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STABILITY ANALYSIS OF STOCHASTIC MULTIOBJECTIVE OPTIMIZATION PROBLEMS WITH COMPLEMENTARITY CONSTRAINTS*

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Abstract: We study the stability of the optimal solutions and the stationary points of the stochastic multiobjective optimization problems with complementarity constraints (SMOPCC) when the underlying probability measure varies in some metric probability space. We show under some moderate conditions that the optimal solutions and the optimal value function are continuous with respect to probability measure. Based on some new results on stochastic generalized equations, we also show that the set-valued mapping of M-stationary points is upper semi-continuous with respect to the variation of the probability measure. A particular focus is given to the empirical probability measure approximation which is also known as the sample average approximation (SAA). We present that the stationary points of SAA problems converge to their true counterparts with probability one (w.p.1.) at exponential rate as the sample size increases.

Key words: *SMOPCC, stability, probability measure, stationary points.*

Mathematics Subject Classification: *90C15, 90C30, 90C31.*

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1 Introduction

In the past decades, it has been popular to study decision-making problems with multiple noncommunicable objectives, for its significant applications in economics, most notably in welfare and utility theory, management and engineering. In practice, Multiobjective optimization problems (MOP) often involve uncertainty when they are applied to describe decision making problems which involve decision makers with hierarchical relationships and future uncertainty. Consequently, stochastic versions of MOP models are needed. Earlier research on stochastic multiobjective optimization problems can be found in [2, 26, 27].

In this paper, we consider the stochastic multiobjective optimization problems with complementarity constraints (SMOPCC):

$$\begin{array}{ll} \min & \mathbb{E}_P[f(z,\xi(\omega))] \\ \text{s.t.} & z \in Z, \\ & 0 \le \mathbb{E}_P[G(z,\xi(\omega))] \perp \mathbb{E}_P[H(z,\xi(\omega))] \ge 0, \end{array} \tag{1.1}$$

where Z is a nonempty, closed and convex subset of $\mathbb{R}^n$, f, G and H are, respectively, continuously differentiable functions from $\mathbb{R}^n \times \mathbb{R}^q$ to $\mathbb{R}^s, \mathbb{R}^m, \mathbb{R}^m$, $\xi : \Omega \to \Xi$ is a vector

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of random variables defined on probability $(\Omega, \mathcal{F}, P)$ with support set $\Xi \subset \mathbb{R}^q$, and $\mathbb{E}_P[\cdot]$ denotes the expected value with respect to probability measure P, and '$\perp$' denotes the perpendicularity of two vectors.

From the perspective of stochastic optimization, SMOPCC is an extension of deterministic multiobjective optimization problems with equilibrium constraints [18, 19, 31] by considering the case that some random data involved in the objective and the constraints. From the perspective of multiobjective optimization, SMOPCC is an extension of stochastic mathematical programs with complementarity constraints (SMPCC) by extending the single valued objective to multiple objective. The SMPCC model plays a very important role in many fields such as engineering design, economic equilibrium, multilevel game and it has been receiving much attention in the recent optimization world, see [16, 28, 29] and the survey paper [10] for some recent developments in SMPCC. Moreover, SMOPCC is related to stochastic equilibrium problems with equilibrium constraints (SEPEC) [7]. SEPEC model characterizes the behavior of the decision makers when they are looking for an equilibria. SMOPCC model is suited for SEPEC when decision makers who are looking for Pareto type optimality, see [7, 33] and the references therein for more results on SEPECs.

Our focus here is on the stability analysis of the SMOPCC. Specifically, we look into the change of the optimal value, the optimal solutions and the stationary points as the underlying probability measure varies under some appropriate metric. This kind of research is numerically motivated in practice due to lack of complete information of the distribution of the random variables, it is often difficult to obtain a closed form of the expected values of the random functions in the objective and constraints and subsequently numerical schemes are proposed to approximate the expected values. The stability results presented here may provide a unified theoretical framework for various numerical approximation schemes of the expected values of the underlying functions in the SMOPCC. Indeed, such a stability analysis has been well-known for stochastic programs with single valued objective, equality and/or inequality constraints although it is new for SMOPCC; see for instance [8, 21, 22] and [3, 4, 15] for the recent development when this kind of stability analysis is applied to stochastic mathematical programs with dominance constraints.

We can not apply the existing stability results in [21, 22] as the multiple objective and the complementarity constraints. The trouble is that the "optimal solution" defined for SMOPCC is different with optimization problems with single valued objective. Moreover, the reformulation of the complementarity constraints as a system of equalities or inequalities does not guarantee certain constraint qualifications (such as linear independence constraint qualification, Mangasarian-Fromovitz constraint qualification) which are often necessary for stability analysis. This motivates us to undertake an independent stability analysis.

As far as we concerned, the contributions of the paper can be summarized as follows:

1. We present some continuity results of the optimal solutions and the optimal value function of SMOPCC with respect to the probability measure. Our results are related to the recently work [1,6] where a special approximation, empirical probability measure approximation, are considered.

2. By employing some new results on stochastic generalized equations and the reformulation of stationary points of SMOPCC, we study the upper semi-continuity of the stationary points with respect to the probability measure.

3. Similar to the work [1, 6], we utilize the sample average method to approximate the expectation. Under some moderate conditions, we show that, with probability approaching one exponentially fast with the increase of sample size, a stationary point to

the SAA problem becomes a stationary point to its true counterpart. Our result is a complementarity of [6] where the exponential rate convergence of the optimal solutions is studied.

The rest of the paper is organized as follows. In section 2, we provide some basic definitions and the stability of stochastic generalized equations. In section 3, we study the stability of the optimal solutions, the optimal value and the stationary points with respect to the probability measure. Specifically, we present some continuity results of the optimal solution and the optimal valued function first and then investigate the stationary point of SMOPCC by employing some new results on stochastic generalized equations. In section 4, we focus on empirical probability measure approximation and some exponential convergence results are provided.

