Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 12, No. 4, pp. 757–774, October 2016

OPTIMAL CONTROL OF THE FITZHUGH-NAGUMO NEURONS SYSTEMS IN GENERAL FORM*

Jie Sun and Wanli Yang

Abstract: We investigate the problem of optimal control of the FitzHugh-Nagumo system in general form. The well-posedness of the optimal control problem, the regularity of the solution of the state systems, and the differentiability of the control-to-state mapping are proved. The necessary optimality conditions are established by standard adjoint calculus.

Key words: *optimal control, FitzHugh-Nagumo neurons systems, reaction-diffusion system, necessary optimality conditions.*

Mathematics Subject Classification: *35J65, 58E05.*

---

1 Introduction

It is well known that the FitzHug-Nagumo model is a simplified version of the Hodgkin-Huxley model, which models activation and deactivation dynamics of a spiking neuron. We investigate the problem of optimal control for the following FitzHugh-Nagumo neurons system in general form

$$
\begin{aligned}
\partial y/\partial t - d_0 \triangle y + R(y) + a_1 u + a_2 v = &\quad w(x,t), && (x,t) \in Q_T,\\
\partial_n y = &\quad 0, && (x,t) \in \Sigma_T,\\
y(x,0) = &\quad y_0(x), && (x,t) \in \Omega,\\
\partial u/\partial t + bu + p_1(y) = &\quad 0, && (x,t) \in Q_T,\\
u(x,0) = &\quad u_0(x), && (x,t) \in \Omega,\\
\partial v/\partial t + cv + p_2(y) = &\quad 0, && (x,t) \in Q_T,\\
v(x,0) = &\quad v_0(x), && (x,t) \in \Omega,
\end{aligned}
\tag{1.1}
$$

where $d_0 > 0, a_i (i = 1,2), b, c,$ are real constants, $p_i(y)(i = 1,2.) \in C^1(\mathbb{R}^1), \triangle$ is the Laplace operator, and $w(x,t) \in L^\infty(Q_T)$ is the control function. The given initial data $y_0(x), u_0(x), v_0(x) \in L^\infty(\Omega)$; Ω is a bounded open Lipschitz domain of $\mathbb{R}^n, n \in \{1,2,3\}$; $T > 0$ is a fixed time horizon; and we use the notation $Q_T := \Omega \times (0,T)$ and $\Sigma_T := \partial\Omega \times (0,T)$. By ∂_n we denote the outward normal derivative on $\partial\Omega$.

We are concerned with the optimal control of system (1.1) with the following objective functional

$$\min \quad J(w) := I(w) + \mu j(w), \tag{1.2}$$

*This work was partially supported by Australia Research Council under Grant DP160102819.

where

$$
\begin{aligned}
I(w) \quad = \quad & \frac{1}{2}\int_0^T\int_\Omega \Big[C_Q^y(y_w(x,t)-y_Q(x,t))^2+C_Q^u(u_w(x,t)-u_Q(x,t))^2 \\
& +C_Q^v(v_w(x,t)-v_Q(x,t))^2\Big]dxdt+\frac{1}{2}\int_\Omega\Big[C_T^y(y_w(x,T)-y_T(x))^2 \\
& +C_T^u(u_w(x,T)-u_T(x))^2+C_T^v(v_w(x,T)-v_T(x))^2\Big]dxdt \qquad (1.3) \\
& +\frac{\kappa}{2}\int_0^T\int_\Omega (w)^2(x,t)dxdt, \\
j(w) \quad = \quad & \int_0^T\int_\Omega \mid w(x,t)\mid dxdt,
\end{aligned}
$$

where $C_Q^y, C_Q^u, C_Q^v, C_T^y, C_T^u, C_T^v$, and κ are nonnegative constants, and the state (y_w, u_w, v_w) is the unique solution of systems (1.1) for the given control w. The given desired terminal states $y_Q, u_Q, v_Q, y_T, u_T, v_T$ are elements of $L^2(Q), L^2(\Omega)$, respectively.

The control functions w are taken from the set of admissible controls defined as

$$U_{ad} := \{w(x,t)\in L^\infty(Q_T) \mid a(x,t)\le w(x,t)\le b(x,t) \quad \forall (x,t)\in Q_T \text{ (a.e.)}\}, \qquad (1.4)$$

where the functions $a(x,t), b(x,t)$ are given in $L^\infty(Q_T)$ such that $a(x,t)\le b(x,t)$ holds almost everywhere in Q_T.

It might be helpful to mention certain applications of the system (1.1). If $a_1=-a_2=1$ and $R(y), p_1(y), p_2(y), w(x,t)$ take the following form

$$R(y)=\frac{y^3}{3}-y,\ p_1(y)=-c_1y-\delta_1,\ p_2(y)=c_2y-\delta_2,\ w(x,t)=\frac{A}{|\Omega|}\cos(|\Omega|t),$$

where $c_i, \delta_i(\ i=1,2.)$ and A are positive constants, and $|\Omega|$ is the measure of Ω, then FitzHugh-Nagumo neurons systems (1.1) is a set of coupled differential equations which arises in computational neuroscience. The function $w(x,t)$ represents the external stimulus. The variable y represents the potential difference between the dendritic spine head and its surrounding medium, u is the recovery variable, and v represents the slowly moving current in the dendrite. In such a model, y and u together make up a fast subsystem relative to v. In recent years, there has been extensive interest for the study of the synchronization of chaotic systems and optimal control under partial differential equation constraints (e.g., Jiang [9], Jiang et al [10], Hintz et al [7], Thompson [16], Mishra et al [12] and references therein). Of particular interest is Mishra's paper [12], in which a nonlinear controller has been designed to synchronize a coupled modified FitzHugh-Nagumo model and the dynamical characteristics of that model under external stimulation are discussed.

In another application, $R(y), p_1(y)$, and a_2 have the following form:

$$R(y)=k(y-y_1)(y-y_2)(y-y_3),\ p_1(y)=-\gamma y,\ a_2=0, \qquad (1.5)$$

where k,γ are positive numbers. Then system (1.1) is a simplified version of the Hodgkin-Huxley model, which can reproduce most of qualitative features of the latter model. The variable y is the electrical potential across the axonal membrane and u is a recovery variable associated to the permeability of the membrane to the principal ionic components of the transmembrane current. The right-hand term $w(x,t)$ of the first equation in system (1.1) is the medicine actuator (the control variable), see [5, 6] for more details. It is natural to consider a control problem for this model. In this case, problem (1.1)–(1.3) under conditions (1.5) becomes the so-called Nagumo model. Existence and uniqueness theorems for the Nagumo system have already been proved by several authors. In particular, we mention the paper [8] by Jackson on the FitzHugh-Nagumo system with non-smooth data. See also the books [13, 15]. In [3], Casas et al proved the differentiability of the control-to-state mapping for both dynamical systems by an L^∞-approach, showed the well-posedness of the optimal control problems, derived first-order necessary optimality conditions, and proved the sparsity of optimal controls.

Our paper makes the following contributions. First, we prove existence and uniqueness of a solution to the FitzHugh-Nagumo systems in the more general form (1.1) (namely, All $p_1(y), p_2(y), R(y)$ and $w(x,t)$ are in general form); second, we show the second-order Fréchet differentiability of the control-to-state mapping; and third, we derive first-order necessary optimality conditions of sparse optimal controls for the more general control problem (1.2).

2 Well-Posedness of the State Equation

Throughout this paper, we make the following assumptions.
There exist positive constants $C_i (i = 0, 1, 2, 3, 4.)$ for all $y \in \mathbb{R}^1$, such that

(H)$_1$ $R'(y) \geq C_0$.

(H)$_2$ $|p_1(y)| \leq C|y| + C_2, \quad |p_2(y)| \leq C_3|y| + C_4$.

