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DUALITY OF MATHEMATICAL PROGRAMMING PROBLEMS WITH EQUILIBRIUM CONSTRAINTS

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Abstract: In this paper, we consider the mathematical programming problem with equilibrium constraints (MPEC). We introduce Wolfe type dual and Mond-Weir type dual models for the MPEC and establish weak and strong duality theorems relating to the MPEC and the two dual models under convexity and generalized convexity assumptions.

Key words: *mathematical programming with equilibrium constraints, Wolfe type dual, Mond-Weir dual, convexity*

Mathematics Subject Classification: *90C30, 90C46*

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1 Introduction

In this paper, we consider the following program, known across the literature as a mathematical program with equilibrium constraints (MPEC):

$$
\begin{array}{llll}
\text{(MPEC)} & \min & f(z) & \\
& \text{subject to :} & g(z) \leq 0, \quad h(z) = 0, & \\
& & G(z) \geq 0, \quad H(z) \geq 0, \quad \langle G(z), H(z) \rangle = 0, &
\end{array}
$$

where, $f : \mathbb{R}^n \to \mathbb{R}, g : \mathbb{R}^n \to \mathbb{R}^k, h : \mathbb{R}^n \to \mathbb{R}^p, G : \mathbb{R}^n \to \mathbb{R}^l$ and $H : \mathbb{R}^n \to \mathbb{R}^l$. Assume that f, g, h, G and H are continuously differentiable on $\mathbb{R}^n$.

Note that if $h(z) := 0, G(z) := 0, H(z) := 0$, then, MPEC coincides with the standard nonlinear programming problem, which is well studied in the literature, see eg. Mangasarian [14].

Luo *et al.* [13] presented a comprehensive study of MPEC. Fukushima and Pang [7] studied some feasibility issues in MPEC. Outrata [17] established necessary optimality conditions for a class of MPEC, provided the complementarity problem is strongly regular at the solution. Outrata *et al.* [18] derived necessary optimality conditions for those MPECs which can be treated by the implicit programming approach and proposed solution method based on the bundle technique of nonsmooth optimization. Further, Kocvara and Outrata [12] presented a new theoretical framework for the implicit programming approach which is useful

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for numerical solution of problems like MPEC and MPCC. Scheel and Scholtes [22] studied mathematical program with complementarity constraints and introduced several stationary concepts. Henrion and Surowiec [10] compared two different calmness conditions on MPEC and derived first order necessary optimality conditions via tools of generalized differentiation introduced by Mordukhovich. Henrion *et al.* [11] derived an inner approximation to the Frechet normal cone to the graph of solution mapping and used this inner approximation to check strong stationarity via the weaker concept of M-stationarity.

Ye [24] considered optimization problem with complementarity constraints and derived necessary and sufficient optimality conditions involving the proximal coderivatives. Ye [25] considered a mathematical program with variational inequality constraints and an abstract constraint and established Fritz-John type and Kuhn-Tucker type necessary optimality conditions involving Mordukhovich coderivatives. Further, Ye [25] introduced several constraint qualifications for the Kuhn-Tucker type necessary optimality conditions involving Mordukhovich coderivative. Flegel and Kanzow [4] used a completely different approach and obtained short and elementary proof of the optimality conditions for MPEC using the standard Fritz-John conditions. Ye [26] considered MPEC and introduced various stationary conditions and established that it is sufficient for global or locally optimal under some generalized convexity assumption and obtained new constraint qualifications. Further, Flegel and Kanzow [5] introduced a new Abadie-type constraint qualification and a new Slater-type constraint qualification for the MPEC and proved that new Slater-type CQ implies new Abadie-type CQ. Moreover, Flegel and Kanzow [5] introduced a new optimality condition which holds under new Abadie-type CQ. Flegel and Kanzow [6] established that M-stationarity is a first order optimality condition under a very weak Abadie-type constraint qualification and obtained very strong exact penalization result. Recently, Guo and Lin [8] have investigated the weakest constraint qualifications for the Bouligand and Mordukhovich stationary points and relation among the existing constraint qualifications for mathematical programs with equilibrium constraints (MPEC). Chieu and Lee [3] have studied the relations among the system of constraint qualifications tailored for mathematical programs with equilibrium constraints (MPEC) and their local preservation property.

MPEC form a relatively new and interesting subclass of nonlinear programming problems. Chemical process industries require the solution of nonlinear problems as part of current process synthesis, design, optimization and control activities. The use of equilibrium constraints in modeling process engineering problems is a relatively new and exciting field of research, see Raghunathan and Biegler [21]. Chemical processes involve systems that are governed by chemical equilibrium. Raghunathan and Biegler [21] developed an algorithmic framework to solve general mathematical programming problems with equilibrium constraints using nonlinear programming algorithms.

Hydro-economic river basin models (HERBM) based on mathematical programming are conventionally formulated as explicit aggregate optimization problems with a single, aggregate objective function. This model implicitly assumes that decisions on water allocation are made via central planning or functioning markets such as to maximize the social welfare. However, in the absence of perfect water markets, individual optimal decisions by water users differ from the social optimum. Britz *et al.* [1] proposed hydro-economic river basin models based on multiple optimization problems with equilibrium constraints, that yield more realistic results on comparing with water management institutions.

