



SOLUTION-EXISTENCE AND ALGORITHMS WITH THEIR CONVERGENCE RATE FOR STRONGLY PSEUDOMONOTONE EQUILIBRIUM PROBLEMS*

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Abstract: We show solution-existence and develop algorithms for solving strongly pseudomonotone equilibrium problems in real Hilbert spaces. We study convergence rate for the proposed algorithms. Application to variational inequalities is discussed.

Key words: *strongly pseudomonotone equilibria, solution existence, algorithm, convergence rate.*

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1 Introduction

Throughout the paper, we suppose that \mathcal{H} is a real Hilbert space endowed with weak topology defined by the inner product $\langle \cdot, \cdot \rangle$ and its reduced norm $\|\cdot\|$. Let $C \subseteq \mathcal{H}$ be a nonempty closed convex subset and $f : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying $f(x, x) = 0$ for every $x \in C$. As usual we call such a bifunction an *equilibrium bifunction*. We consider the following equilibrium problem defined as

$$\text{Find } x^* \in C : f(x^*, x) \geq 0 \quad \forall x \in C. \quad (EP)$$

This inequality was first used by Nikaido and Isoda in [20] for noncooperative game. After the publication of the paper by Blum and Oettli [5], Problem (EP) has attracted many attention and a large number of articles on this problem have been published (see. e.g. the monograph [14] and survey paper [4] and the references therein).

An interesting feature of Problem (EP) is that, although having a very simple formulation, it gives a unified formulation for some important problems such as optimization problems, saddle point, variational inequalities, Kakutani fixed point and Nash equilibria, in the sense that it includes these problems as particular cases (see for instance [4, 5, 7, 18]).

An important approach for solving Problem (EP) is the auxiliary principle. This principle was first proposed by Cohen for optimization problem. Then it was used for variational inequalities [7, 11], and further extended to monotone equilibrium problems [17, 19, 21]. In an algorithm based upon the auxiliary principle, at each iteration k , it requires to solve a strongly convex minimization subproblem. Under suitable conditions, such an algorithm is

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convergent for strongly monotone problems, however it may fail to converge for monotone ones. In the latter case the extragradient (double projection) method, first proposed by Korpelevich [15], can be used to ensure the convergence (see e.g. [23, 24, 26]).

The concept of strongly pseudomonotone operator, to our best knowledge, has been introduced by Fagouq in [8] and recently studied in [13]. This notion then has been extended to bifunctions.

The aim of this paper is first to show the solution existence, then to use the auxiliary problem principle to develop three algorithms for solving strongly pseudomonotone equilibrium problem (EP) and to investigate their convergence rate. Thanks to strong pseudomonotonicity, the proposed algorithms require, at each iteration, to solve only one strongly convex program, rather than two programs as in an extragradient algorithm for monotone and pseudomonotone equilibrium problems. Moreover, linear convergence is obtained for the first algorithm, and in the last algorithm, the moving direction does not take only the objective bifunction, but also the feasible domain into account.

The paper is organized as follows. The next section contains preliminaries. The third section is devoted to existence solution for strongly pseudomonotone equilibrium problems. In the last section we describe three algorithms for solving strongly pseudomonotone equilibrium problems and discuss their convergence rate.

2 Preliminaries

We recall the following well-known definition on monotonicity (see e.g. [2]).

Definition 2.1. A bifunction $f : C \times C \rightarrow \mathbb{R}$ is said to be

- (i) strongly monotone on C with modulus $\beta > 0$ (shortly β -strongly monotone) if

$$f(x, y) + f(y, x) \leq -\beta\|y - x\|^2, \quad \forall x, y \in C;$$

- (ii) monotone on C if

$$f(x, y) + f(y, x) \leq 0, \quad \forall x, y \in C;$$

- (iii) strongly pseudomonotone on C with modulus $\beta > 0$ (shortly β -strongly pseudomonotone) if

$$f(x, y) \geq 0 \implies f(y, x) \leq -\beta\|y - x\|^2, \quad \forall x, y \in C;$$

- (iv) pseudomonotone on C if

$$f(x, y) \geq 0 \implies f(y, x) \leq 0, \quad \forall x, y \in C.$$

Note that a strongly pseudomonotone bifunction may not be monotone (see the example at the end of Section 4).