To prove the existence and uniqueness of the solution (y, u, v) of the state systems (1.1), we first transform (1.1) to an integro-differential equation.

2.1 Transformation of the State Equation

The last two equations of systems (1.1) can be solved by

$$\begin{aligned} u(x,t) &= e^{-bt}u_0(x) + \int_0^t e^{-b(t-s)}p_1(y(x,s))ds \\ &= e^{-bt}u_0(x) + K_1(y(x,t)), \\ v(x,t) &= e^{-ct}v_0(x) + \int_0^t e^{-c(t-s)}p_2(y(x,s))ds \\ &= e^{-ct}v_0(x) + K_2(y(x,t)), \end{aligned} \tag{2.1}$$

where the integral operators $K_i, i = 1, 2$, are defined as

$$\begin{aligned} K_1(y(x,t)) &= \int_0^t e^{-b(t-s)}p_1(y(x,s))ds, \\ K_2(y(x,t)) &= \int_0^t e^{-c(t-s)}p_2(y(x,s))ds. \end{aligned} \tag{2.2}$$

Inserting (2.1) in the first equation of systems (1.1), we obtain the following integro-differential equation

$$\begin{aligned} &\partial y/\partial t - d_0 \triangle y + R(y) + a_1K_1(y(x,t)) + a_2K_2(y(x,t)) \\ &\quad = w(x,t) - a_1e^{-bt}u_0(x) - a_2e^{-ct}v_0(x), \ \ (x,t) \in Q_T. \end{aligned} \tag{2.3}$$

Note that $R(y)$ is not assumed to be monotone, we substitute

$$y(x,t) := e^{\eta t} z(x,t) \tag{2.4}$$

with a sufficiently large real parameter η. This leads to a new equation for z

$$\begin{aligned} &\partial z/\partial t - d_0 \triangle z + e^{-\eta t} R(e^{\eta t} z) + \eta z + a_1 e^{-\eta t} K_{1\eta}(e^{\eta t} z(x,t)) + a_2 e^{-\eta t} K_{2\eta}(e^{\eta t} z(x,t)) \\ &\quad = e^{-\eta t}[w(x,t) - a_1 e^{-bt} u_0(x) - a_2 e^{-ct} v_0(x)], \end{aligned} \tag{2.5}$$

with the given initial and boundary conditions, where the operators $(K_{1\eta})$ and $(K_{2\eta})$ are respectively defined as

$$\begin{aligned} K_{1\eta}(e^{\eta t} z(x,t)) &= \int_0^t e^{-b(t-s)} p_1(e^{\eta s} z(x,s)) ds, \\ K_{2\eta}(e^{\eta t} z(x,t)) &= \int_0^t e^{-c(t-s)} p_2(e^{\eta s} z(x,s)) ds. \end{aligned} \tag{2.6}$$

For convenience we write

$$w := e^{-\eta t}[w(x,t) - a_1 e^{-bt} u_0(x) - a_2 e^{-ct} v_0(x)]. \tag{2.7}$$

Thus, systems (1.1) becomes

$$\begin{aligned} &\partial z/\partial t - d_0 \triangle z + e^{-\eta t} R(e^{\eta t} z) + \eta z \\ &\quad + a_1 e^{-\eta t} K_{1\eta}(e^{\eta t} z(x,t)) + a_2 e^{-\eta t} K_{2\eta}(e^{\eta t} z(x,t)) = w, \quad (x,t) \in Q_T, \\ &\partial_n z = 0, \quad (x,t) \in \Sigma_T; \\ &z(x,0) = y_0(x), \quad (x,t) \in \Omega. \end{aligned} \tag{2.8}$$

2.2 A Priori Estimates

Let

$$W(0,T) = \Big\{ y \in L^2(0,T; H^1(\Omega)) \mid \frac{\partial y}{\partial t} \in L^2(0,T; H^1(\Omega)^*) \Big\}. \tag{2.9}$$

We give the definition of a *weak solution* of equation (2.8).

Definition 2.1. A function $y \in W(0,T)$ is said to be a *weak solution* of equation (2.8) if

$$\begin{aligned} &\int_0^T \partial z/\partial t \varphi + d_0 \int_0^T \int_\Omega \{\nabla z \nabla \varphi + [e^{-\eta t} R(e^{\eta t} z) + \eta z] \varphi dx dt \\ &\quad + \int_0^T \int_\Omega [a_1 e^{-\eta t} K_{1\eta}(e^{\eta t} z(x,t)) + a_2 e^{-\eta t} K_{2\eta}(e^{\eta t} z(x,t))] \varphi dx dt \\ &\quad = \int_0^T \int_\Omega w \varphi dx dt, \end{aligned} \tag{2.10}$$

holds for all $\varphi \in W(0,T)$. Here, $z(x,t) := e^{-\eta t} y(x,t)$, and $y(x,0) = y_0$.

We next estimate the norm of the operator $K_{1\eta}, K_{2\eta}$. We have that

$$
\begin{aligned}
|e^{-\eta t}K_{1\eta}(e^{\eta t}z(x,t))| &= e^{-\eta t}\left|\int_0^t e^{-b(t-s)}p_1(e^{\eta s}z(x,s))ds\right| \\
&\le e^{-(b+\eta)t}\int_0^t e^{bs}|p_1(e^{\eta s}z(x,s))|ds \\
&\le e^{-(b+\eta)t}\int_0^t e^{bs}(Ce^{\eta s}|z|+C_2)ds \\
&\le e^{-(b+\eta)t}\left[C\Big(\int_0^t e^{2(b+\eta)s}ds\Big)^{1/2}\left(\int_0^t z^2ds\right)^{1/2}+\frac{C_2}{b}(e^{bt}-1)\right] \\
&\le e^{-(b+\eta)t}\left[C\sqrt{\frac{1}{2(b+\eta)}[e^{2(b+\eta)t}-1]}\left(\int_0^t z^2ds\right)^{1/2}+\frac{C_2}{b}(e^{bt}-1)\right] \\
&\le C\sqrt{\frac{1}{2(b+\eta)}}\left(\int_0^t z^2ds\right)^{1/2}+\frac{C_2}{b}e^{-(b+\eta)t}(e^{bt}-1).
\end{aligned}
\tag{2.11}
$$

Thus, for sufficiently large η, we have that

$$
\begin{aligned}
\int_0^T\int_\Omega |e^{-\eta t}K_{1\eta}(e^{\eta t}z(x,t))|^2 &\le \int_0^T\int_\Omega 2\left[\frac{C_1^2}{2(b+\eta)}\int_0^t z(x,t)^2+\frac{C_2^2}{b^2}e^{-2\eta t}\right]dxdt \\
&\le \frac{C_1^2}{(b+\eta)}\int_0^T\int_\Omega\left[\int_0^T z(x,t)^2dt\right]dxdt+\frac{C_2^2}{b^2\eta}|\Omega| \\
&\le \frac{C_1^2T}{(b+\eta)}\|z\|^2_{L^2(Q_T)}+\frac{C_2^2}{b^2\eta}|\Omega|.
\end{aligned}
\tag{2.12}
$$

Similarly, we can get the estimate of operator $K_{2\eta}$. In conclusion, we have

Lemma 2.2. *Under the assumption (H)$_2$, if (y,u,v) is a solution of systems (1.1), $y(x,t) := e^{\eta t}z(x,t)$, then for sufficiently large η we have that*

$$
\begin{aligned}
\int_0^T\int_\Omega |e^{-\eta t}K_{1\eta}(e^{\eta t}z(x,t))|^2 &\le \frac{C_1^2}{(b+\eta)}\|z\|^2_{L^2(Q_T)}+\frac{C_2^2}{b^2\eta}|\Omega|, \\
\int_0^T\int_\Omega |e^{-\eta t}K_{2\eta}(e^{\eta t}z(x,t))|^2 &\le \frac{C_3^2}{(c+\eta)}\|z\|^2_{L^2(Q_T)}+\frac{C_4^2}{c^2\eta}|\Omega|.
\end{aligned}
\tag{2.13}
$$

On the L^2 estimate of solution (y,u,v), we have the following result.