2 Preliminaries

For vectors $a, b \in \mathbb{R}^n$, $a^T b$ denotes the scalar product, $\|\cdot\|$ denotes the Euclidean norm of a vector, $\|\cdot\|_F$ denotes the Frobenius norm of a matrix, $\mathcal{B}$ denotes the closed unit ball in the respective space. $d(z,\mathcal{D}) := \inf_{z'\in\mathcal{D}} \|z-z'\|$ denotes the distance from a point z to a set $\mathcal{D}$. For two compact sets $\mathcal{C}$ and $\mathcal{D}$,

$$\mathbb{D}(\mathcal{C},\mathcal{D}) := \sup_{z\in\mathcal{C}} d(z,\mathcal{D})$$

denotes the deviation of $\mathcal{C}$ from $\mathcal{D}$ and $\mathbb{H}(\mathcal{C},\mathcal{D}) := \max(\mathbb{D}(\mathcal{C},\mathcal{D}), \mathbb{D}(\mathcal{D},\mathcal{C}))$ denotes the Hausdorff distance between $\mathcal{C}$ and $\mathcal{D}$. Moreover, $\mathcal{C}+\mathcal{D}$ denotes the Minkowski addition of the two sets, that is, $\{C+D : C\in\mathcal{C}, D\in\mathcal{D}\}$. For a function $g : \mathbb{R}^s \to \mathbb{R}^{s'}$, we use $\nabla g(z)$ to denote the transposed Jacobian of g at z. If $g(z)$ is a scalar-valued function, $\nabla g(z)$ denotes the gradient of g at point z. Finally, for a set $\{(x,y) = z : z \in Z\}$, $\Pi_x Z = \{x : \exists\, y \text{ such that } (x,y) = z \in Z\}$.

Let $\Psi : X \rightrightarrows Y$ be a set-valued mapping. Ψ is said to be *closed* at $\bar{x}$ if $x_k \in X$, $x_k \to \bar{x}$, $y_k \in \Psi(x_k)$ and $y_k \to \bar{y}$ implies $\bar{y} \in \Psi(\bar{x})$. Ψ is said to be *upper semi-continuous* (usc for short) at $\bar{x} \in X$ if for every $\epsilon > 0$, there exists a constant $\delta > 0$ such that

$$\Psi(\bar{x} + \delta\mathcal{B}) \subset \Psi(\bar{x}) + \epsilon\mathcal{B}.$$

Ψ is said to be *lower semi-continuous* (lsc for short) at $\bar{x} \in X$ if for every $\epsilon > 0$, there exists a constant $\delta > 0$ such that

$$\Psi(\bar{x}) \subset \Psi(\bar{x} + \delta\mathcal{B}) + \epsilon\mathcal{B}.$$

Ψ is said to be *continuous* at $\bar{x}$ if it is both usc and lsc at the point. Ψ is said to be *metrically regular* at $\bar{x}$ for $\bar{y}$ if there exist constant $\alpha > 0$, neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$d(x, \Psi^{-1}(y)) \le \alpha d(y, \Psi(x)), \quad \forall x \in U, \ \forall y \in V.$$

2.1 Optimal solutions and stationary points of SMOPCC

Let $\mathcal{K} \subseteq \mathbb{R}^m$ be a closed and convex cone with $\mathrm{int}\mathcal{K} \neq \emptyset$. The cone $\mathcal{K}$ introduces the order relation $\le_{\mathcal{K}}$ in $\mathbb{R}^n$ defined as follows: for $x', x'' \in \mathbb{R}^m$

$$x' \leqq_{\mathcal{K}} x'' \Leftrightarrow x'' - x' \in \mathcal{K},$$
$$x' <_{\mathcal{K}} x'' \Leftrightarrow x'' - x' \in \mathrm{int}\mathcal{K}.$$

Definition 2.1 ([17]). The feasible point $\bar{z}$ is said to be an optimal solution of problem (1.1) for the given domination cone $\mathcal{K}$ if does not exist feasible point z such that $f(z) \leqq_{\mathcal{K}} f(\bar{z})$; $\bar{z}$ is said to be a weak optimal solution of problem (1.1) if does not exist feasible point z such that $f(z) <_{\mathcal{K}} f(\bar{z})$.

Definition 2.2. [11] We call the feasible point $\bar{z}$ a *Clarke or C-stationary point* of problem (1.1) for the given domination cone $\mathcal{K}$ if there exist multipliers $(\tau, u, v) \in \mathbb{R}^s \times \mathbb{R}^m \times \mathbb{R}^m$ such that $\tau \in \mathcal{K}$ and

$$0 \in \nabla \mathbb{E}_P[f(\bar{z}, \xi)]\tau - \nabla \mathbb{E}_P[G(\bar{z}, \xi)]u - \nabla \mathbb{E}_P[H(\bar{z}, \xi)]v + \mathcal{N}_Z(\bar{z}), \tag{2.1}$$

$$\sum_{i=1}^{s} \tau_i = 1, \tag{2.2}$$

$$u_i = 0, \quad i \notin \mathcal{I}_{\mathbb{E}_P[G]}(\bar{z}), \tag{2.3}$$

$$v_i = 0, \quad i \notin \mathcal{I}_{\mathbb{E}_P[H]}(\bar{z}), \tag{2.4}$$

$$u_i v_i \geq 0, \quad i \in \mathcal{I}_{\mathbb{E}_P[G]}(\bar{z}) \cap \mathcal{I}_{\mathbb{E}_P[H]}(\bar{z}), \tag{2.5}$$

where $\mathcal{N}_Z(\bar{z})$ denotes the convex normal cone of convex set Z at $\bar{z}$,

$$\mathcal{I}_{\mathbb{E}_P[G]}(\bar{z}) := \{i \, : \, \mathbb{E}_P[G_i(\bar{z}, \xi)] = 0, \;\; i = 1, \ldots, m\}$$

and

$$\mathcal{I}_{\mathbb{E}_P[H]}(\bar{z}) := \{i \, : \, \mathbb{E}_P[H_i(\bar{z}, \xi)] = 0, \;\; i = 1, \ldots, m\}.$$

We call $\bar{z} \in \mathcal{F}$ a *Mordukhovich or M-stationary point* of problem (1.1) if (2.1)–(2.4) hold and $\min(u_i, v_i) > 0$ or $u_i v_i = 0$ hold for each $i \in \mathcal{I}_{\mathbb{E}_P[G]}(\bar{z}) \cap \mathcal{I}_{\mathbb{E}_P[H]}(\bar{z})$.

We call $\bar{z} \in \mathcal{F}$ a *Strongly or S-stationary point* of problem (1.1) if (2.1)–(2.4) hold and $u_i \geq 0$ and $v_i \geq 0$ hold for each $i \in \mathcal{I}_{\mathbb{E}_P[G]}(\bar{z}) \cap \mathcal{I}_{\mathbb{E}_P[H]}(\bar{z})$.