The following blanket assumptions will be used for the bifunction $f : C \times C \rightarrow \mathbb{R}$:

(A1) $f(\cdot, y)$ is upper semicontinuous for each $y \in C$;

(A2) $f(x, \cdot)$ is closed, convex and subdifferentiable on C for each $x \in C$;

(A2a) $f(x, \cdot)$ is closed, convex on C for each $x \in C$.

Note that under Assumption (A2a) the function $f(x, \cdot)$ may not be subdifferentiable on C , but it is ϵ -subdifferentiable on C for every $\epsilon > 0$.

The following Lipschitz-type condition introduced in [17] will be used in the sequel.

$$\exists L_1, L_2 > 0 : f(x, y) + f(y, z) \geq f(x, z) - L_1 \|x - y\|^2 - L_2 \|y - z\|^2, \quad \forall x, y, z \in C \quad (2.1)$$

It is clear that for optimization problem $\min_{x \in C} \varphi(x)$, the bifunction $f(x, y) := \varphi(y) - \varphi(x)$ has property (2.1) for any function φ defined on C .

Furthermore, for the variational inequality case when $f(x, y) := \langle F(x), y - x \rangle$ with $F : C \rightarrow \mathcal{H}$, it is not hard to show (see e.g. [23]) that if F is Lipschitz on C with constant $L > 0$, then for any $\mu > 0$ one has

$$f(x, y) + f(y, z) \geq f(x, z) - \frac{L\mu}{2} \|x - y\|^2 - \frac{L}{2\mu} \|y - z\|^2, \quad \forall x, y, z \in C,$$

that is f satisfies the Lipschitz-type condition (2.1) with $L_1 = \frac{L\mu}{2}$ and $L_2 = \frac{L}{2\mu}$.

3 Solution Existence

In this section we show that a strongly pseudomonotone equilibrium problem always admits a solution. The following lemma, that will be used to prove Proposition 3.2 below, is a direct consequence of Theorem 3.2 in [3].

Lemma 3.1. *Let $f : C \times C \rightarrow \mathbb{R}$ be an equilibrium bifunction such that $f(\cdot, y)$ is hemicontinuous for each $y \in C$ and $f(x, \cdot)$ is lower semicontinuous convex for each $x \in C$. Suppose that the following coercivity condition holds*

$$\exists \text{ compact set } W : (\forall x \in C \setminus W, \exists y \in C : f(x, y) < 0).$$

Then the equilibrium problem (EP) has a solution.

The following result seems has not been appeared in the literature.

Proposition 3.2. *Suppose that f is β -strongly pseudomonotone on C , then under Assumptions (A1) and (A2a), Problem (EP) has a unique solution.*

Proof. First, suppose that C is unbounded. Then by Lemma 3.1 it is sufficiency to prove the following coercivity condition:

$$\exists \text{ closed ball } B : (\forall x \in C \setminus B, \exists y \in C \cap B : f(x, y) < 0). \quad (C0)$$

Indeed, otherwise, for every closed ball B_r around 0 with radius r , there exists $x^r \in C \setminus B_r$ such that $f(x, y) \geq 0 \forall y \in C \cap B_r$.

Fix $r_0 > 0$, then for every $r > r_0$, there exists $x^r \in C \setminus B_r$ such that $f(x^r, y^0) \geq 0$ with $y^0 \in C \cap B_{r_0}$. Thus, since f is β -strongly pseudomonotone, we have

$$f(y^0, x^r) + \beta \|x^r - y^0\|^2 \leq 0 \quad \forall r. \quad (3.1)$$

On the other hand, since C is convex and $f(y^0, \cdot)$ is convex on C , for $\epsilon_r := 1/r$, it is well known from convex analysis that there exists $x^0 \in C$ such that $\partial_2^{\epsilon_r} f(y^0, x^0) \neq \emptyset$, where $\partial_2^{\epsilon_r} f(y^0, x^0)$ stands for the ϵ_r -subdifferential of the convex function $f(y^0, \cdot)$ at x^0 . Take $w^* \in \partial_2^{\epsilon_r} f(y^0, x^0)$, by definition of ϵ_r -subgradient one has

$$f(y^0, x) + 1/r \geq \langle w^*, x - x^0 \rangle + f(y^0, x^0) \quad \forall x.$$