Lemma 2.3. *(L^2-a-priori estimate) Under assumptions (H)$_1$ and (H)$_2$, for sufficiently large η, there exists a positive constant C with the following properties: If $\eta \ge \eta_0$ and $z \in W(0,T)\cap L^\infty(Q_T)$ is any weak solution of system (1.1), then there holds for all $w \in L^2(Q_T)$ and $y_0 \in L^2(\Omega)$ that*

$$
\|z\|_{L^2(0,T;H^1(\Omega))} \le C\left(\|w\|_{L^2(Q_T)}+\|y_0\|_{L^2(\Omega)}+|R(0)|+\frac{C_2^2}{b^2\eta}|\Omega|+\frac{C_4^2}{c^2\eta}|\Omega|\right). \tag{2.14}
$$

Proof. Let
$$R_\eta(t,z) = e^{-\eta t}R(e^{\eta t}z) + \frac{\eta}{3}z. \tag{2.15}$$

Then (2.8) becomes

$$\begin{aligned}
\partial z/\partial t - d_0\triangle z + R_\eta(t,z) + \left[\frac{1}{6}\eta z + a_1 e^{-\eta t}K_{1\eta}(e^{\eta t}z(x,t))\right]& & &\\
+\left[\frac{1}{6}\eta z + a_2 e^{-\eta t}K_{2\eta}(e^{\eta t}z(x,t))\right] + \frac{1}{3}\eta z &= w, & &(x,t)\in Q_T,\\
\partial_n z &= 0, & &(x,t)\in \Sigma_T,\\
z(x,0) &= y_0(x), & &(x,t)\in \Omega.
\end{aligned} \tag{2.16}$$

From $(\mathrm{H})_1$, we have that
$$\frac{\partial}{\partial z}R_\eta(t,z) = C_0 + \frac{\eta}{3}. \tag{2.17}$$

Thus, for sufficiently large $\eta, R_\eta(t,z)$ is a monotone function, we obtain

$$\begin{aligned}
&\frac{1}{2}\|z(T)\|^2_{L^2(\Omega)} + \int_0^T\int_\Omega (|\nabla z|^2 + \frac{\eta}{3}z^2)dxdt + \int_0^T\int_\Omega (R_\eta(t,z) - R_\eta(t,0))(z-0)\\
&\quad + \left[\frac{\eta}{6} - \frac{C_1^2}{(b+\eta)}\right]\|z\|^2_{L^2(Q_T)} + \left[\frac{\eta}{6} - \frac{C_3^2}{(c+\eta)}\right]\|z\|^2_{L^2(Q_T)} - \left[\frac{C_2^2}{b^2\eta}|\Omega| + \frac{C_4^2}{c^2\eta}|\Omega|\right]\|z\|^2_{L^2(Q_T)}\\
&\le \int_0^T\int_\Omega |w - R_\eta(t,0)||z|dxdt + \frac{1}{2}\|z(0)\|^2_{L^2(\Omega)} + \frac{C_2^2}{b^2\eta}|\Omega| + \frac{C_4^2}{c^2\eta}|\Omega|
\end{aligned} \tag{2.18}$$

By the monotonicity of $R_\eta(t,z)$ for sufficiently large η, (2.18) becomes

$$\begin{aligned}
&\frac{1}{2}\|z(T)\|^2_{L^2(\Omega)} + \int_0^T\int_\Omega (|\nabla z|^2 + \frac{\eta}{6}z^2)dxdt\\
&\quad\le \int_0^T\int_\Omega |w - R_\eta(t,0)||z|dxdt + \frac{1}{2}\|z(0)\|^2_{L^2(\Omega)} + \frac{C_2^2}{b^2\eta}|\Omega| + \frac{C_4^2}{c^2\eta}|\Omega|.
\end{aligned} \tag{2.19}$$

Young's inequality yields that for sufficiently large η

$$\int_0^T\int_\Omega \Big(|\nabla z|^2 + \frac{\eta}{7}z^2\Big)dxdt \le C\left[\|w - e^{-\eta}R(t,0)\|^2_{L^2(Q_T)} + \|z(0)\|^2_{L^2(\Omega)} + \frac{C_2^2}{b^2\eta}|\Omega| + \frac{C_4^2}{c^2\eta}|\Omega|\right]. \tag{2.20}$$

An application of the triangle inequality and $e^{-\eta} \le 1$ yields that

$$\|z\|_{L^2(0,T;H^1(\Omega))} \le C\left(\|w\|_{L^2(Q_T)} + \|y_0\|_{L^2(\Omega)} + |R(0)| + \frac{C_2^2}{b^2\eta}|\Omega| + \frac{C_4^2}{c^2\eta}|\Omega|\right).$$

□

In the following, we will replace this L^2-estimate by an L^∞-estimate.

Lemma 2.4 (L^∞-a-priori estimate)**.** *Assume that $w \in L^p(Q_T)$ with $p > 5/2$ and $y_0 \in L^\infty(\Omega)$. If $\eta \ge \eta_0$ and $z \in W(0,T)\cap L^\infty(Q_T)$ is any weak solution of the system (1.1), then there is a positive constance C_∞, such that*

$$\|z\|_{L^\infty(Q_T)} \le C_\infty(\|w\|_{L^p(Q_T)} + |R(0)| + \|y_0\|_{L^\infty(\Omega)} + 1). \tag{2.21}$$

Proof. Under hypothesis $(\mathrm{H})_2$, for all $t \in [0,T]$ we easily verify

$$\begin{aligned}\left\|e^{-\eta t}\int_0^t e^{-b(t-s)}p_1(e^{\eta s}z)ds\right\|_{H^1(\Omega)} &\leq \int_0^t e^{-(b+\eta)t+bs} \parallel p_1(e^{\eta s})z \parallel_{H^1(\Omega)} ds \\ &\leq \int_0^t Ce^{-(b+\eta)(t-s)} \parallel z \parallel_{H^1(\Omega)} ds + \int_0^t C_2 e^{-(b+\eta)t+bs}ds \\ &\leq \frac{C}{\sqrt{2(b+\eta)}} \parallel z \parallel_{L^2(0,T;H^1(\Omega))} + \frac{C_2 e^{-\eta t}(1-e^{-bt})}{b}.\end{aligned} \tag{2.22}$$

Similar to (2.22), for all $t \in [0,T]$ we obtain

$$\left\|e^{-\eta t}\int_0^t e^{-c(t-s)}p_2(e^{\eta s}z)ds\right\|_{H^1(\Omega)} \leq \frac{C_3}{\sqrt{2(c+\eta)}} \parallel z \parallel_{L^2(0,T;H^1(\Omega))} + \frac{C_4 e^{-\eta t}(1-e^{-ct})}{c}. \tag{2.23}$$

The continuous embedding of $H^1(\Omega)$ in $L^6(\Omega)$ for $n \leq 3$ yields

$$\begin{aligned}\|K_{1\eta}z\|_{L^6(Q_T)} \leq C\|K_{1\eta}z\|_{C([0,T];L^6(\Omega))} &\leq C\|K_{1\eta}z\|_{C([0,T];H^1(\Omega))} \\ &\leq \frac{C}{\sqrt{2(b+\eta)}} \parallel z \parallel_{L^2(0,T;H^1(\Omega))} + \frac{C_2 e^{-\eta t}(1-e^{-bt})}{b}\end{aligned} \tag{2.24}$$

and

$$\|K_{2\eta}z\|_{L^6(Q_T)} \leq \frac{C_3}{\sqrt{2(c+\eta)}} \parallel z \parallel_{L^2(0,T;H^1(\Omega))} + \frac{C_4 e^{-\eta t}(1-e^{-ct})}{c}. \tag{2.25}$$