Indeed, Definition 2.2 is the Fritz-John type of stationarity conditions. We call them C-, M- or S-stationary points by following the notation of SMPCC. Under some constraint qualifications such as generalized Mangasarian-Fromovitz constraint qualification [11], the stationary points defined above characterize the local solution of SMPOCC, that is, the local optimal solutions of SMPOCC satisfy the optimality condition above. In what follows, the domination cone $\mathcal{K}$ is fixed as $\mathcal{K} = \mathbb{R}^m_+$ which implies that the order defined above is the Pareto order. Then the optimal solution is Pareto optimal solution and the stationary points are Pareto C-, M- or S-stationary points. Moreover, we should note the condition $\tau \in \mathcal{K}$ in Definition 2.2 should be replaced by $\tau_i \geq 0$, $i = 1, \ldots, s$.

2.2 Stochastic generalized equations

Consider the following stochastic generalized equation (SGE):

$$0 \in \mathbb{E}_P[\Gamma(x, \xi)] + \mathcal{G}(x), \tag{2.6}$$

and its perturbation

$$0 \in \mathbb{E}_Q[\Gamma(x, \xi)] + \mathcal{G}(x). \tag{2.7}$$

where $\Gamma : \mathcal{X} \times \Xi \rightrightarrows \mathcal{Y}$ and $\mathcal{G} : \mathcal{X} \rightrightarrows \mathcal{Y}$ are closed set-valued mappings, $\mathcal{X}$ and $\mathcal{Y}$ are subsets of Banach spaces X and Y, $\xi : \Omega \to \Xi$ is a random vector with support set $\Xi \subset \mathbb{R}^d$. $\mathbb{E}_P[\cdot]$ ($\mathbb{E}_Q[\cdot]$) denotes the expected value with respect to probability measure P (Q).

Let $\Gamma(x,\xi)$ be defined as above and $\sigma(\Gamma(x,\cdot),u)$ be its support function. Let $\mathcal{X}$ be a compact subset of X. Define

$$\mathbf{G} := \{g(\cdot) : g(\xi) := \sigma(\Gamma(x,\xi),u),\ \text{for}\, x \in \mathcal{X},\ \|u\| \leq 1\}.$$

Then $\mathbf{G}$ consists of all functions generated by the support function $\sigma(\Gamma(x,\cdot),u)$ over the set $\mathcal{X} \times \{u : \|u\| \leq 1\}$. Define

$$\mathbf{D}(Q,P) := \sup_{g(\xi)\in\mathbf{G}} \big(\mathbb{E}_Q[g(\xi)] - \mathbb{E}_P[g(\xi)]\big)$$

and

$$\mathbf{H}(Q,P) := \max\big(\mathbf{D}(Q,P), \mathbf{D}(P,Q)\big).$$

Neither $\mathbf{H}$ nor $\mathbf{D}$ is a metric but one may enlarge the set $\mathbf{G}$ so that $\mathbf{H}(Q,P) = 0$ implies $Q = P$. We call $\mathbf{H}(Q,P)$ a *pseudometric.* It is also known as a distance of probability measures having ζ-structure, see [34].

The following stability results of SGE have been provided by Liu et al [12].

Lemma 2.3. *Consider the stochastic generalized equation (2.6) and its perturbation (2.7). Let $\mathcal{X}$ be a compact subset of X, and $S(P)$ and $S(Q)$ denote the set of solutions of (2.6) and (2.7) restricted to $\mathcal{X}$ respectively. Assume: (a) Y is a Euclidean space and Γ is a set-valued mapping taking convex and compact set-values in $\mathcal{Y}$; (b) Γ is upper semi-continuous with respect to x for every $\xi \in \Xi$ and bounded by a P-integrable function $\kappa(\xi)$ for $x \in \mathcal{X}$; (c) $\mathcal{G}$ is upper semi-continuous; (d) $S(Q)$ is nonempty for $Q \in \mathbb{P}(\Omega)$ and $\mathbf{D}(Q,P)$ sufficiently small. Then the following assertions hold:*

(i) *For any $\epsilon > 0$, let*

$$R(\epsilon) := \inf_{x\in\mathcal{X},\ d(x,S(P))\geq\epsilon} d(0, \mathbb{E}_P[\Gamma(x,\xi)] + \mathcal{G}(x)). \tag{2.8}$$

Then

$$\mathbb{D}(S(Q), S(P)) \leq R^{-1}(2\mathbf{D}(Q,P)),$$

where $R^{-1}(\epsilon) := \min\{t \in \mathbb{R}_+ : R(t) = \epsilon\}$, and $R^{-1}(\epsilon) \to 0$ as $\epsilon \downarrow 0$.

(ii) *For any $\epsilon > 0$, there exists a $\delta > 0$ such that if $\mathbf{D}(Q,P) \leq \delta$, then $\mathbb{D}(S(Q), S(P)) \leq \epsilon$.*

(iii) *If $x^* \in S(P)$ and $\Phi(x) := \mathbb{E}_P[\Gamma(x,\xi)] + \mathcal{G}(x)$ is metrically regular at x^* for 0 with regularity modulus α, then there exists neighborhood U_{x^*} of x^* such that*

$$d(x, S(P)) \leq \alpha\mathbf{D}(Q,P) \tag{2.9}$$

for $x \in S(Q) \cap U_{x^*}$; if Φ is strongly metrically regular at x^* for 0 with the same regularity modulus and neighborhood, then

$$\|x - x^*\|_X \leq \alpha\mathbf{D}(Q,P) \tag{2.10}$$

for $x \in S(Q)$ close to $\Phi^{-1}(0)$.

3 Stability of SMOPCC

Let $\mathbb{P}(\Omega)$ denote the set of all Borel probability measures. Assuming $Q \in \mathbb{P}(\Omega)$ is close to P under some metric to be defined shortly, we investigate the following optimization problem:

$$\begin{array}{ll} \min & \mathbb{E}_Q[f(z,\xi)] \\ \text{s.t.} & z \in Z, \\ & 0 \le \mathbb{E}_Q[G(z,\xi)] \perp \mathbb{E}_Q[H(z,\xi)] \ge 0, \end{array} \tag{3.1}$$

which is regarded as a perturbation of problem (1.1). Specifically, we study the relationship between the perturbed problem (3.1) and the original problem (1.1) in terms of the optimal value, the optimal solutions and the stationary points when Q is close to P.

Let us start by introducing a distance function for the set $\mathbb{P}(\Omega)$, which is appropriate for our analyzing. Define the set of functions:

$$\mathbf{G_o} := \{g(\cdot) = f_i(z,\cdot) : z \in Z, i = 1,\ldots,s\} \cup \{g(\cdot) = G_i(z,\cdot) : z \in Z, i = 1,\ldots,m\} \\ \cup\{g(\cdot) = H_i(z,\cdot) : z \in Z, i = 1,\ldots,m\}.$$

The distance function for the elements in set $\mathbb{P}(\Omega)$ is defined as:

$$\mathbf{D_o}(P,Q) := \sup_{g(\xi)\in\mathbf{G_o}} \big|\mathbb{E}_P[g(\xi)] - \mathbb{E}_Q[g(\xi)]\big|.$$

This kind of distance is first studied by Römisch in the 1980s, see the excellent review by Römisch [22] for more details.