With $x = x^r$ it yields

$$\begin{aligned} f(y^0, x^r) + \beta \|x^r - y^0\|^2 + 1/r &\geq f(y^0, x^0) + \langle w^*, x^r - x^0 \rangle + \beta \|x^r - y^0\|^2 \\ &\geq f(y^0, x^0) - \|w^*\| \|x^r - x^0\| + \beta \|x^r - y^0\|^2. \end{aligned}$$

Letting $r \rightarrow \infty$, since $\|x^r\| \rightarrow \infty$, we obtain $f(y^0, x^r) + \beta \|x^r - y^0\|^2 \rightarrow \infty$ which contradicts (3.1). Thus the coercivity condition (C0) must hold true. Then by virtue of Lemma 3.1, Problem (EP) admits a solution.

In the case when C is bounded, the proposition is a consequence of Ky Fan's theorem [9].

The uniqueness of the solution is immediate from the strong pseudomonotonicity of f . \square

We recall [10] that an operator $F : C \rightarrow \mathcal{H}$ is said to be *strongly pseudomonotone* on C with modulus $\beta > 0$, shortly β -strongly pseudomonotone, if

$$\langle F(x), y - x \rangle \geq 0 \Rightarrow \langle F(y), y - x \rangle \geq \beta \|y - x\|^2 \quad \forall x, y \in C.$$

In order to apply the above proposition to the variational inequality problem

$$\text{Find } x^* \in C : \langle F(x^*), y - x^* \rangle \geq 0 \quad \forall y \in C, \quad (VI)$$

where F is a strongly pseudomonotone operator on C , we define the bifunction f by taking

$$f(x, y) := \langle F(x), y - x \rangle. \quad (3.2)$$

It is obvious that x^* is a solution of (VI) if and only if it is a solution of Problem (EP) with f defined by (3.2). Moreover, it is easy to see that F is β -strongly pseudomonotone and upper semicontinuous on C if and only if so is f . The following solution existence result is an immediate consequence of Proposition 3.2.

Corollary 3.3. *Suppose that F is hemicontinuous and strongly pseudomonotone on C . Then variational inequality problem (VI) has a unique solution.*

4 Algorithms and Their Convergence Rate

Following the auxiliary problem principle, for each $x \in C$, we define the mapping s by taking

$$s(x) := \operatorname{argmin}_{y \in C} \left\{ \rho f(x, y) + \frac{1}{2} \|y - x\|^2 \right\} \quad (4.1)$$

where $\rho > 0$. Since $f(x, \cdot)$ is closed, convex on the closed, convex set C , the mapping s is well-defined.

The following well-known lemma will be used in the sequel.

Lemma 4.1 ([17]). *Let s be defined by (4.1). Then, under Assumptions (A1), (A2), x^* is a solution of (EP) if and only if $x^* = s(x^*)$.*

4.1 A Linearly Convergent Algorithm

We recall that a sequence $\{z^k\}$ strongly linearly converges to z^* if there exists a number $t \in (0, 1)$ and an index k_0 such that $\|z^{k+1} - z^*\| \leq t \|z^k - z^*\|$ for every $k \geq k_0$.

Proposition 4.2. *Suppose that f is strongly pseudomonotone on C with modulus β . Then under Assumptions (A1), (A2) and the Lipschitz-type condition (2.1), for any starting point $x^0 \in C$, the sequence $\{x^k\}_{k \geq 0}$ defined by*

$$x^{k+1} := \operatorname{argmin}_{y \in C} \left\{ \rho f(x^k, y) + \frac{1}{2} \|y - x^k\|^2 \right\} \tag{4.2}$$

satisfies

$$[1 + 2\rho(\beta - L_2)] \|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 \tag{4.3}$$

provided $0 < \rho \leq \frac{1}{2L_1}$, where x^* denotes the unique solution of (EP).