Assume now that $u \in L^p(Q_T)$ with $p > 5/2$ and set $q := \min\{p, 6\}$. In (2.8), we shift the term $K_{1\eta}z,\ K_{2\eta}z$ to the right-hand side and consider the associated semilinear equation

$$\begin{array}{rcll} \partial z/\partial t - d_0\triangle z + e^{-\eta t}R(e^{\eta t}z) + \eta z &=& & \\ w - a_1 e^{-\eta t}K_{1\eta}(e^{\eta t}z(x,t)) - a_2 e^{-\eta t}K_{2\eta}(e^{\eta t}z(x,t)), & & & (x,t) \in Q_T, \\ \partial_n z &=& 0, & (x,t) \in \Sigma_T; \\ z(x,0) &=& y_0(x), & (x,t) \in \Omega. \end{array} \tag{2.26}$$

We can invoke the known L^∞-estimates for semilinear parabolic equations for the given q (cf. the treatment of quasilinear equations in [11]) and obtain that

$$\begin{aligned}\|z\|_{L^\infty(Q_T)} &\leq C(\|w - a_1 e^{-\eta t}K_{1\eta}(e^{\eta t}z) - a_2 e^{-\eta t}K_{2\eta}(e^{\eta t}z) \\ &\qquad -e^{-\eta t}R(0)\|_{L^q(Q_T)}) + \|y_0\|_{L^\infty(\Omega)} \\ &\leq C(\|w\|_{L^q(Q_T)} + a_1\|e^{-\eta t}K_{1\eta}(e^{\eta t}z)\|_{L^6(Q_T)} \\ &\qquad +a_2\|e^{-\eta t}K_{2\eta}(e^{\eta t}z)\|_{L^6(Q_T)} + |R(0)| + \|y_0\|_{L^\infty(\Omega)}) \\ &\leq C(\|w\|_{L^q(Q_T)} + |R(0)| + \|y_0\|_{L^\infty(\Omega)} + \parallel z \parallel_{L^2(0,T;H^1(\Omega))} + 1).\end{aligned} \tag{2.27}$$

By Lemma 2.3, (2.27) implies

$$\begin{aligned}\|z\|_{L^\infty(Q_T)} &\leq C\Big(\|w\|_{L^q(Q_T)} + |R(0)| + \|y_0\|_{L^\infty(\Omega)} \\ &\qquad + C\Big(\|w\|_{L^2(Q_T)} + \|y_0\|_{L^2(\Omega)} + |R(0)| + \frac{C_2^2}{b_1^2\eta}|\Omega| + \frac{C_4^2}{c_1^2\eta}|\Omega|\Big) + 1\Big) \\ &\leq C(\|w\|_{L^p(Q_T)} + |R(0)| + \|y_0\|_{L^\infty(\Omega)} + 1).\end{aligned} \tag{2.28}$$

□

2.3 Solvability of the State Equation

Now we keep the given control u, together with y_0 being fixed and set

$$M_\infty := C_\infty(\|w\|_{L^p(Q_T)} + |R(0)| + \|y_0\|_{L^\infty(\Omega)} + 1). \tag{2.29}$$

Here, C_∞ is a constance in Lemma 2.4. We define the following auxiliary function cutting off R_η

$$\begin{array}{rcll} \hat{R}_\eta(t,z) &=& R_\eta(t,M_\infty), & \text{if } z \geq M_\infty; \\ \hat{R}_\eta(t,z) &=& R_\eta(t,z), & \text{if } \mid z \mid < M_\infty; \\ \hat{R}_\eta(t,z) &=& R_\eta(t,-M_\infty), & \text{if } z \leq -M_\infty. \end{array} \tag{2.30}$$

Theorem 2.5 (Existence and uniqueness)**.** *Assume that $w \in L^p(Q_T)$ with $p > 5/2$ and $y_0 \in L^\infty(\Omega)$. Then the integro-differential equation (2.8) has a unique solution $z \in W(0,T) \cap L^\infty(Q_T) \cap C(\overline{\Omega} \times (0,T])$ such that*

$$\|z\|_{L^\infty(Q_T)} + \|z\|_{W(0,T)} \leq C(\|w\|_{L^p(Q_T)} + \|y_0\|_{L^\infty(\Omega)} + |R(0)| + 1). \tag{2.31}$$

If y_0 is continuous in $\bar{\Omega}$, then the solution z belongs to $C(\bar{Q}_T)$.

Proof. (i). *Existence of a solution*
For given $h \in L^2(Q_T)$, we consider the equation

$$\begin{array}{rcll} \partial z/\partial t - d_0 \triangle z + \hat{R}_\eta(t,z) + \dfrac{2\eta}{3} z &=& & \\ w - a_1 e^{-\eta t} K_{1\eta}(e^{\eta t} h(x,t)) - a_2 e^{-\eta t} K_{2\eta}(e^{\eta t} h(x,t)), & & & (x,t) \in Q_T, \\ \partial_n z &=& 0, & (x,t) \in \Sigma_T; \\ z(x,0) &=& y_0(x), & (x,t) \in \Omega. \end{array} \tag{2.32}$$

Let us denote by F the solution mapping of (2.32) $F : h \mapsto z$ acting in $L^2(Q_T)$. Since (2.32) is a monotone linear system, the mapping F is well defined. Consider the following equation with $a_1 = a_2 = 0$ in (2.8)

$$\begin{array}{rcll} \partial z/\partial t - d_0 \triangle z + e^{-\eta t} R(e^{\eta t} z) + \eta z &=& w, & \text{in } Q_T, \\ \partial_n z &=& 0, & \text{in } \Sigma_T; \\ z(x,0) &=& y_0(x), & \text{in } \Omega. \end{array} \tag{2.33}$$

From Lemma 2.3, there exist positive constants C_* such that

$$\|z\|_{L^2(0,T;H^1(\Omega))} \leq C_*(\|w\|_{L^2(Q_T)} + \|y_0\|_{L^2(\Omega)} + |R(0)|). \tag{2.34}$$

We define

$$M_0 := C_*(\|w\|_{L^2(Q_T)} + \|y_0\|_{L^2(\Omega)} + |R(0)|). \tag{2.35}$$

We assume that $\| h \|_{L^2(Q_T)} \leq 2M_0$. Then we have

$$\begin{aligned} \|F(h)\|_{L^2(Q_T)} &= \|z\|_{L^2(Q_T)} \leq \|z\|_{L^2(0,T;H^1(\Omega))} \\ &\leq C_*(\|w\|_{L^2(Q_T)} + a_1 \|e^{-\eta t} K_{1\eta}(e^{\eta t} h(x,t))\|_{L^2(Q_T)} \\ &\quad + a_2 \|e^{-\eta t} K_{2\eta}(e^{\eta t} h(x,t))\|_{L^2(Q_T)} + \|y_0\|_{L^2(\Omega)} + |R(0)|) \\ &\leq M_0 + a_1 \|e^{-\eta t} K_{1\eta}(e^{\eta t} h(x,t))\|_{L^2(Q_T)} + a_2 \|e^{-\eta t} K_{2\eta}(e^{\eta t} h(x,t))\|_{L^2(Q_T)} \\ &\leq M_0 + a_1 \frac{C_1^2}{(b+\eta)} \|z\|^2_{L^2(Q_T)} + \frac{C_2^2}{b^2 \eta} |\Omega| + a_2 \frac{C_3^2}{(c+\eta)} \|z\|^2_{L^2(Q_T)} + \frac{C_4^2}{c^2 \eta} |\Omega|. \end{aligned} \tag{2.36}$$

If η is sufficiently large, then F maps $B_{2M_0}(0)$, the closed ball of $L_2(Q_T)$ around zero with radius $2M_0$, into itself, and $\|F(h)\| \le C$, for all $h \in B_{2M_0}(0)$ (see [3] or [11]). By Aubin's lemma, bounded sets of $W(0;T)$ are relatively compact in $L^2(Q_T)$. Hence the mapping F is compact. By Schauder's theorem, F has a fixed point in $B_{2M_0}(0)$, this is a solution to (2.36).