Throughout this section, we use the following notation:

$$\begin{array}{rcl} \mathcal{F}(Q) & := & \big\{z \in Z : 0 \le \mathbb{E}_Q[G(z,\xi)] \perp \mathbb{E}_Q[H(z,\xi)] \ge 0\big\}, \\ E(Q) & := & \big\{\mathbb{E}_Q[f(z,\xi)] : z \in \mathcal{F}(Q)\big\}, \\ \vartheta(Q) & := & \big\{\bar{y} : \text{there exists no } y \in E(Q) \text{ such that } y \leqq_{\mathcal{K}} \bar{y}\big\}, \\ \bar{\vartheta}(Q) & := & \big\{\bar{y} : \text{there exists no } y \in E(Q) \text{ such that } y <_{\mathcal{K}} \bar{y}\big\}, \\ S_o(Q) & := & \big\{z \in \mathcal{F}(Q) : \mathbb{E}_Q[f(z,\xi)] \in \vartheta(Q)\big\}, \\ \bar{S}_o(Q) & := & \big\{z \in \mathcal{F}(Q) : \mathbb{E}_Q[f(z,\xi)] \in \bar{\vartheta}(Q)\big\}, \\ \mathbb{P}_{\mathbf{G_o}}(\Omega) & := & \left\{Q \in \mathbb{P}(\Omega) : -\infty < \inf\limits_{g(\xi)\in\mathbf{G_o}} \mathbb{E}_Q[g(\xi)] \text{ and } \sup\limits_{g(\xi)\in\mathbf{G_o}} \mathbb{E}_Q[g(\xi)] < \infty\right\}. \end{array}$$

It is easy to see that for $P, Q \in \mathbb{P}_{\mathbf{G_o}}(\Omega)$, $\mathbf{D_o}(P,Q) < \infty$.

3.1 Optimal solutions

Assumption 3.1. There exist a neighborhood U_P of P, positive constants β and δ such that for any $Q \in U_P$ and $z \in Z \cap \mathcal{B}(\mathcal{F}(Q),\delta)$,

$$d(z,\mathcal{F}(Q)) \quad \le \quad \beta \left\|\min\{\mathbb{E}_Q[G(z,\xi)], \mathbb{E}_Q[H(z,\xi)]\}\right\|, \tag{3.2}$$

where $\mathcal{B}(U,\delta)$ denotes the δ neighborhood of set U.

Assumption 3.2. There exist a neighborhood U_P of P, positive constants β and δ such that for any $Q \in U_P$ and $z \in Z \cap \mathcal{B}(\mathcal{F}(Q),\delta)$,

$$d(z,\mathcal{F}(Q)) \quad \le \quad \beta \left\|(-\mathbb{E}_Q[G(z,\xi)], -\mathbb{E}_Q[H(z,\xi)], \mathbb{E}_Q[G(z,\xi)] \circ \mathbb{E}_Q[H(z,\xi)])_+\right\|, \tag{3.3}$$

where $(a)_+ := \max\{a, 0\}$ for a vector "a" and the maximum is taken componentwise and "$\circ$" denotes the Hadamard product.

In the literature [20, 32], inequality (3.2) is known as *natural type error bound* whereas inequality (3.3) is known as *S-type error bound* of the complementarity constraint. See [14] for more details of the two kinds of error bounds for stochastic complementary problems. Moreover, we refer readers interested in the topic to monograph [5] and a survey paper by Pang [20] on error bound of variational inequalities and complementarity problems.

Proposition 3.3 ([14]). *Assume Assumption 3.1 or Assumption 3.2 hold. Suppose that there exist a neighborhood $\tilde{U}_P$ of P and a nonnegative function $\kappa(\xi)$ such that $\max(\|G(z,\xi)\|, \|H(z,\xi)\|) \leq \kappa(\xi)$ and $\mathbb{E}_Q[\kappa(\xi)] < \infty$ for $Q \in \tilde{U}_P$ and $z \in Z$. Then, there exist a neighborhood U^* of P and a positive constant β^* such that the feasible set mapping $\mathcal{F}(Q)$ is Lipschitz continuous with modulus β^* on U^*, that is*

$$\mathbb{H}\left(\mathcal{F}(Q_1), \mathcal{F}(Q_2)\right) \leq \beta^* \mathbf{D_o}(Q_1, Q_2), \qquad \forall Q_1, Q_2 \in U^*.$$

The following definition is introduced by Lemaire [9].

Definition 3.4. Let $\{(f^N, \mathcal{F}^N)\}$ be a sequence of multiobjective optimization problems where $f^N : \mathbb{R}^n \to \mathbb{R}^s$ and $\mathcal{F}^N \subset \mathbb{R}^n$ denote the objective and the set of feasible points respectively. The sequence $\{(f^N, \mathcal{F}^N)\}$ is said to be converging to the multiobjective optimization problem $(f, \mathcal{F})$, denoted by $(f^N, \mathcal{F}^N) \to (f, \mathcal{F})$, if the following two conditions hold:

$\forall z \in \mathcal{F}$, $\exists z^N \in \mathcal{F}^N$ such that $\lim_{N\to\infty} z^N = z$ and

$$\limsup_{N\to\infty} f_i^N(z^N) \leq f_i(z) \quad \text{for all } i = 1, \ldots, s,$$

$\forall z \in Z$, $\forall z^N \to z$ such that

$$\liminf_{N\to\infty} \bar{f}_i^N(z^N) \geq \bar{f}_i(z) \quad \text{for all } i = 1, \ldots, s,$$

where

$$\bar{f}(z) = \begin{cases} +\infty, & \text{if } z \notin \mathcal{F} \\ f(z), & \text{if } z \in \mathcal{F} \end{cases}, \qquad \bar{f}^N(z) = \begin{cases} +\infty, & \text{if } z \notin \mathcal{F}^N \\ f^N(z), & \text{if } z \in \mathcal{F}^N \end{cases}.$$

We should notice that $(f^N, \mathcal{F}^N) \to (f, \mathcal{F})$ implies epi-convergence of $f_i^N + \delta_{\mathcal{F}^N}$ to $f_i + \delta_{\mathcal{F}}$ for $i = 1, \ldots, s$, where $\delta_{\mathcal{D}}$ denotes the index function of set $\mathcal{D}$, that is, $\delta_{\mathcal{D}}(z) = 0$ iff $z \in \mathcal{D}$, otherwise $\delta_{\mathcal{D}}(z) = +\infty$. In general, it is a stronger notion of convergence since the approximating sequence is supposed to be independent of $i = 1, \ldots, s$.