Proof. For each $k \geq 0$, for simplicity of notation, let

$$f_k(x) := \rho f(x^k, x) + \frac{1}{2} \|x - x^k\|^2. \tag{4.4}$$

By Assumption (A2), the function f_k is strongly convex with modulus 1 and subdifferentiable, which implies

$$f_k(x^{k+1}) + \langle g^k, x - x^{k+1} \rangle + \frac{1}{2} \|x - x^{k+1}\|^2 \leq f_k(x), \quad \forall x \in C \tag{4.5}$$

for any $g^k \in \partial f_k(x^{k+1})$. Since x^{k+1} is defined by (4.2), using the optimality condition for convex programming, we have $0 \in \partial f_k(x^{k+1}) + N_C(x^{k+1})$, which implies that there exists $-g^k \in N_C(x^{k+1})$ such that $\langle g^k, x - x^{k+1} \rangle \geq 0, \forall x \in C$. Hence, from (4.5), it follows that

$$f_k(x^{k+1}) + \frac{1}{2} \|x - x^{k+1}\|^2 \leq f_k(x), \quad \forall x \in C. \tag{4.6}$$

Replacing $x = x^*$ in (4.6) and using the definition (4.4) of f_k we get

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 + 2\rho[f(x^k, x^*) - f(x^k, x^{k+1})] - \|x^{k+1} - x^k\|^2. \tag{4.7}$$

Applying the Lipschitz-type condition (2.1) with $x = x^k, y = x^{k+1}, z = x^*$, we obtain

$$\begin{aligned} f(x^k, x^{k+1}) + f(x^{k+1}, x^*) &\geq f(x^k, x^*) - L_1 \|x^k - x^{k+1}\|^2 - L_2 \|x^{k+1} - x^*\|^2 \\ \Rightarrow f(x^k, x^*) - f(x^k, x^{k+1}) &\leq f(x^{k+1}, x^*) + L_1 \|x^{k+1} - x^k\|^2 + L_2 \|x^{k+1} - x^*\|^2 \end{aligned} \tag{4.8}$$

Since x^* is a solution of (EP), $f(x^*, x^{k+1}) \geq 0$. Then, by the strong pseudomonotonicity of f , we have

$$f(x^{k+1}, x^*) \leq -\beta \|x^{k+1} - x^*\|^2. \tag{4.9}$$

From (4.8) and (4.9), it follows that

$$\begin{aligned} f(x^k, x^*) - f(x^k, x^{k+1}) &\leq -\beta \|x^{k+1} - x^*\|^2 + L_1 \|x^k - x^{k+1}\|^2 + L_2 \|x^{k+1} - x^*\|^2 \\ &= -(\beta - L_2) \|x^{k+1} - x^*\|^2 + L_1 \|x^{k+1} - x^k\|^2. \end{aligned} \tag{4.10}$$

Replacing (4.10) to (4.7), by the choice of ρ , we can write

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &\leq \|x^k - x^*\|^2 + 2\rho[-(\beta - L_2) \|x^{k+1} - x^*\|^2 + L_1 \|x^{k+1} - x^k\|^2] \\ &\quad - \|x^{k+1} - x^k\|^2 \\ \Leftrightarrow [1 + 2\rho(\beta - L_2)] \|x^{k+1} - x^*\|^2 &\leq \|x^k - x^*\|^2 - (1 - 2\rho L_1) \|x^{k+1} - x^k\|^2 \\ &\leq \|x^k - x^*\|^2. \end{aligned}$$

The proposition is thus proved. □

Based upon Proposition 4.2 we can develop the following linearly convergent algorithm for strongly pseudomonotone problems satisfying the Lipschitz-type condition (2.1). As usual, we call a point $x \in C$ an ε -solution to (EP) if $\|x^k - x^*\| \leq \varepsilon$, where x^* is the exact solution of (EP).

Algorithm 4.3. Choose a tolerance $\varepsilon \geq 0$ and $0 < \rho < \frac{1}{2L_1}$.

Take $x^0 \in C$ and $k = 0$.

Step 1: Solve the strongly convex program

$$\min \left\{ \rho f(x^k, y) + \frac{1}{2} \|y - x^k\|^2 : y \in C \right\}$$

to obtain its unique solution x^{k+1} .

Step 2: If $\frac{\alpha}{1-\alpha} \|x^{k+1} - x^k\| \leq \varepsilon$ where $\alpha := \frac{1}{\sqrt{1+2\rho(\beta-L_2)}}$, then terminate: x^{k+1} is an ε -solution to (EP). Otherwise let $k \leftarrow k+1$ and go to *Step 1*.

Note that for variational inequality (VI), when $f(x, y) := \langle F(x), y - x \rangle$, solving the strongly convex program in Step 1 amounts to computing the projection of the vector $x^k - \frac{1}{\rho} F(x^k)$ onto C , that is $x^{k+1} = P_C(x^k - \frac{1}{\rho} F(x^k))$.