By Lemma 2.4, the solution v satisfies the L^∞-estimate (2.21) provided that η is taken sufficiently large. In this case, $\hat{R}_\eta(t,z) = R_\eta(t,z)$ is satisfied, so z is a solution of (2.8).

(*ii*). *Uniqueness of the solution.*

Suppose that z_1 and z_2 are solutions of (2.8) and set $z := z_1 - z_2$. Subtracting the associated equations and applying the mean value theorem to the appearing difference $R_\eta(t,z_1) - R_\eta(t,z_2)$, we see that v solves

$$\begin{array}{rcll} \partial z/\partial t - d_0\triangle z + \Big(\dfrac{\partial}{\partial z}\hat{R}_\eta(t,z(\theta_1)) + \dfrac{2}{3}\eta\Big)z & & & \\ +a_1 e^{-\eta t}\Big(\dfrac{\partial}{\partial z}K_{1\eta}(e^{\eta t}z(\theta_2))\Big)z + a_2 e^{-\eta t}\Big(\dfrac{\partial}{\partial z}K_{2\eta}(e^{\eta t}z(\theta_3))\Big)z &=& 0, & (x,t)\in Q_T, \\ \partial_n z &=& 0, & (x,t)\in \Sigma_T, \\ z(x,0) &=& 0, & (x,t)\in \Omega, \end{array} \tag{2.37}$$

where

$$z_{\theta_i} = z_1 + \theta_i(z_2 - z_1), i = 1,2,3. \tag{2.38}$$

Equation (2.37) is a linear equation with non-negative coefficient. Applying the same technique as in the proof of Lemma 2.3, we can find that $z = 0$, hence, $z_1 = z_2$ showing the uniqueness.

(*iii*). *Continuity properties of* z.

On the continuity properties of z, we refer to [1, 14]. □

2.4 Differentiability of the Control-to-State Mapping

To show the differentiability of the control-to-state mapping $w \to y$, we first state an analog of Theorem 2.5 for a linear system without proof.

Lemma 2.6. *If η is taken sufficiently large, $c_0 \in L^\infty(Q_T)$ is almost everywhere nonnegative, $w \in L^2(Q_T)$ and $y_0 \in L^2(\Omega)$, then the linear integro-differential system*

$$\begin{array}{rcll} \partial z/\partial t - d_0\triangle z + c_0(x,t)z + \eta z + a_1(K_{1\eta})z + a_2(K_{2\eta})z &=& w, & (x,t)\in Q_T \\ \partial_n z &=& 0, & (x,t)\in \Sigma_T; \\ z(x,0) &=& y_0(x), & (x,t)\in \Omega \end{array} \tag{2.39}$$

has a unique solution $z \in W(0,T)$. There is $C > 0$ depending neither on y_0 nor on c_0 such that

$$\|z\|_{W(0,T)} \le C(\|w\|_{L^2(Q_T)} + \|y_0\|_{L^2(\Omega)}). \tag{2.40}$$

For the differentiability, we have the following results.

Lemma 2.7. *For all $p > 5/2$ and all sufficiently large η, the solution mapping $\Gamma_\eta : w \to z$ for equation (2.8) is of class $C^2 : L^p(Q_T) \to W(0,T) \cap L^\infty(Q_T) \cap C(\overline{\Omega} \times (0,T])$.*

Proof. First, we consider the semilinear parabolic differential equation of monotone type

$$\begin{aligned} \partial z/\partial t - d_0 \triangle z + R_\eta(z) + \frac{2}{3}\eta z = w, (x,t) \in Q_T \\ \partial_n z = 0, (x,t) \in \Sigma_T; \\ z(x,0) = y_0(x), (x,t) \in \Omega. \end{aligned} \tag{2.41}$$

where

$$R_\eta(z) = e^{-\eta t} R(e^{\eta t} z) + \frac{1}{3}\eta z. \tag{2.42}$$

From (2.41), for sufficiently large η, $R_\eta(t,z)$ is a monotone function, we obtain

$$\frac{\partial R_\eta(z)}{\partial z} \geq 0. \tag{2.43}$$

For each $w \in L^p(Q_T), y_0 \in L^\infty(\Omega)$ and $\eta \geq \eta_0$, this equation has a unique solution $z_w \in V_\infty := W(0,T) \cap L^\infty(Q_T) \cap C(\overline{\Omega} \times (0,T])$. Let G_η denote the associated solution mapping

$$G_\eta : L^p(Q_T) \ni w \mapsto z_w \in V_\infty. \tag{2.44}$$

It is known that G_η is twice continuously Fréchet-differentiable. For this differentiability property and the concrete form of the first- and second-order derivatives, we refer to [2].

By Theorem 2.5, there exists a constant C such that

$$\|z_w\|_{L^\infty(Q_T)} \leq C(\|w\|_{L^2(Q_T)} + \|y_0\|_{L^\infty(\Omega)}) \tag{2.45}$$

holds. We return to the nonlinear equation (2.8) in the form

$$\begin{aligned} \partial z/\partial t - d_0 \triangle z + R_\eta(z) + \frac{2}{3}\eta z + a_1(K_{1\eta})z(x,t) + a_2(K_{2\eta})z(x,t) &= w, & (x,t) \in Q_T, \\ \partial_n z &= 0, & (x,t) \in \Sigma_T, \\ z(x,0) &= y_0(x), & (x,t) \in \Omega. \end{aligned} \tag{2.46}$$

Obviously, by using the mapping G_η for (2.41), z solves (2.46) if only if

$$z - G_\eta\left(w - a_1(K_{1\eta})z(x,t) - a_2(K_{2\eta})z(x,t)\right) =: F(z,w) = 0, \tag{2.47}$$

and we have

$$\frac{\partial}{\partial z}F(z_0,w_0) = I + G_\eta^{'}(w_0 - a_1(K_{1\eta})z_0 - a_2(K_{2\eta})z_0)[a_1(K_{1\eta}) + a_2(K_{2\eta})]. \tag{2.48}$$

From Lemma 2.2, the norms of $\|K_{1\eta}\|_{L^2(Q_T)}$ and $\|K_{2\eta}\|_{L^2(Q_T)}$ tend to zero as $\eta \to \infty$, hence

$$\|G_\eta^{'}(w_0 - a_1(K_{1\eta})z_0 - a_2(K_{2\eta})z_0)[a_1(K_{1\eta}) + a_2(K_{2\eta})]\|_{L^2(Q_T)} < 1 \tag{2.49}$$

holds for all sufficiently large η. Therefore $\frac{\partial}{\partial z}F(z_0,w_0)$ is continuously invertible for sufficiently large η. By the implicit function theorem, the mapping $\Gamma_\eta : w \to z$ is also of class C^2 from $L^p(Q_T)$ to $L^\infty(Q_T)$ for sufficiently large η. □

From the above result, it is easy to prove our main result concerning differentiability.