Theorem 3.5. *Assume the conditions of Proposition 3.3. Then, $\vartheta(Q)$ and $S_o(Q)$ are lower semi-continuous at point P. Moreover, if $S_o(P) = \bar{S}_o(P)$, $\vartheta(Q)$ and $S_o(Q)$ are continuous at point P.*

Proof. If the optimal solution set-valued mapping $S_o(Q)$ is semi-continuous at point P, and $\mathbb{E}_Q[f(z,\xi)]$ is continuous at point P uniformly on $z \in Z$, $\vartheta(Q)$ is semi-continuous at point P. By the definition of $\mathbf{D_o}(\cdot,\cdot)$,

$$\sup_{z\in Z} \|\mathbb{E}_Q[f(z,\xi)] - \mathbb{E}_P[f(z,\xi)]\| \leq \mathbf{D_o}(Q, P), \tag{3.4}$$

which shows the uniform continuity of $\mathbb{E}_Q[f(z,\xi)]$. Then, we just need to study the continuous properties of $S_o(Q)$.

Lower semi-continuity. Since the conditions of Proposition 3.3 hold, there exist a neighborhood $U^*$of P and a positive constant β^* such that

$$\mathbb{H}\left(\mathcal{F}(Q_1), \mathcal{F}(Q_2)\right) \le \beta^* \mathbf{D_o}(Q_1, Q_2), \qquad \forall\, Q_1, Q_2 \in U^*.$$

Then, taking advantage of (3.4),

$$(\mathbb{E}_Q[f(\cdot,\xi)], \mathcal{F}(Q)) \to (\mathbb{E}_P[f(\cdot,\xi)], \mathcal{F}(P))$$

with $\mathbf{D_o}(Q,P)$ tends to zero. Together with the compactness of Z, we get the lower semi-continuity of $S_o(Q)$ at P by [9, Theorem 2.3].

Upper semi-continuity. Let $Q^N \in \mathbb{P}_{\mathbf{G_o}}(\Omega)$ be any sequence of probability measure such that $\mathbf{D_o}(Q^N, P)$ tends to zero. For $\forall z^N \in S_o(Q^N)$, we just need to show that the accumulation points of sequence $\{z^N\}$ are contained in $S_o(P)$. Taking a subsequence if necessary, we may assume that $z^N \to \bar{z}$ as N tends to infinity. Assume a contradiction that $\bar{z} \notin S_o(P)$, then there exists $\hat{z} \in \mathcal{F}(P)$ such that

$$\mathbb{E}_P[f(\hat{z},\xi)] \leqq_{\mathcal{K}} \mathbb{E}_P[f(\bar{z},\xi)].$$

Since $S_o(P) = \bar{S}_o(P)$, the formula above can be strengthened as

$$\mathbb{E}_P[f(\hat{z},\xi)] <_{\mathcal{K}} \mathbb{E}_P[f(\bar{z},\xi)].$$

Taking advantage of the continuity of the feasible set $\mathcal{F}(Q)$ and $\mathbb{E}_Q[f(z,\xi)]$ on Q, there exists a sequence $\{\hat{z}^N\}$ such that $\hat{z}^N \to \hat{z}$ and for N large enough,

$$\mathbb{E}_{Q^N}[f(\hat{z}^N,\xi)] \leqq_{\mathcal{K}} \mathbb{E}_{Q^N}[f(z^N,\xi)],$$

which contradicts the fact that $z^N \in S_o(Q^N)$. The proof is complete. □

3.2 Stationary points

By introducing some slack and auxiliary variables, the first order optimality condition of problem (1.1) which characterizes the Pareto M-stationarity can be reformulated as a constrained generalized equation [11]:

$$0 \in \mathbb{E}_P[\Phi(z,\tau,\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3,\beta_4,u,v,\xi)] + \mathcal{N}_Z(z) \times \mathbf{0}, \tag{3.5}$$

where $(z,\tau,\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3,\beta_4,u,v) \in Z \times \mathcal{W}$,

$$\Phi(z,\tau,\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3,\beta_4,u,v,\xi) = \begin{pmatrix} \nabla f(z,\xi)\tau - \nabla G(z,\xi)u - \nabla H(z,\xi)v \\ \tau^T\mathbf{1} - 1 \\ \alpha_1 - G(z,\xi) \\ \alpha_2 - H(z,\xi) \\ \alpha_1^T\alpha_2 \\ u \circ \alpha_1 \\ v \circ \alpha_2 \\ \beta_1 - u \circ v \\ \beta_3^T\beta_4 \\ \beta_2 - \beta_3 - u \\ \beta_2 - \beta_4 - v \end{pmatrix},$$

$$\mathcal{W}=\Big\{w\,\Big|\,w=(\tau,\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3,\beta_4,u,v),\tau\geq 0,\alpha_1,\alpha_2\geq 0;\beta_i\geq 0\,(i=1,2,3,4)\Big\},$$

$\mathbf{0}(\mathbf{1})$ denotes the vector whose each component is 0 (1). This means that $z\in Z$ is an M-stationary point if and only if there exist multiplier $w\in\mathcal{W}$ such that (z,w) is a solution of the stochastic generalized equation (3.5) and hence studying the stability of the stationary point amounts to that of the generalized equation. Similarly, the stationary point of problem (3.1) can be characterized by the following stochastic generalized equation:

$$0\in\mathbb{E}_Q[\Phi(z,\tau,\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3,\beta_4,u,v,\xi)]+\mathcal{N}_Z(z)\times\mathbf{0}. \tag{3.6}$$

Firstly, we need to introduce a distance which is appropriate for characterizing stationary point. Let $\mathcal{W}^*$ be a compact subset of $\mathcal{W}$, we denote

$$\begin{aligned}
\mathbf{G_s} &:= \{g(\xi):g(\xi):=\sigma(\Phi(z,w,\xi),\nu),\ \text{for}\,(z,w)\in Z\times\mathcal{W}^*,\ \|\nu\|\leq 1\},\\
\mathbf{D_s}(Q,P) &:= \sup_{g(\xi)\in\mathbf{G_s}}\big(\mathbb{E}_Q[g(\xi)]-\mathbb{E}_P[g(\xi)]\big),\\
\mathbf{H_s}(Q,P) &:= \max\big(\mathbf{D_s}(Q,P),\mathbf{D_s}(P,Q)\big).
\end{aligned}$$

It is ready to study the stability of stationary points of problem (1.1).

