdot)$ *is continuous with compact values on* Λ;

(ii) $F(\cdot,\cdot,\cdot)$ *is C-lower semicontinuous on* $B\times B\times\Lambda$;

(iii) *For any given* $\mu\in\Lambda$,

$$(\inf_{z\in F(x,y,\mu)} f(z))(\inf_{z\in F(y,x,\mu)} f(z))\leq 0,\forall x,y\in A(\mu);$$

(iv) *The assumption* (S) *holds for* f.

Then, $V^f(\cdot)$ *is l.s.c on* Λ.

Proof. Suppose that there exists $\mu_0\in\Lambda$ such that $V^f(\cdot)$ is not l.s.c at μ_0. Then, there exist a net $\{\mu_\alpha\}$ with $\mu_\alpha\to\mu_0$ and $x_0\in V^f(\mu_0)$ such that for any $x_\alpha\in V^f(\mu_\alpha), x_\alpha\not\to x_0$.

Since $x_0\in A(\mu_0)$ and $A(\cdot)$ is l.s.c at μ_0, there exists $\{\bar{x}_\alpha\}$ with $\bar{x}_\alpha\in A(\mu_\alpha)$ such that

$$\bar{x}_\alpha\to x_0.$$

For any $y_\alpha\in V^f(\mu_\alpha)\subset A(\mu_\alpha)$, since $A(\cdot)$ is u.s.c with compact values at μ_0, there exist $y_0\in A(\mu_0)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that

$$y_\beta\to y_0.$$

It follows from $y_\beta\in V^f(\mu_\beta)$ and $\bar{x}_\beta\in A(\mu_\beta)$ that

$$\inf_{z\in F(y_\beta,\bar{x}_\beta,\mu_\beta)} f(z)\geq 0. \tag{3.5}$$

Now, we claim that $\inf_{z\in F(y_0,x_0,\mu_0)} f(z)\geq 0$. If not, there exists $z_0\in F(y_0,x_0,\mu_0)$ such that $f(z_0)<0$. Set $U=\{z\in Y:|f(z)|<\frac{1}{2}|f(z_0)|\}$. It is clear that U is a neighborhood of 0 in Y. By assumption (ii), for above U, there exists a neighborhood $U(y_0,x_0,\mu_0)$ of (y_0,x_0,μ_0) in $A(\Lambda)\times A(\Lambda)\times\Lambda$ such that

$$F(y',x',\mu')\cap(z_0+U-C)\neq\emptyset,\forall(y',x',\mu')\in U(y_0,x_0,\mu_0). \tag{3.6}$$

It follows from $(y_\beta,\bar{x}_\beta,\mu_\beta)\to(y_0,x_0,\mu_0)$ and (3.6) that there exists some β_0, such that

$$F(y_\beta,\bar{x}_\beta,\mu_\beta)\cap(z_0+U-C)\neq\emptyset,\forall\beta\geq\beta_0. \tag{3.7}$$

Therefore, there exists $z_\beta\in F(y_\beta,\bar{x}_\beta,\mu_\beta)$ such that $z_\beta\in z_0+U-C$, i.e., there exist $u_\beta\in U$ and $c_\beta\in C$ such that $z_\beta=z_0+u_\beta-c_\beta$. Thus,

$$f(z_\beta)=f(z_0+u_\beta-c_\beta)<f(z_0)+\frac{1}{2}|f(z_0)|-f(c_\beta)\leq\frac{1}{2}f(z_0)<0.$$

However, by (3.5), we have

$$f(z_\beta) \geq \inf_{z \in F(y_\beta, \bar{x}_\beta, \mu_\beta)} f(z) \geq 0,$$

which is a contradiction. Hence, we have

$$\inf_{z \in F(y_0, x_0, \mu_0)} f(z) \geq 0. \tag{3.8}$$

Then, the rest of the proof is similar to Lemma 3.2. Hence, we can obtain the result. □

Lemma 3.6. *Let $f \in B_e^*, \epsilon \geq 0$. Suppose that the following conditions are satisfied:*

(i) *For any given $\mu \in \Lambda$,*

$$(\inf_{z \in F(x,y,\mu)} f(z) + \epsilon)(\inf_{z \in F(y,x,\mu)} f(z) + \epsilon) \leq 0, \forall x, y \in A(\mu);$$

(ii) *The assumption (S) holds for f.*

Then, $V_\cdot^f(\cdot)$ is a singleton on $\epsilon \times \Lambda$.