The following convergence result is immediate from Proposition 4.2.

Theorem 4.4. Suppose that $L_2 < \beta$ and $0 < \rho \leq \frac{1}{2L_1}$. Then the sequence $\{x^k\}$ generated by Algorithm 4.3 converges linearly to the unique solution x^* of (EP) and we have the estimation

$$\|x^{k+1} - x^*\| \leq \frac{\alpha^{k+1}}{1-\alpha} \|x^1 - x^0\|, \quad \forall k \geq 0 \quad (4.11)$$

where $\alpha := \frac{1}{\sqrt{1+2\rho(\beta-L_2)}} \in (0, 1)$.

4.2 An Algorithm without Knowledge of Lipschitz Constants

Algorithm 4.3 has a disadvantage that, for determining the regularization ρ , it requires knowing Lipschitz constants in advance. Algorithm 4.5 below can avoid this disadvantage. However it should be mentioned that although this algorithm does not require to know the Lipschitz constants, it now requires the use of stepsizes converging to 0, which may be viewed as a practical disadvantage.

Algorithm 4.5. *Initialization:* Choose a tolerance $\varepsilon \geq 0$ and a sequence $\{\rho_k\}_{k \geq 0} \subset (0, \infty)$ of positive numbers satisfying

$$\sum_{k=0}^{\infty} \rho_k = \infty, \quad \lim_{k \rightarrow \infty} \rho_k = 0$$

Take $x^0 \in C$ and $k = 0$.

Step 1: Solve the strongly convex program

$$\min_{y \in C} \left\{ \rho_k f(x^k, y) + \frac{1}{2} \|y - x^k\|^2 \right\}$$

to obtain its unique solution x^{k+1} .

If $\|x^{k+1} - x^k\| \leq \epsilon$, terminate. Otherwise, increase k by 1 and go back to *Step 1*.

The convergence of $\{x^k\}$ can be stated as follows.

Theorem 4.6. *Suppose that f is β -strongly pseudomonotone on C and satisfies Assumptions (A1), (A2), and the Lipschitz-type condition (2.1) with $L_2 < \beta$. Let $\{x^k\}_{k \geq 0}$ be the sequence generated by Algorithm 4.5 and x^* be the unique solution of (EP). Then there exists an index $k_0 \in \mathbb{N}$ such that for each $k > k_0$, one has*

$$\|x^{k+1} - x^*\| \leq \frac{1}{\sqrt{\prod_{i=k_0}^k [1 + 2\rho_k(\beta - L_2)]}} \|x^{k_0} - x^*\|. \tag{4.12}$$

In addition, it holds that

$$\lim_{k \rightarrow \infty} \frac{1}{\sqrt{\prod_{i=k_0}^k [1 + 2\rho_k(\beta - L_2)]}} = 0, \tag{4.13}$$

and therefore $\{x^k\}$ converges strongly to x^* .

Proof. Using the same argument as in the above proof, for each k we have

$$[1 + 2\rho_k(\beta - L_2)] \|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - (1 - 2\rho_k L_1) \|x^{k+1} - x^k\|^2$$

Since $\lim_{k \rightarrow \infty} \rho_k = 0$, there exists $k_0 \in \mathbb{N}$ such that $1 - 2\rho_k L_1 > 0, \forall k \geq k_0$. Hence

$$[1 + 2\rho_k(\beta - L_2)] \|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 \quad \forall k \geq k_0,$$

which implies

$$\|x^{k+1} - x^*\| \leq \frac{1}{\sqrt{1 + 2\rho_k(\beta - L_2)}} \|x^k - x^*\| \quad \forall k \geq k_0.$$

Hence

$$\|x^{k+1} - x^*\| \leq \frac{1}{\sqrt{\prod_{i=k_0}^k [1 + 2\rho_i(\beta - L_2)]}} \|x^{k_0} - x^*\|.$$

To see (4.13), we let $\alpha_k := 2\rho_k(\beta - L_2) > 0$, then

$$\sum_{k=k_0}^{\infty} \alpha_k = 2(\beta - L_2) \sum_{k=k_0}^{\infty} \rho_k = \infty,$$

which implies

$$\frac{1}{\prod_{i=k_0}^k (1 + \alpha_i)} \leq \frac{1}{1 + \sum_{i=k_0}^k \alpha_i} \rightarrow 0, \text{ as } k \rightarrow \infty.$$