Theorem 2.8 (Differentiable of the control-to-state mapping). *The solution mapping associated with systems (1.1)*

$$G : w \mapsto (y(w), u(w), v(w))(L^p(Q_T) \to (W(0,T) \cap L^\infty(Q_T) \cap C(\overline{\Omega} \times (0,T]))^3) \tag{2.50}$$

is twice continuously Fréchet-differentiable and

(1) *The derivative* $(y_h(w), u_h(w), v_h(w)) := G'(w)h$ *equal to the pair* (y, u, v) *solving the following system*

$$\begin{aligned} \partial y/\partial t - d_0 \triangle y + R'(y(w))y + a_1 u + a_2 v =&\ h, && (x,t) \in Q_T,\\ \partial_n y =&\ 0, && (x,t) \in \Sigma_T;\\ y(x,0) =&\ 0, && (x,t) \in \Omega;\\ \partial u/\partial t + bu + b_0 y =&\ 0, && (x,t) \in Q_T,\\ u(x,0) =&\ 0, && (x,t) \in \Omega;\\ \partial v/\partial t + cv + c_0 y =&\ 0, && (x,t) \in Q_T,\\ v(x,0) =&\ 0, && (x,t) \in \Omega. \end{aligned} \tag{2.51}$$

(2) *The second derivative* $(y_{h_1h_2}(w), u_{h_1h_2}(w), v_{h_1h_2}(w)) := G''(w)[h_1, h_2]$ *equal to the pair* (y, u, v) *solving the following system*

$$\begin{aligned} \partial y/\partial t - d_0 \triangle y + R'(y(w))y + a_1 u + a_2 v =&\ 0, && (x,t) \in Q_T,\\ \partial_n y =&\ 0, && (x,t) \in \Sigma_T;\\ y(x,0) =&\ 0, && (x,t) \in \Omega;\\ \partial u/\partial t + bu + b_0 y =&\ 0, && (x,t) \in Q_T,\\ u(x,0) =&\ 0, && (x,t) \in \Omega;\\ \partial v/\partial t + cv + c_0 y =&\ 0, && (x,t) \in Q_T,\\ v(x,0) =&\ 0, && (x,t) \in \Omega; \end{aligned} \tag{2.52}$$

where $b_0 = p_1'(y(w)), c_0 = p_2'(y(w))$.

3 Well-Posedness of the Optimal Control Problems and First-Order Necessary Optimality Conditions

3.1. Solvability of the general optimal control problem

For the optimal control problems (1.1)-(1.3), we have the following results.

Theorem 3.1 (Existence of an optimal solution). *The optimal control problem (1.2) with constraint (1.1) has at least one optimal solution* $\bar{w}$ *with associated optimal state*

$$(\bar{y}(\bar{w}), \bar{u}(\bar{w}), \bar{v}(\bar{w})) := G(\bar{w}). \tag{3.1}$$

Proof. The set U_{ad} is non-empty and weakly compact in $L^p(Q_T)$. Moreover, the reduced objective functional $J(w)$ is weakly lower semi-continuous in $L^p(Q_T)$ for $p > 2$ because of the compactness of the mapping

$$w \in L^p(Q_T) \to (y_w, u_w, v_w) \in (L^p(Q_T))^3 \tag{3.2}$$

and the convexity of the terms involving the control. Notice also that the mapping

$$G : u \mapsto (y_w, u_w, v_w) \tag{3.3}$$

is of class C^2. The result now follows by standard arguments. □

3.2. First-Order Necessary Optimality Conditions

Let $w \in U_{ad}$ be a locally optimal control with associated state (y_w, u_w, v_w). Since any global solution is also a local one, we formulate the optimality conditions for local solutions. The triple (y_w, u_w, v_w) and w has to satisfy a variational inequality including the sub-differential $\partial j(\bar{w})$. We recall that

$$\partial j(\bar{w}) = \left\{ \lambda \in L^\infty(Q_T) : \ j(w) \geq j(\bar{w}) + \int_0^T \int_\Omega \lambda(w - \bar{w}) dxdt, \forall w \in L^\infty(Q_T) \right\}. \tag{3.4}$$

Obviously,

$$\begin{aligned} \lambda &= 1, \text{if } \bar{w}(x,t) > 0, \\ \lambda &\in [-1,1], \text{if } \bar{w}(x,t) = 0, \\ \lambda &= -1, \text{if } \bar{w}(x,t) < 0. \end{aligned} \tag{3.5}$$

From the theory of standard variational inequality, we can obtain the following results.

Lemma 3.2. *If $(\overline{y}, \overline{u}, \overline{v}, \overline{w})$ is a local solution to the optimal control problem(1.1)-(1.3), then there exists a function $\bar{\lambda} \in \partial j(\overline{w})$ with μ such that*

$$I'(\bar{w})(w - \bar{w}) + \int_0^T \int_\Omega \mu \bar{\lambda}(w - \bar{w}) dxdt \geq 0, \forall w \in U_{ad}. \tag{3.6}$$

For any $w \in U_{a,d}$, let $h = w - \overline{w}$. Then, (y_h, u_h, v_h) solves the following linear system

$$\begin{aligned} \partial y/\partial t - d_0 \triangle y + R'(\overline{y})y + a_1 u + a_2 v = &\ h, \quad (x,t) \in Q_T, \\ \partial_n y = &\ 0, \quad (x,t) \in \Sigma_T; \\ y(x,0) = &\ 0, \quad (x,t) \in \Omega; \\ \partial u/\partial t + bu + b_0 y = &\ 0, \quad (x,t) \in Q_T, \\ u(x,0) = &\ 0, \quad (x,t) \in \Omega; \\ \partial v/\partial t + cv + c_0 y = &\ 0, \quad (x,t) \in Q_T, \\ v(x,0) = &\ 0, \quad (x,t) \in \Omega. \end{aligned} \tag{3.7}$$

We define the following adjoint system for a pair of adjoint states $(\varphi_1, \varphi_2, \varphi_3) \in (W(0,T))^3$.

$$\begin{aligned} -\partial\varphi_1/\partial t - d_0\triangle\varphi_1 + R'(\overline{y})\varphi_1 + b_0\varphi_2 + c_0\varphi_3 &= C_Q^y(\overline{y} - y_Q), (x,t) \in Q_T, \\ \partial_n\varphi_1 &= 0, (x,t) \in \Sigma_T; \\ \varphi_1(x,T) &= C_T^y(x)(\overline{y}(x,T) - y_T(x)), (x,t) \in \Omega; \\ -\partial\varphi_2/\partial t + b\varphi_2 + a_1\varphi_1 &= C_Q^u(\overline{u} - u_Q), (x,t) \in Q_T, \\ \varphi_2(x,T) &= C_T^u(x)(\overline{u}(x,T) - u_T(x)), (x,t) \in \Omega; \\ -\partial\varphi_3/\partial t + c\varphi_3 + a_2\varphi_1 &= C_Q^v(\overline{v} - v_Q), (x,t) \in Q_T, \\ \varphi_3(x,T) &= C_T^v(x)(\overline{v}(x,T) - v_T(x)), (x,t) \in \Omega. \end{aligned} \tag{3.8}$$

Lemma 3.3. *Let $(\varphi_1, \varphi_2, \varphi_3) \in (W(0,T))^3$ be the unique solution of the adjoint system (3.8). Then there holds*

$$\begin{aligned} \int_0^T \int_\Omega h\varphi_1 dxdt &= \int_0^T \int_\Omega [C_Q^y(\overline{y} - y_Q)y + C_Q^u(\overline{u} - u_Q)u + C_Q^v(\overline{v} - v_Q)v]dxdt \\ &= \int_\Omega C_T^y(x)(\overline{y}(x,T) - y_T(x))y(\cdot,T)dx \\ &+ \int_\Omega C_T^u(x)(\overline{u}(x,T) - u_T(x))u(\cdot,T)dx \\ &+ \int_\Omega C_T^v(x)(\overline{v}(x,T) - v_T(x))v(\cdot,T)]dx. \end{aligned} \tag{3.9}$$