Proof. By virtue of the assumption and the previous proof idea, we can easily get the conclusion. □

Lemma 3.7. *Let $\epsilon \geq 0$. Suppose that for each $\mu \in \Lambda, x \in A(\mu), F(x, \cdot, \mu) + \epsilon e$ is nearly C-subconvexlike on $A(\mu)$. Then,*

$$V_\epsilon^W(\mu) = \bigcup_{f \in B_e^*} V_\epsilon^f(\mu).$$

Proof. (i) Let $x \in \bigcup_{f \in B_e^*} V_\epsilon^f(\mu)$. Then there exists $f' \in B_e^*$ such that $x \in V_\epsilon^{f'}(\mu)$. Hence, $x \in A(\mu)$ and $\inf_{z \in F(x,y,\mu)} f'(z) + \epsilon \geq 0, \forall y \in A(\mu)$, i.e., for any $y \in A(\mu), z \in F(x, y, \mu), f'(z) + \epsilon \geq 0$, which deduces that $z \notin -\epsilon e - \text{int}C$. Thus, by the arbitrariness of z, we obtain $F(x, y, \mu) \cap (-\epsilon e - \text{int}C) = \emptyset, \forall y \in A(\mu)$. Hence, $x \in V_\epsilon^W(\mu)$.

(ii) Let $x \in V_\epsilon^W(\mu)$. Then $x \in A(\mu)$ and $F(x, y, \mu) \cap (-\epsilon e - \text{int}C) = \emptyset, \forall y \in A(\mu)$. Thus,

$$(F(x, A(\mu), \mu) + C) \cap (-\epsilon e - \text{int}C) = \emptyset,$$

i.e.,

$$(F(x, A(\mu), \mu) + C + \epsilon e) \cap (-\text{int}C) = \emptyset.$$

Hence, we have

$$\text{cl cone}(F(x, A(\mu), \mu) + C + \epsilon e) \cap (-\text{int}C) = \emptyset. \tag{3.9}$$

Since for each $\mu \in \Lambda, x \in A(\mu), F(x, \cdot, \mu) + \epsilon e$ is nearly C-subconvexlike on $A(\mu)$. Then, cl cone$(F(x, A(\mu), \mu) + C + \epsilon e)$ is a convex set, by (3.9) and separation theorem, there exist a continuous linear functional $g \in Y^* \setminus \{0\}$ and a real number r such that

$$g(\hat{c}) < r \leq g(z + c + \epsilon e),$$

for all $\hat{c} \in -\text{int}C, z \in F(x, A(\mu), \mu)$ and $c \in C$. Since C is a cone and $\hat{c} \in -\text{int}C$ can be taken arbitrarily close to $0 \in Y$. By the continuity of g, we have $r \geq 0$. Therefore, from Lemma 2.10, we have $g \in C^* \setminus \{0\}$.

Since $0 \in C$, then

$$z + \epsilon e \in F(x, A(\mu), \mu) + \epsilon e + C, \ \forall z \in F(x, A(\mu), \mu).$$

Thus, we get

$$g(z) + g(\epsilon e) = g(z) + \epsilon g(e) \geq 0, \ \forall z \in F(x, A(\mu), \mu).$$

Since $e \in \text{int}C$ and $g \in C^* \setminus \{0\}$, then $g(e) > 0$. Let $f = \frac{g}{g(e)}$, we have $f \in B_e^*$ and

$$f(z) + \epsilon f(e) = f(z) + \epsilon \geq 0, \ \forall z \in F(x, A(\mu), \mu),$$

i.e.,

$$\inf_{z \in F(x,y,\mu)} f(z) + \epsilon \geq 0, \ \forall y \in A(\mu),$$

which implies that $x \in V_\epsilon^f(\mu) \subset \bigcup_{f \in B_e^*} V_\epsilon^f(\mu)$. □

Remark 3.8. (i) When $\epsilon = 0$, Lemma 3.7 collapses to Lemma 3.2 in [31]. (ii) Because the assumption of nearly C-subconvexlikeness is weaker than C-convexlikeness (and C-convexity), Lemma 3.7 also improves Lemma 3.1 in [9] (Lemma 3.1 (ii) of [30], Lemma 2.3 of [6]).