Theorem 3.6. *Assume: (a') there exist neighborhoods U_P of P and a compact set $\mathcal{W}^*\subset\mathcal{W}$ such that the set of stationary points of problem (3.1), denoted by $S_s(Q)$, is not empty and for every $Q\in U_P$, the solution set of stochastic generalized equation (3.6) is bounded by $Z\times\mathcal{W}^*$;(b') the Lipschitz modulus of $f(z,\xi)$, $G(z,\xi)$ and $H(z,\xi)$ are bounded by a positive constant L for any $\xi\in\Xi$, (c') $G_i(z,\cdot)$ and $H_i(z,\cdot)$ are bounded for any $z\in Z$ and $i=1,\ldots,m$. Then the conclusions (i)-(iii) of Lemma 2.3 hold for $S_s(P)$ and $S_s(Q)$.*

Proof. The thrust of the proof is to apply Lemma 2.3 to generalized equation (3.5) and its perturbation (3.6). To this end, we verify the hypotheses of Lemma 2.3.

Observe first that $\Phi(\cdot)$ is single valued, it is convex and compact valued and hence verifies (a). The upper semi-continuity of $\Phi(\cdot)$ and its integrable boundedness follows from the fact that all the involved functions are continuously differentiable, (b') and (c') hold and hence verifies (b). The condition (c) follows from the upper semi-continuity of normal cone, while (d) coincides with (a'). The proof is complete. □

Remark 3.7. (i) Some words on $\mathbf{D_s}$. Denote

$$\begin{aligned}
\Phi^*(z,w,\xi) &:= \begin{pmatrix}\nabla f(z,\xi)\tau-\nabla G(z,\xi)u-\nabla H(z,\xi)v\\ G(z,\xi)\\ H(z,\xi)\end{pmatrix},\\
\mathbf{G_s^*} &:= \{g(\xi):g(\xi):=\sigma(\Phi^*(z,w,\xi),\nu),\ \text{for}\,(z,w)\in Z\times\mathcal{W}^*,\ \|\nu\|\leq 1\},\\
\mathbf{D_s^*}(Q,P) &:= \sup_{g(\xi)\in\mathbf{G_s^*}}\big(\mathbb{E}_Q[g(\xi)]-\mathbb{E}_P[g(\xi)]\big).
\end{aligned}$$

By some easy analysis, we have $\mathbf{D_s}=\mathbf{D_s^*}$. Note that $\Phi^*(\cdot)$ is a single valued function, we denote $\mathbf{G_o^\#}:=\mathbf{G_o}\cup\{g(\cdot)=\nabla(f_\iota(z,\cdot))_k,(\nabla G_i(z,\cdot))_k,(\nabla H_j(z,\cdot))_k:z\in Z,\ \iota=1,\ldots,s,1\leq i\leq m,1\leq j\leq m,k=1,\ldots,n\}$ and

$$\mathbf{D}^\#(P,Q):=\sup_{g(\xi)\in\mathbf{G_o^\#}}\big|\mathbb{E}_P[g(\xi)]-\mathbb{E}_Q[g(\xi)]\big|.$$

By letting

$$u^*=\sup_{u\in\Pi_u\mathcal{W}^*}\|u\|,\qquad v^*=\sup_{v\in\Pi_v\mathcal{W}^*}\|v\|,$$

we have

$$
\begin{aligned}
\mathbf{D}_{\mathbf{s}}^*(Q,P) &= \sup_{\|\nu\|\le 1,\ z\in Z} |\mathbb{E}_Q[\Phi^*(z,w,\xi)^T\nu] - \mathbb{E}_P[\Phi^*(z,w,\xi)^T\nu]| \\
&\le \sup_{z\in Z} \|\mathbb{E}_Q[\Phi^*(z,w,\xi)] - \mathbb{E}_P[\Phi^*(z,w,\xi)]\| \\
&\le \sup_{z\in Z} \Big(\|\nabla\mathbb{E}_Q[f(z,\xi)] - \nabla\mathbb{E}_P[f(z,\xi)]\|_F + u^* \|\nabla\mathbb{E}_Q[G(z,\xi)] - \nabla\mathbb{E}_P[G(z,\xi)]\|_F \\
&\quad + v^* \|\nabla\mathbb{E}_Q[H(z,\xi)] - \nabla\mathbb{E}_P[H(z,\xi)]\|_F \\
&\quad + \|\mathbb{E}_Q[G(z,\xi)] - \mathbb{E}_P[G(z,\xi)]\| + \|\mathbb{E}_Q[H(z,\xi)] - \mathbb{E}_P[H(z,\xi)]\| \Big) \\
&\le \gamma^*\mathbf{D}^{\#}(P,Q),
\end{aligned}
\tag{3.7}
$$

where

$$\gamma^* = ns + nm(u^* + v^*) + 2m, \tag{3.8}$$

n, s, m are the dimensions of variable z, functions f and G respectively.

(ii) The boundedness condition (a'). The boundedness of Lagrange multipliers has been well discussed in the past decades, see [23,30] for the developments on optimization problems with equilibrium constraint. If the abstract constraints Z of problem (1.1) is characterized by some equalities and/or inequalities, the boundedness condition in Theorem 3.6 can be ensured by MPCC Mangasarian-Fromovitz constraint qualification holding at every feasible point of problem (1.1), see [13] for a similar discussion.

(iii) Lin et al. [11] have studied the convergence of stationary points of SMPOCC when the true probability measure is approximated by empirical probability measure. Theorem 3.6 extends their results to a general case. Interested readers see the section 5 of [11] for the applications of convergence results.

4 Empirical Probability Measure

In practice, the distribution of ξ is often unknown or it is numerically too expensive to calculate the expected values. Instead it might be possible to obtain a sample of the random vector ξ from past data. A well-known approximation method in stochastic programming based on sampling is sample average approximation (SAA), that is, if we have an independent identically distributed (iid) sample $\xi^1, \ldots, \xi^N$ of random vector ξ, then we may use the empirical probability measure

$$P^N := \frac{1}{N}\sum_{k=1}^{N} \mathbb{I}_{\xi^k}(\omega),$$

to approximate the original probability measure P, where

$$\mathbb{I}_{\xi^k}(\omega) := \begin{cases} 1, & \text{if } \xi(\omega) = \xi^k, \\ 0, & \text{if } \xi(\omega) \ne \xi^k. \end{cases}$$

Denote

$$f^N(z) := \mathbb{E}_{P^N}[f(z,\xi)], \quad G^N(z) := \mathbb{E}_{P^N}[G(z,\xi)], \quad H^N(z) := \mathbb{E}_{P^N}[H(z,\xi)].$$

Then the corresponding sample average problem is

$$\begin{array}{ll} \min & f^N(z) \\ \text{s.t.} & z \in Z, \\ & 0 \le G^N(z) \perp H^N(z) \ge 0. \end{array} \tag{4.1}$$

Under some moderate conditions, it is easy to show by [24, Propostion 7] that $\mathbf{D_o}(P^N, P)$ tends to zero with probability one. Therefore Theorem 3.5 implies immediately almost sure convergence of the optimal value and the optimal solutions of the SAA problem to their true counterparts. Our focus here is on the quantitative behavior of the stationary points as the sample size increases. Next, we study the exponential rate of convergence of the stationary points.