Proof. We multiply the first equation in (3.7) with φ_1, the fourth equation with φ_2, and the sixth equation with φ_3, integrate the three equations over Q_T and integrate by parts for the term containing Δy. We then add these equations to obtain

$$\begin{aligned} \int_0^T \int_\Omega h\varphi_1 &= \int_0^T \Big[\frac{\partial y}{\partial t}\varphi_1 + \frac{\partial u}{\partial t}\varphi_2 + \frac{\partial v}{\partial t}\varphi_3\Big]dt \\ &+ \int_0^T \int_\Omega \{\nabla y \nabla\varphi_1 + [R'(\overline{y})y + a_1u + a_2v]\varphi_1\}dxdt \\ &+ \int_0^T \int_\Omega \{[bu + b_0y]\varphi_2 + [cv + c_0y]\varphi_3\}dxdt. \end{aligned} \tag{3.10}$$

Next, we multiple the first equation of (3.8) with y, the fourth equation with u, and the sixth equation with v, integrate the three equations over Q_T and integrate by parts in the term containing Δy. We then add these equations to obtain

$$\begin{aligned} &\int_0^T \int_\Omega [C_Q^y(\overline{y} - y_Q)y + C_Q^u(\overline{u} - u_Q)u + C_Q^v(\overline{v} - v_Q)v]dxdt \\ &= -\int_\Omega [\varphi_1(\cdot,T)y(\cdot,T) + \varphi_2(\cdot,T)u(\cdot,T) + \varphi_3(\cdot,T)v(\cdot,T)]dx \\ &+ \int_0^T \Big[\frac{\partial y}{\partial t}\varphi_1 + \frac{\partial u}{\partial t}\varphi_2 + \frac{\partial v}{\partial t}\varphi_3\Big]dt \\ &+ \int_0^T \int_\Omega \{\nabla y \nabla\varphi_1 + [R'(\overline{y})y + a_1u + a_2v]\varphi_1\}dxdt \\ &+ \int_0^T \int_\Omega \{[bu + b_0y]\varphi_2 + [cv + c_0y]\varphi_3\}dxdt. \end{aligned} \tag{3.11}$$

From (3.10) and (3.11), we have that

$$\begin{aligned}
&\int_0^T \int_\Omega [C_Q^y(\overline{y}-y_Q)y + C_Q^u(\overline{u}-u_Q)u + C_Q^v(\overline{v}-v_Q)v]dxdt \\
&\quad = -\int_\Omega [\varphi_1(\cdot,T)y(\cdot,T) + \varphi_2(\cdot,T)u(\cdot,T) + \varphi_3(\cdot,T)v(\cdot,T)]dx \\
&\qquad + \int_0^T \int_\Omega h\varphi_1 dxdt.
\end{aligned} \tag{3.12}$$

This is equivalent to the statement of the lemma. □

Theorem 3.4. *(Necessary optimality conditions). Let $(\overline{y},\overline{u},\overline{v},\overline{w})$ be a local solution to the optimal control problem. Then, there exists a unique triple $(\varphi_1,\varphi_2,\varphi_3) \in (W(0,T))^3$ of adjoint states solving the adjoint system and a function $\lambda \in L^\infty(Q_T)$ such that*

$$\int_0^T \int_\Omega (\varphi_1(x,t) + k\overline{w}(x,t) + \mu\overline{\lambda}(x,t))(w(x,t)-\overline{w}(x,t)))dxdt \geq 0, \forall w \subseteq U_{a,d}. \tag{3.13}$$

Proof. The theorem follows from Lemma 3.2 and Lemma 3.3. □

In case that $k > 0$, from [1,17], the following standard projection formula can be obtained

$$\overline{w}(x,t) = P_{[a,b]}\left\{-\frac{1}{k}(\varphi_1(x,t) + \mu\overline{\lambda}(x,t))\right\}, \forall (x,t) \in Q_T \ a.e. \tag{3.14}$$

Further, we have the following results from [3].

Theorem 3.5. *Assume that $k > 0, \mu > 0$, Then, for almost all $(x,t) \in Q_T$, there holds*

$$\overline{w}(x,t) = 0 \Leftrightarrow \{|\overline{\varphi}_1| \leq \mu, if a < 0; \overline{\varphi}_1 \geq -\mu, if a = 0\}. \tag{3.15}$$

and

$$\overline{\lambda}(x,t) = P_{[-1,1]}\Big\{-\frac{1}{\mu}(\varphi_1(x,t)\Big\}. \tag{3.16}$$

4 Applications

In this section, we present some examples of the optimal control of the FitzHugh-Nagumo neurons systems with general form (1.1).

Example 1 ([3, 4, FitzHugh-Nagumo system]). Let $R(y), p_1(y), p_2(y)$ take the following form

$$\begin{aligned}
&R(y) = \alpha_1 y^3 + \alpha_2 y^2 + \alpha_3 y + \alpha_4, \\
&p_1(y) = -\gamma y + \delta, \\
&\quad \alpha_1 > 0, \alpha_i \in \mathbb{R}^1 (i = 2,3,4.), \gamma > 0, \delta > 0, a_2 = 0.
\end{aligned} \tag{4.1}$$

In this case, the state systems is as following.

$$\begin{array}{rcll}
\partial y/\partial t - d_0\triangle y + (\alpha_1 y^3 + \alpha_2 y^2 + \alpha_3 y + \alpha_4) + a_1 u & = & w(x,t), & (x,t) \in Q_T, \\
\partial_n y & = & 0, & (x,t) \in \Sigma_T; \\
y(x,0) & = & y_0(x), & (x,t) \in \Omega; \\
\partial u/\partial t + bu - \gamma y + \delta & = & 0, & (x,t) \in Q_T, \\
u(x,0) & = & u_0(x), & (x,t) \in \Omega;
\end{array} \tag{4.2}$$

the objective functional is

$$\begin{aligned} I(w) : &= \frac{1}{2}\int_0^T \int_\Omega [C_Q^y(y_w(x,t) - y_Q(x,t))^2 + C_Q^u(u_w(x,t) - u_Q(x,t))^2]dxdt \\ &+ \frac{1}{2}\int_\Omega [C_T^y(y_w(x,T) - y_T(x))^2 + C_T^u(u_w(x,T) - u_T(x))^2]dxdt \\ &+ \frac{\kappa}{2}\int_0^T \int_\Omega (w)^2(x,t)dxdt \\ j(w) :&= \int_0^T \int_\Omega \mid w(x,t) \mid dxdt. \end{aligned} \tag{4.3}$$

Obviously, hypotheses $(\mathrm{H})_1$ and $(\mathrm{H})_2$ are satisfied, from Theorem 3.1 and Theorem 3.4, we have that

Theorem 4.1. *The optimal control problem (4.2) with (4.3) has at least one optimal solution $\bar{w}$ with associated optimal state*

$$(\bar{y}(\bar{w}), \bar{u}(\bar{w})) := G(\bar{w}). \tag{4.4}$$

If $(\overline{y}, \overline{u}, \overline{w})$ is a local solution to the optimal control problem, then there exists a unique pair $(\varphi_1, \varphi_2) \in (W(0,T))^2$ of adjoint states solving the adjoint system and a function $\lambda \in L^\infty(Q_T)$ such that

$$\int_0^T \int_\Omega (\varphi_1(x,t) + k\overline{w}(x,t) + \mu\overline{\lambda}(x,t))(w(x,t) - \overline{w}(x,t)))dxdt \geq 0, \forall w \subseteq U_{a,d}. \tag{4.5}$$

The adjoint system is as following.

$$\begin{array}{rcll} -\partial\varphi_1/\partial t - d_0\triangle\varphi_1 + R'(\overline{y})\varphi_1 + \gamma\varphi_2 &=& C_Q^y(\overline{y} - y_Q), & (x,t) \in Q_T, \\ \partial_n\varphi_1 &=& 0, & (x,t) \in \Sigma_T; \\ \varphi_1(x,T) &=& C_T^y(x)(\overline{y}(x,T) - y_T(x)), & (x,t) \in \Omega; \\ -\partial\varphi_2/\partial t + b\varphi_2 + a_1\varphi_1 &=& C_Q^u(\overline{u} - u_Q), & (x,t) \in Q_T, \\ \varphi_2(x,T) &=& C_T^u(x)(\overline{u}(x,T) - u_T(x)), & (x,t) \in \Omega. \end{array} \tag{4.6}$$

Remark 4.2. In fact, if $R(y), p_1(y), p_2(y)$ take the following form

$$\begin{aligned} &R(y) = \alpha_1 y^{2k+1} + \alpha_2 y^{2k} + \cdots + \alpha_{2k+1}y + \alpha_{2k+2}, \\ &p_1(y) = -\gamma y + \delta, \\ &\quad \alpha_1 > 0, \alpha_i \in \mathbb{R}^1(i = 2, 3, \ldots 2k+2.), \gamma > 0, \delta > 0, a_2 = 0. \end{aligned} \tag{4.7}$$

Then Theorem 4.1 still holds.