Thus from (4.12) we see that $x^k \rightarrow x^*$ as $k \rightarrow \infty$. □

The following example shows that Algorithm 4.5 is not linearly convergent. Let $C = \mathcal{H} = \mathbb{R}$ and $f(x, y) = x(y - x)$. Clearly, $f(x, y)$ is 1-strongly monotone on C and satisfies the Lipschitz-type condition with $L_1 = L_2 = \frac{1}{2}$. Problem (EP) has a unique solution

$x^* = 0$. Let $\{\rho_k\}_{k \geq 0} \subset (0, 1)$ such that $\rho_k \rightarrow 0$ as $k \rightarrow \infty$. Starting from any point $x^0 \neq 0$. According to the algorithm

$$\begin{aligned} x^{k+1} &= \operatorname{argmin}_{y \in C} \left\{ \rho_k f(x^k, y) + \frac{1}{2} \|y - x^k\|^2 \right\} \\ &= \operatorname{argmin}_{y \in C} \left\{ \rho_k x^k (y - x^k) + \frac{1}{2} \|y - x^k\|^2 \right\} = (1 - \rho_k) x^k. \end{aligned}$$

which together with $\lim_{k \rightarrow \infty} \rho_k = 0$ and $x^k \neq 0$ for all $k \in \mathbb{N}$, imply that $\{x^k\}$ does not converge linearly to the unique solution $x^* = 0$.

4.3 An Algorithm without Lipschitz Condition

Algorithm 4.5 above does not require to know the Lipschitz constants in advance, but its convergence needs the Lipschitz-type condition. In this subsection we propose a strongly convergent algorithm which does not require f to satisfy the Lipschitz-type condition.

The following well-known lemma will be used to prove the convergence result.

Lemma 4.7. *Suppose that $\{\alpha_k\}_0^\infty$ is an infinite sequence of positive numbers satisfying*

$$\alpha_{k+1} \leq \alpha_k + \xi_k \quad \forall k,$$

with $\sum_{k=0}^\infty \xi_k < \infty$. Then the sequence $\{\alpha_k\}$ is convergent.

Algorithm 4.8. *Initialization:* Set $x^1 \in C$, choose a tolerance $\epsilon \geq 0$ and a sequence $\{\rho_k\}$ of positive numbers such that

$$\sum_{k=1}^\infty \rho_k = \infty, \quad \sum_{k=1}^\infty \rho_k^2 < \infty. \quad (4.14)$$

Let $k := 1$.

Step 1 (Finding a moving direction) Find $g^k \in \mathcal{H}$ such that

$$f(x^k, y) + \langle g^k, x^k - y \rangle \geq -\rho_k \quad \forall y \in C, \quad (4.15)$$

- a) If $g^k = 0$ and $\rho_k \leq \epsilon$, terminate: x^k is an ϵ -solution.
- b) If $g^k = 0$ and $\rho_k > \epsilon$, go back to Step 1 where k is replaced by $k + 1$.
- c) Otherwise, execute Step 2.

Step 2 (Projection)

Compute $x^{k+1} := P_C(x^k - \rho_k g^k)$.

- a) If $x^{k+1} = x^k$ and $\rho_k \leq \epsilon$, terminate: x^k is an ϵ -solution.
- b) Otherwise, go back to Step 1 where k is replaced by $k + 1$.

Theorem 4.9. *Suppose that Assumptions (A1) and (A2a) are satisfied. Then*

- (i) *if the algorithm terminates at iteration k , x^k is an ϵ -solution.*
- (ii) *It holds that*

$$\|x^{k+1} - x^*\|^2 \leq (1 - 2\beta\rho_k) \|x^k - x^*\|^2 + 2\rho_k^2 + \rho_k^2 \|g^k\|^2 \quad \forall k, \quad (4.16)$$

where x^* is the unique solution of (EP). Furthermore, if the algorithm does not terminate, then the sequence $\{x^k\}$ strongly converges to the solution x^* provided $\{g^k\}$ is bounded.