We said that z^N is an M-stationary point of (4.1) if there exists multiplier $w^N \in \mathcal{W}$ such that (z^N, w^N) is the solution of the generalized equation

$$0 \in \Phi^N(z^N, \tau^N, \alpha_1^N, \alpha_2^N, \beta_1^N, \beta_2^N, \beta_3^N, \beta_4^N, u^N, v^N) + \mathcal{N}_Z(z^N) \times \mathbf{0}, \tag{4.2}$$

where

$$\Phi^N(z, \tau, \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4, u, v) = \begin{pmatrix} \nabla f^N(z)\tau - \nabla G^N(z)u - \nabla H^N(z)v \\ (\tau)^T \mathbf{1} - 1 \\ \alpha_1 - G^N(z) \\ \alpha_2 - H^N(z) \\ \alpha_1^T \alpha_2 \\ u \circ \alpha_1 \\ v \circ \alpha_2 \\ \beta_1 - u \circ v \\ \beta_3^T \beta_4 \\ \beta_2 - \beta_3 - u \\ \beta_2 - \beta_4 - v \end{pmatrix}.$$

Assumption 4.1. Let functions f, G, H be differentiable and $\theta(z, \xi)$ denote any element in the collection of functions

$$\{(\nabla f_i(z,\xi))_k, (\nabla G_j(z,\xi))_k, (\nabla H_j(z,\xi))_k, G_j(z,\xi), H_j(z,\xi), \\ i = 1, \ldots, s, j = 1, \ldots, m, k = 1, \ldots, n\}.$$

Then $\theta(z, \xi)$ possesses the following properties:

(a) for every $z \in Z$ the moment generating function $\mathbb{E}[e^{(\theta(z,\xi) - \mathbb{E}_P[\theta(z,\xi)])t}]$ of the random variable $\theta(z, \xi) - \mathbb{E}_P[\theta(z, \xi)]$ is finite valued for t close to 0;

(b) there exist a (measurable) function $\kappa_2(\xi)$ and a constant $\gamma_2 > 0$, such that

$$|\theta(z, \xi) - \theta(z', \xi)| \le \kappa_2(\xi) \|z - z'\|^{\gamma_2},$$

for all $\xi \in \Xi$ and $z', z \in Z$;

(c) the moment generating function $M_{\kappa_2}(t)$ of $\kappa_2(\xi)$, is finite valued for all t in a neighborhood of zero.

Assumption 4.1 (a) holds if the support set Ξ is a compact set. Assumption 4.1 (b) requires $\theta(z, \xi)$ to be globally Hölder continuous with respect to z. Assumption 4.1 (c) is a bounded condition. See [15, 25] for a similar discussion.

Theorem 4.2. *Let z^N be a sequence of M-stationary points of problem (4.1), w^N be the corresponding multiplier and (z^*, w^*) be a limiting point of sequence $\{(z^N, w^N)\}$. Suppose that Assumption 4.1 holds and*

$$\Upsilon(z,\tau,\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3,\beta_4,u,v) := \mathbb{E}_P[\Phi(z,\xi,\tau,\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3,\beta_4,u,v)] + \mathcal{N}_Z(z)\times \mathbf{0}$$

is metrically regular at point (z^*, w^*) for 0. Then for any small positive number ϵ, there exist positive constants $C(\epsilon) > 0$, $\beta(\epsilon) > 0$ independent of N such that for N sufficiently large

$$\text{Prob}\{d(z^N, S_s(P)) \geq \lambda\epsilon\} \leq C(\epsilon)e^{-N\beta(\epsilon)}, \tag{4.3}$$

where $\lambda := \alpha\gamma^*$, α is the metric modulus and γ^* is defined by (3.8).

Proof. Under Assumption 4.1 and the metric regularity condition, conclusion (iii) of Lemma 2.3 holds. Then, there exists neighborhood U_{z^*} of z^* such that

$$d(z, S_s(P)) \leq \alpha \mathbf{D_s}(P^N, P), \quad z \in U_{z^*}.$$

Subsequently, by formula (3.7) in Remark 3.7,

$$d(z, S_s(P)) \leq \alpha\gamma^* \mathbf{D}^{\#}(P^N, P),$$

where γ^* is defined by (3.8). Then,

$$\text{Prob}\{d(z, S_s(P)) \geq \lambda\epsilon\} \leq \text{Prob}\{\mathbf{D}^{\#}(P^N, P) \geq \epsilon\}. \tag{4.4}$$

By the definition of $\mathbf{D}^{\#}(P, Q)$,

$$\begin{aligned}
&\text{Prob}\{\mathbf{D}^{\#}(P^N, P) \geq \epsilon\}\\
&\leq \text{Prob}\Bigg\{\sup_{z\in Z}\Bigg(\|\mathbb{E}_{P^N}[f(z,\xi)] - \mathbb{E}_P[f(z,\xi)]\| + \|\mathbb{E}_{P^N}[G(z,\xi)] - \mathbb{E}_P[G(z,\xi)]\|\\
&\qquad + \|\mathbb{E}_{P^N}[H(z,\xi)] - \mathbb{E}_P[H(z,\xi)]\| + \|\mathbb{E}_{P^N}[\nabla_z f(z,\xi)] - \mathbb{E}_P[\nabla_z f(z,\xi)]\|_F\\
&\qquad\quad + \|\mathbb{E}_{P^N}[\nabla_z G(z,\xi)] - \mathbb{E}_P[\nabla_z G(z,\xi)]\|_F + \|\mathbb{E}_{P^N}[\nabla_z H(z,\xi)]\\
&\qquad\qquad - \mathbb{E}_P[\nabla_z H(z,\xi)]\|_F\Bigg) \geq \epsilon\Bigg\}\\
&\leq \text{Prob}\{\sup_{z\in Z} |\mathbb{E}_{P^N}[f(z,\xi)] - \mathbb{E}_P[f(z,\xi)]| \geq \epsilon/6\}\\
&\quad + \text{Prob}\{\sup_{z\in Z} \|\mathbb{E}_{P^N}[G(z,\xi)] - \mathbb{E}_P[G(z,\xi)]\| \geq \epsilon/6\}\\
&\quad + \text{Prob}\{\sup_{z\in Z} \|\mathbb{E}_{P^N}[H(z,\xi)] - \mathbb{E}_P[H(z,\xi)]\| \geq \epsilon/6\}\\
&\quad + \text{Prob}\{\sup_{z\in Z} \|\mathbb{E}_{P^N}[\nabla_z f(z,\xi)] - \mathbb{E}_P[\nabla_z f(z,\xi)]\|_F \geq \epsilon/6\}\\
&\quad + \text{Prob}\{\sup_{z\in Z} \|\mathbb{E}_{P^N}[\nabla_z G(z,\xi)] - \mathbb{E}_P[\nabla_z G(z,\xi)]\|_F \geq \epsilon/6\}\\
&\quad + \text{Prob}\{\sup_{z\in Z} \|\mathbb{E}_{P^N}[\nabla_z H(z,\xi)] - \mathbb{E}_P[\nabla_z H(z,\xi)]\|_F \geq \epsilon/6\}.
\end{aligned}$$