Now, we establish the lower semicontinuity of the approximate solution mapping $V_\epsilon^W(\cdot)$ to (PSWVEP).

Theorem 3.9. *Let $f \in B_e^*, \epsilon \geq 0$. Suppose that the following conditions are satisfied:*

(i) $A(\cdot)$ *is continuous with compact values on* Λ;

(ii) $F(\cdot, \cdot, \cdot)$ *is lower semicontinuous on* $B \times B \times \Lambda$;

(iii) *For any given* $\mu \in \Lambda$,

$$(\inf_{z \in F(x,y,\mu)} f(z) + \epsilon)(\inf_{z \in F(y,x,\mu)} f(z) + \epsilon) \leq 0, \forall x, y \in A(\mu);$$

(iv) *For each* $\mu \in \Lambda$ *and for each* $x \in A(\mu), F(x, \cdot, \mu) + \epsilon e$ *is nearly* C*-subconvexlike on* $A(\mu)$;

(v) *The assumption* (S) *holds for* f.

Then, $V_\cdot^W(\cdot)$ *is lower semicontinuous on* $\epsilon \times \Lambda$.

Proof. Since for each $\mu \in \Lambda$ and for each $x \in A(\mu), F(x, \cdot, \mu) + \epsilon e$ is nearly C-subconvexlike on $A(\mu)$, it follows from Lemma 3.7 that

$$V_\epsilon^W(\mu) = \bigcup_{f \in B_e^*} V_\epsilon^f(\mu).$$

From Lemma 3.2, we know that for each $f \in B_e^*, V_\cdot^f(\cdot)$ is l.s.c on $\epsilon \times \Lambda$. Thus, by virtue of Lemma 2.9, we obtain that $V_\cdot^W(\cdot)$ is l.s.c on $\epsilon \times \Lambda$. □

Remark 3.10. Our main result Theorem 3.9 is different from the ones in [31, 32] in the following two aspects:

(i) We extend the lower semicontinuity of the solution mapping of (PSWVEP) to the lower semicontinuity of the approximate solution mapping of (PSWVEP);

(ii) We use the assumption (ii) instead of the upper semicontinuity and compactness of F in [31, 32].

Now, we give an example to illustrate that the assumption (v) of Theorem 3.9 is essential.

Example 3.11. Let $X = Z = \mathbb{R}, Y = \mathbb{R}^2, C = \mathbb{R}^2_+, \Lambda = [-1, 0]$ and let $\epsilon = \frac{1}{2}, e = \left(\frac{1}{4}, \frac{1}{3}\right), f = (0, 3)$. Let $A : \Lambda \to 2^X$ defined by $A(\mu) = [0, 1]$. For each $\mu \in \Lambda, x, y \in A(\mu)$, define set-valued mapping $F : X \times X \times \Lambda \to 2^Y$ by

$$\begin{aligned}&F(x, y, \mu)\\ &= \Big\{(a, b)|(a, b) = t\Big((\mu^2+2\mu+3)(y-x-1)-\frac{1}{8}, \frac{1}{3}\mu x(y-x)-\frac{1}{6}\Big)+(1-t)\Big(-\frac{1}{8}, -\frac{1}{6}\Big), \forall t \in [0, 1]\Big\}.\end{aligned}$$

Obviously, $A(\cdot)$ is continuous with nonempty compact values on Λ, and conditions (ii), (iii) and (iv) of Theorem 3.9 are satisfied.

It follows from a direct computation that

$$V^W_\epsilon(\mu) = \begin{cases} [0, 1], & \text{if} \quad \mu = 0, \\ \{0, 1\}, & \text{if} \quad \mu \in [-1, 0). \end{cases}$$

Hence, $V^W_\epsilon(\mu)$ is even not l.s.c at $\mu = 0$. The reason is that the assumption (v) does not hold for $f = (0, 3)$ and $\mu = 0$.