Proof. (i) If the algorithm terminates at Step 1, then $g^k = 0$ and $\rho_k \leq \epsilon$. Then, by (4.15), $f(x^k, y) \geq -\rho_k \geq -\epsilon$ for every $y \in C$. Hence, x^k is an ϵ -solution. If the algorithm terminates at Step 2, by the same way, one can see that x^k is an ϵ -solution

(ii) Since $x^{k+1} = P_C(x^k - \rho_k g^k)$, one has

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &\leq \|x^k - \rho_k g^k - x^*\|^2 \\ &= \|x^k - x^*\|^2 - 2\rho_k \langle g^k, x^k - x^* \rangle + \rho_k^2 \|g^k\|^2. \end{aligned} \tag{4.17}$$

Applying (4.15) with $y = x^*$ we obtain

$$f(x^k, x^*) + \langle g^k, x^k - x^* \rangle \geq -\rho_k,$$

which implies

$$-\langle g^k, x^k - x^* \rangle \leq f(x^k, x^*) + \rho_k. \tag{4.18}$$

Then it follows from (4.17) that

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 + 2\rho_k (f(x^k, x^*) + \rho_k) + \rho_k^2 \|g^k\|^2. \tag{4.19}$$

Since x^* is the solution, $f(x^*, x^k) \geq 0$, it follows from the β -strong pseudomonotonicity of f that

$$f(x^k, x^*) \leq -\beta \|x^k - x^*\|^2.$$

Combining the last inequality with (4.19) we obtain

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &\leq \|x^k - x^*\|^2 - 2\beta\rho_k \|x^k - x^*\|^2 + 2\rho_k^2 + \rho_k^2 \|g^k\|^2 \\ &= (1 - 2\beta\rho_k) \|x^k - x^*\|^2 + 2\rho_k^2 + \rho_k^2 \|g^k\|^2. \end{aligned} \tag{4.20}$$

Now suppose that the algorithm does not terminate, and that $\|g^k\| \leq C$ for every k . Then it follows from (4.20) that

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &\leq (1 - 2\beta\rho_k) \|x^k - x^*\|^2 + (2 + C^2)\rho_k^2 \\ &= \|x^k - x^*\|^2 - \lambda_k \|x^k - x^*\|^2 + (2 + C^2)\rho_k^2, \end{aligned} \tag{4.21}$$

where $\lambda_k := 2\beta\rho_k$. Since $\sum_{k=1}^\infty \rho_k^2 < \infty$, by virtue of Lemma 4.7, we can conclude that the sequence $\{\|x^k - x^*\|^2\}$ is convergent. In order to prove that the limit of this sequence is 0, we apply inequality (4.21) for $k = 1, \dots, j + 1$ and sum up it from 1 to $j + 1$ to obtain

$$\|x^{j+1} - x^*\|^2 \leq \|x^1 - x^*\|^2 - \sum_{k=1}^j \lambda_k \|x^k - x^*\|^2 + (2 + C^2) \sum_{k=1}^j \rho_k^2,$$

which implies

$$\|x^{j+1} - x^*\|^2 + \sum_{k=1}^j \lambda_k \|x^k - x^*\|^2 \leq \|x^1 - x^*\|^2 + (2 + C^2) \sum_{k=1}^j \rho_k^2. \tag{4.22}$$

Since $\lambda_k := 2\beta\rho_k$, we have

$$\sum_{k=1}^\infty \lambda_k = 2\beta \sum_{k=1}^\infty \rho_k = \infty. \tag{4.23}$$

Note that $\{x^j\}$ is bounded and that $\sum_{k=0}^\infty \rho_k^2 < \infty$ we can deduce from (4.22) and (4.23) that $\|x^j - x^*\|^2 \rightarrow 0$ as $j \rightarrow \infty$. \square

The algorithm described above can be regarded as an extension of the one in [25] in a Hilbert space setting. The main difference lies in the determination of g^k given by formula (4.15). This formula is motivated by the projection-descent method in optimization, where a moving direction must be both descent and feasible. Such a direction thus involves both the objective function and the feasible domain. In fact moving directions defined by (4.15) rely not only upon the gradient or a subgradient as in [25] and other projection algorithms for equilibrium problems, but also upon the feasible set.

Remark 4.10. (i) It is obvious that if g^k is a ρ_k -subgradient of the convex function $f(x^k, \cdot)$ at x^k , then g^k satisfies (4.15).