Taking advantage of [25, Theorem 5.1], there exist positive constants $(C_i(\epsilon/6), \beta_i(\epsilon/6)), i =$

$1, \ldots, 6$, such that

$$\mathrm{Prob}\{\sup_{z\in Z}\left\|f^N(z)-\mathbb{E}_P[f(z,\xi)]\right\|\geq \epsilon/6\}\leq C_1(\epsilon/6)e^{-N\beta_1(\epsilon/6)},$$
$$\mathrm{Prob}\{\sup_{z\in Z}\left\|G^N(z)-\mathbb{E}_P[G(z,\xi)]\right\|\geq \epsilon/6\}\leq C_2(\epsilon/6)e^{-N\beta_2(\epsilon/6)},$$
$$\mathrm{Prob}\{\sup_{z\in Z}\left\|H^N(z)-\mathbb{E}_P[H(z,\xi)]\right\|\geq \epsilon/6\}\leq C_3(\epsilon/6)e^{-N\beta_3(\epsilon/6)},$$
$$\mathrm{Prob}\{\sup_{z\in Z}\left\|\nabla f^N(z)-\mathbb{E}_P[\nabla_z f(z,\xi)]\right\|_F\geq \epsilon/6\}\leq C_4(\epsilon/6)e^{-N\beta_4(\epsilon/6)},$$
$$\mathrm{Prob}\{\sup_{z\in Z}\left\|\nabla G^N(z)-\mathbb{E}_P[\nabla_z G(z,\xi)]\right\|_F\geq \epsilon/6\}\leq C_5(\epsilon/6)e^{-N\beta_5(\epsilon/6)},$$
$$\mathrm{Prob}\{\sup_{z\in Z}\left\|\nabla H^N(z)-\mathbb{E}_P[\nabla_z H(z,\xi)]\right\|_F\geq \epsilon/6\}\leq C_6(\epsilon/6)e^{-N\beta_6(\epsilon/6)}.$$

Combining (4.4) and the estimations above, (4.3) holds with

$$C(\epsilon)=C_1(\epsilon/6)+C_2(\epsilon/6)+C_3(\epsilon/6)+C_4(\epsilon/6)+C_5(\epsilon/6)+C_6(\epsilon/6)$$

and

$$\beta(\epsilon)=\min\{\beta_1(\epsilon/6),\beta_2(\epsilon/6),\beta_3(\epsilon/6),\beta_4(\epsilon/6),\beta_5(\epsilon/6),\beta_6(\epsilon/6)\}.$$

The proof is complete. □

In [6], Fliege and Xu present the exponential rate convergence of the optimal solutions of Stochastic MOP. As a complementary work, we study the exponential rate of the M-stationary point. Similar results can be obtained for C- and S-stationary points. If we strengthen the conditions in Assumption 4.1 to the following:

> there exists a constant $\varrho>0$ such that for every $z\in Z$,
> $$\mathbb{E}[e^{(\theta(z,\xi)-\mathbb{E}_P[\theta(z,\xi)])t}]\leq e^{\varrho^2t^2/2},\forall t\in \mathbb{R},$$
> the constant γ_2 equals to 1.

Then by formula (5.13) in [25], we have

$$\mathrm{Prob}\{\sup_{z\in Z}\left|\theta^N(z)-\mathbb{E}_P[\theta(z,\xi)]\right| \geq \epsilon\}\leq e^{-N\varrho^*}+2\left[\frac{O(1)\sup\limits_{z',z''\in Z}\|z'-z''\|\mathbb{E}[\kappa_2(\xi)]}{\epsilon}\right]^n e^{(-N\frac{\epsilon^2}{32\varrho^2})},$$

where $O(1)$ is a generic constant and ϱ^* is a positive constant given by Cramér's Large Deviation Theorem, see [25, Theorem 5.1] and the following comments for more details. Then

$$\begin{aligned}
&\mathrm{Prob}\{\sup_{z\in Z}\left\|f^N(z)-\mathbb{E}_P[f(z,\xi)]\right\|\geq \epsilon/6\}\\
&\leq \mathrm{Prob}\Big\{\sup_{z\in Z}\sup_{1\leq i\leq s}\left|f_i^N(z)-\mathbb{E}_P[f_i(z,\xi)]\right|\geq\frac{\epsilon}{6s}\Big\}\\
&\leq e^{-N\varrho^*}+2\left[\frac{O(1)\sup\limits_{z',z''\in Z}\|z'-z''\|\mathbb{E}[\kappa_2(\xi)]}{\epsilon/6s}\right]^n e^{(-N\frac{(\epsilon/6s)^2}{32\varrho^2})},
\end{aligned}$$

where s is the dimension of function f. Similarly, we can estimate the rest five probabilities in the proof of Theorem 4.2. Then Theorem 4.2 holds with

$$C(\epsilon) = 6 \times 2 \left[\frac{O(1) \sup_{z',z'' \in Z} \|z' - z''\| \mathbb{E}[\kappa_2(\xi)]}{\epsilon/6\sigma n} + 1 \right]^n,$$

$$\beta(\epsilon) = \min \left\{ \varrho^*, \frac{(\epsilon/6\sigma n)^2}{32\varrho^2} \right\},$$

where $\sigma = \max\{s, m\}$, m and n are the dimensions of function G and variable z respectively.

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