Now, we show that $V^W_\epsilon(\mu)$ is even not l.s.c at $\mu = 0$. Indeed, there exists $\frac{1}{10} \in V^W_\epsilon(0)$ and there exists a neighborhood $(\frac{1}{20}, \frac{3}{20})$ of $\frac{1}{10}$, for any neighborhood $U(0)$ of 0, there exists $-1 < \tilde{\mu} < 0$ such that $\tilde{\mu} \in U(0)$ and

$$V^W_\epsilon(\tilde{\mu}) \cap \Big(\frac{1}{20}, \frac{3}{20}\Big) = \emptyset.$$

;nition 2.3, we know that $V^W_\epsilon(\cdot)$ is not l.s.c at $\mu = 0$. Therefore, the assumption (v) in Theorem 3.9 is essential.

When $\epsilon = 0$, by virtue of Corollary 3.5 and Theorem 3.9, we can obtain the following result.

Corollary 3.12. *Let $f \in C^* \setminus \{0\}$. Suppose that the following conditions are satisfied:*

(i) *$A(\cdot)$ is continuous with compact values on Λ;*

(ii) *$F(\cdot, \cdot, \cdot)$ is C-lower semicontinuous on $B \times B \times \Lambda$;*

(iii) *For any given $\mu \in \Lambda$,*

$$(\inf_{z \in F(x,y,\mu)} f(z))(\inf_{z \in F(y,x,\mu)} f(z)) \leq 0, \forall x, y \in A(\mu);$$

(iv) *For each $\mu \in \Lambda$ and for each $x \in A(\mu), F(x, \cdot, \mu)$ is nearly C-subconvexlike on $A(\mu)$;*

(v) *The assumption (S) holds for f.*

Then, $V^W(\cdot)$ is lower semicontinuous on Λ.

Remark 3.13. In Corollary 3.12, under new assumptions, we obtain the lower semicontinuity of the solution mapping to parametric set-valued weak vector equilibrium problems, where the monotonicity, the information about solution mappings and the upper semicontinuity and compactness of F are not required. Thus, our result improve the corresponding ones in [6, 9, 17, 18, 30, 31]. The following example is given to illustrate these case.

Example 3.14. Let $X = Y = Z = \mathbb{R}, \Lambda = [1, 2], C = \mathbb{R}_+$. Let $A : \Lambda \to 2^X$ defined by $A(\mu) = [0, 1]$ and let $F : X \times X \times \Lambda \to 2^Y$ defined by

$$F(x, y, \mu) = \begin{cases} -\frac{1}{2}y(y - x), & \text{if} \quad x = 0, \\ [-y(y - x), (x + 2 + \mu)^2], & \text{if} \quad x \in (0, 1], \end{cases}$$

for each $\mu \in \Lambda, x, y \in A(\mu)$.

It is easy to verify that all conditions of Corollary 3.12 are satisfied. By a direct computation that we have

$$V^W(\mu) = V^f(\mu) = \{1\}.$$

Clearly, $V^W(\mu)$ is l.s.c on Λ. Hence, Corollary 3.12 holds here.

However, for any $x, y \in A(\mu)$ with $x \neq y$, we have

$$F(x, y, \mu) + F(y, x, \mu) \not\subset -\text{int}C,$$

i.e., $F(\cdot, \cdot, \mu)$ is not C-strictly monotone on $A(\mu) \times A(\mu)$ for any given $\mu \in \Lambda$. Thus, Theorem 4.1 in [17], Theorem 2.1 in [18] and Theorem 3.1 in [9] are not applicable.

However, there exists $x \in A(\mu) \setminus V^f(\mu)$, for any $z \in V^f(\mu)$, such that

$$F(x, z, \mu) + F(z, x, \mu) + B(0, d(x, z)) \not\subset -C.$$

Thus, Theorem 3.4 in [30] and Theorem 3.1 in [6] are not applicable.

Because $F(\cdot, y, \mu)$ is not upper semicontinuous at $x = 0$ for each $\mu \in \Lambda, y \in A(\mu)$. Hence, Theorem 3.3 in [31] is not applicable.

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