When $m_k := \inf_{y \in C} f(x^k, y) > -\infty$, it is easy to see that if g^k is any vector satisfying

$$\langle g^k, y - x^k \rangle \leq m_k + \rho_k := t_k \quad \forall y \in C,$$

i.e., g^k is a vector in t_k -normal set $N_C^{t_k}(x^k)$ of C at x^k , then (4.15) holds true.

(ii) For variational inequality (VI) with $f(x, y)$ defined by (3.2), the formula (4.15) takes the form

$$\langle F(x^k), y - x^k \rangle + \langle g^k, x^k - y \rangle \geq -\rho_k \quad \forall y \in C, \tag{4.24}$$

which means that $g^k - F(x^k) \in N_C^{\rho_k}(x^k)$, where $N_C^{\rho_k}(x^k)$ denotes the ρ_k -normal set of C at x^k , that is

$$N_C^{\rho_k}(x^k) := \{w^k : \langle w^k, y - x^k \rangle \leq \rho_k \quad \forall y \in C\}.$$

Remark 4.11. If f is jointly continuous on an open set $\Delta \times \Delta$ containing $C \times C$, then $\{g^k\}$ is bounded whenever $\epsilon_k \rightarrow 0$ (see e.g. Proposition 3.4 in [26]). In the case of variational inequality (VI) with $f(x, y)$ defined by (3.2), if $g^k = F(x^k)$ and F is continuous, then $\{g^k\}$ is bounded if so is $\{x^k\}$.

By using the same example as at the end of the previous section we can see that Algorithm 4.8 is not linearly convergent.

We close the paper with an example for strongly pseudomonotone bifunction which is not monotone.

For $0 < r < R$, let $C = B(r) := \{x \in \mathcal{H} : \|x\| \leq r\}$ and define f by taking

$$f(x, y) := h(x, y) + (R - \|x\|)g(x, y),$$

where h and g satisfy the following conditions:

- (i) $h(x, y) \leq 0 \quad \forall x, y \in C$ and g is β -strongly monotone on C ;
- (ii) $\exists y^0 \in C : h(0, y^0) + h(y^0, 0) = 0$ and $Rg(0, y^0) + (R - \|y^0\|)g(y^0, 0) > 0$.

To see that f is strongly pseudomonotone on C , we suppose that $f(x, y) \geq 0$. Then, since $h(x, y) \leq 0$, one has $g(x, y) \geq 0$, which, by strong monotonicity of g , implies that $g(y, x) \leq -\beta\|x - y\|^2$. Then, by definition of $f(y, x)$ we have

$$f(y, x) = h(y, x) + (R - \|y\|)g(y, x) \leq -(R - r)\beta\|y - x\|^2 \quad \forall x, y \in C.$$

Hence f is strongly pseudomonotone on C .

To see that f is not monotone on C we use (ii) to get

$$f(0, y^0) + f(y^0, 0) = h(0, y^0) + Rg(0, y^0) + h(y^0, 0) + (R - \|y^0\|)g(y^0, 0) > 0.$$

Thus f is not monotone.

A concrete example for bifunctions g and h that satisfy conditions (i) and (ii) is

$$g(x, y) := \langle x, y - x \rangle + m(\|y\|^2 - \|x\|^2) \quad \text{with } m > 0$$

and

$$h(x, y) := (x - y)^T A(y - x)$$

with $A : \mathcal{H} \rightarrow \mathcal{H}$ being a singular linear operator satisfying $h(x, y) \leq 0$ for every $x, y \in C$. Clearly, g is strongly monotone for every $m > 0$. It is easy to verify that

$$\begin{aligned} Rg(0, y) + (R - \|y\|)g(y, 0) &= [mR - (m + 1)R + (m + 1)\|y\|] \|y\|^2 \\ &= [(m + 1)\|y\| - R]\|y\|^2. \end{aligned}$$

Thus, if $m > \frac{R-r}{r}$, then condition (ii) is satisfied for every $y^0 \in C = B(r)$ with $\|y^0\| > \frac{R}{m+1}$, and $(y^0)^T Ay^0 = 0$.

5 Conclusion

We have shown solution-existence and developed three algorithms for strongly pseudomonotone equilibrium problems with and without Lipschitz-type condition. The proposed algorithms require solving, at each iteration, only one strongly convex program rather than two as in extragradient algorithms for monotone and pseudomonotone equilibrium problems. Convergence rate has been studied.

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