Example 2. Let $R(y), p_1(y), p_2(y)$ take the following form

$$\begin{aligned} R(y) &= \frac{1}{3}y^3 - y, \\ p_1(y) &= -cy - \delta_1, \\ p_2(y) &= c_2 y - \delta_2, \\ \delta_i &> 0, \quad i = 1, 2. \end{aligned} \tag{4.8}$$

We obtain three coupled reaction-diffusion with differential equations which arising computational neuroscience similar to system (1.5) as the following.

$$\begin{array}{rcll} \partial y/\partial t - d_0\Delta y + \frac{y^3}{3} - y + u - v & = & F(t), & (x,t)\in Q_T,\\ \partial_n y & = & 0, & (x,t)\in \Sigma_T;\\ y(x,0) & = & y_0(x), & (x,t)\in\Omega;\\ \partial u/\partial t + bu - cy - \delta_1 & = & 0, & (x,t)\in Q_T,\\ u(x,0) & = & u_0(x), & (x,t)\in\Omega;\\ \partial v/\partial t + b_2 v + c_2 y - \delta_2 & = & 0, & (x,t)\in Q_T,\\ v(x,0) & = & v_0(x), & (x,t)\in\Omega. \end{array} \tag{4.9}$$

Hypotheses $(\mathrm{H})_1$ and $(\mathrm{H})_2$ are satisfied. From Theorem 3.1 and Theorem 3.4 we have that

Theorem 4.3. *The optimal control problem (4.9) with (1.3) has at least one optimal solution $\bar{w}$ with associated optimal state*

$$(\bar{y}(\bar{w}),\bar{u}(\bar{w}),\bar{v}(\bar{w})) := G(\bar{w}). \tag{4.10}$$

If $(\overline{y},\overline{u},\overline{v},\overline{w})$ is a local solution to the optimal control problem, then there exists a unique pair $(\varphi_1,\varphi_2,\varphi_3)\in(W(0,T))^3$ of adjoint states solving the adjoint system and a function $\lambda\in L^\infty(Q_T)$ such that

$$\int_0^T\int_\Omega(\varphi_1(x,t)+k\overline{w}(x,t)+\mu\overline{\lambda}(x,t))(w(x,t)-\overline{w}(x,t)))dxdt\ge 0,\forall w\subseteq U_{a,d}. \tag{4.11}$$

The adjoint system is

$$\begin{array}{rcll} -\partial\varphi_1/\partial t - d_0\triangle\varphi_1 + (\overline{y}^2-1)\varphi_1 + c\varphi_2 - c_2\varphi_3 & = & C_Q^y(\overline{y}-y_Q) & (x,t)\in Q_T,\\ \partial_n\varphi_1 & = & 0 & (x,t)\in\Sigma_T,\\ \varphi_1(x,T) & = & C_T^y(x)(\overline{y}(x,T)-y_T(x)) & (x,t)\in\Omega;\\ -\partial\varphi_2/\partial t + b\varphi_2 + \varphi_1 & = & C_Q^u(\overline{u}-u_Q) & (x,t)\in Q_T,\\ \varphi_2(x,T) & = & C_T^u(x)(\overline{u}(x,T)-u_T(x)) & (x,t)\in\Omega;\\ -\partial\varphi_3/\partial t + b_2\varphi_3 - \varphi_1 & = & C_Q^v(\overline{v}-v_Q) & (x,t)\in Q_T,\\ \varphi_3(x,T) & = & C_T^v(x)(\overline{v}(x,T)-v_T(x)) & (x,t)\in\Omega. \end{array} \tag{4.12}$$

References

[1] E. Casas, Pontryagin's principle for state-constrained boundary control problems of semilinear parabolic equations, *SIAM J. Control Optim.* 35 (1997) 1297–1327.

[2] E. Casas, J.C. de los Reyes and F. Troltzsch, Sufficient second-order optimality conditions for semilinear control problems with pointwise state constraints, *SIAM J. Optim.* 19 (2008) 616–643.

[3] E. Casas, C. Ryll and F. Troltzsch, Sparse optimal control of the Schlgl and FitzHugh-Nagumo systems, *Comp. Methods Appl. Math.* 13 (2013) 415-442.

[4] R. FitzHugh, Impulses and physiological states in theoretical models of nerve membrane,*Biophys. J.* 1 (1961) 445–465.

[5] S.P. Hastings, Some mathematical problems from neurobiology, *Amer. Math. Monthly* 82 (1975) 881–895.

[6] A.L. Hodgkin and A.F. Huxley, A quantitative description of membrane current and its application to conduction and excitation in nerve, *J. Physiol.* 117 (1952) 500–44.

[7] M. Hintz, R. Pinnau and M. Ulbrich, *Optimization with PDF Constraints*, Springer, 2009.

[8] D.E. Jackson, Existence and regularity for the FitzHugh-Nagumo equations with inhomogeneous boundary conditions, *Nonlinear Anal.* 14 (1990) 201–216.

[9] W. Jiang, D. Bin and K. M. Tsang, Chaotic synchronization of neurons coupled with gap junction under external electrical stimulation, *Chaos, Solitons and Fractals* 22 (2004) 469–476.

[10] Y. Jiang, Y. He and J. Sun, Proximal analysis and the minimal time function of a class of semilinear control systems, *J. Optim. Theory. Appl.* 169 (2016) 784–800.

[11] O.A. Ladyzhenskaya and N.N. Uralceva, *Linear and Quasilinear Equations of Parabolic Type,* American Mathematical Society, Providence, 1968.

[12] D. Mishra, A. Yadav, S. Ray and P.K. Kalra, Controlling synchronization of modified FitzHugh-Nagumo neurons under external electrical stimulation, *NeuroQuantology* 1 (2006) 50–67.

[13] J.D. Murray, *Mathematical Biology,* 2nd ed., Biomathematics, 19, Springer, Berlin, 1993.

[14] J-P. Raymond and H. Zidani, Hamiltonian Pontryagin's principles for control problems governed by semilinear parabolic equations, *Applied Math. Optim.* 39 (1999)143–177.

[15] J. Smoller, *Shock Waves and Reaction-Diffusion Equations,* 2nd ed., Grundlehren Math. Wiss., 258, Springer, New York, 1994.

[16] C.J. Thompson, D.C. Bardos, Y.S. Yang and K.H. Joyner, Nonlinear cable models for cells exposed to electric fields I.General Theory and Space-clamped Solutions, *Chaos, Solitons and Fractlas* 10 (1999) 1825–1842.

[17] F. Troltzsch, *Optimal Control of Partial Differential Equations: Theory, Methods and Applications,* Grad. Stud. Math., 112, American Mathematical Society, Providence, 2010.

Manuscript received 25 May 2015
revised 8 October 2015
accepted for publication 10 October 2015

JIE SUN
Department of Mathematics and Statistics
Curtin University, Australia and
School of Business, National University of Singapore
E-mail address: jie.sun@curtin.edu.au

WANLI YANG
Zhuhai College of Jilin University, Zhuhai, PRC
Email address: bj630523@163.com