The concept of duality is of fundamental importance in nonlinear programming problems. Wolfe [23] used the Kuhn-Tucker conditions to formulate a dual program for a nonlinear programming problem with the aim of defining a problem whose objective value gives lower bound on the optimal value of the primal problem and whose optimal solution yields an optimal solution for the primal problem under certain regularity conditions. Wolfe [23] established the weak duality i. e. every feasible solution of the dual has an objective value less then or equal to the objective value for every feasible solution of the primal problem, under the same convexity assumptions as required for the sufficiency of the Kuhn-Tucker conditions. Mond and Weir [16] proposed a new type of dual based on the Wolfe type of dual and established usual duality theorems under weaker convexity assumptions on the functions involved in the objective and constraints. Many authors have studied duality results for nonlinear programming problems in last four decades, see Mishra *et al.* [15] and references cited in these.

To the best of our knowledge, dual problem to a MPEC has not been given in the literature as yet.

In this paper, motivated by Wolfe [23] and Mond and Weir [16], we introduce Wolfe type and Mond-Weir type dual programs to the mathematical programming problem with equilibrium constraints (MPEC). We have established weak and strong duality theorems relating the MPEC and the two dual programs. The paper is organized as follows: in Section 2, we give some preliminaries, definitions and results. In Section 3, we derive weak and strong duality theorems relating to the MPEC and the two dual models under convexity and generalized convexity assumptions.

2 Preliminaries

This section contains some preliminaries which will be used throughout the paper.

Theorem 2.1 ([14])**.** *Let f be a differentiable real valued function defined on a nonempty open convex set $X \subseteq \mathbb{R}^n$. Then, f is convex at $\bar{x} \in X$ iff*

$$f(x) \geq f(\bar{x}) + \langle \nabla f(\bar{x}), x - \bar{x} \rangle, \quad \forall x \in X.$$

Definition 2.2 ([14])**.** Let f be a differentiable real valued function defined on a nonempty open convex set $X \subseteq \mathbb{R}^n$. Then, the function f is said to be pseudoconvex at $\bar{x} \in X$ iff the following implication holds:

$$x, \bar{x} \in X, \ \langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq 0 \Rightarrow f(x) \geq f(\bar{x}).$$

Equivalently,

$$x, \bar{x} \in X, \ f(x) < f(\bar{x}) \Rightarrow \langle \nabla f(\bar{x}), x - \bar{x} \rangle < 0.$$

Remark 2.3 ([14])**.** Every convex function is pseudoconvex, but the converse is not true in general. For example, $f(x) = x + x^3$ is not convex on $\mathbb{R}$, because $\nabla^2 f(x) = 6x < 0$, for $x < 0$. However, f is pseudocovex on $\mathbb{R}$. Since, $\nabla f(\bar{x})(x - \bar{x}) \geq 0 \Rightarrow x - \bar{x} \geq 0 \Rightarrow x + x^3 \geq \bar{x} + \bar{x}^3 \Rightarrow f(x) \geq f(\bar{x})$.

Definition 2.4 ([14])**.** Let f be a differentiable real valued function defined on a nonempty open convex set $X \subseteq \mathbb{R}^n$. Then, the function f is said to be quasiconvex at $\bar{x} \in X$ iff the following implication holds:

$$x, \bar{x} \in X, \quad f(x) \leq f(\bar{x}) \Rightarrow \langle \nabla f(\bar{x}), x - \bar{x} \rangle \leq 0.$$

Remark 2.5 ([14])**.** Every pseudoconvex function is quasiconvex, but converse is not true. For example $f(x) = x^3$, $x \in \mathbb{R}$ is quasiconvex, but not pseudoconvex.

Given a feasible vector $\bar{z}$ for the MPEC, we define the following index sets:

$$
\begin{aligned}
&I_g := \{i : g_i(\bar{z}) = 0\},\\
&\alpha := \alpha(\bar{z}) = \{i : G_i(\bar{z}) = 0, H_i(\bar{z}) > 0\},\\
&\beta := \beta(\bar{z}) = \{i : G_i(\bar{z}) = 0, H_i(\bar{z}) = 0\},\\
&\gamma := \gamma(\bar{z}) = \{i : G_i(\bar{z}) > 0, H_i(\bar{z}) = 0\},
\end{aligned}
$$

where $i \in \{1, 2, ..., l\}$. The set β is known as degenerate set. If β is empty, the vector $\bar{z}$ is said to satisfy the strict complementarity condition. The standard nonlinear programming which has only one dual stationary condition, i.e., the Karush-Kuhn-Tucker condition, but for the MPEC, we have various stationarity concepts.

The following concept of M-stationary point was introduced by Outrata [19].

Definition 2.6 (M-stationary point)**.** A feasible point $\bar{z}$ of MPEC is called the Mordukhovich stationary point if there exists $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{k+p+2l}$, such that following conditions hold:

$$
0 = \nabla f(\bar{z}) + \sum_{i\in I_g} \lambda_i^g \nabla g_i(\bar{z}) + \sum_{i=1}^{p} \lambda_i^h \nabla h_i(\bar{z}) - \sum_{i=1}^{l} [\lambda_i^G \nabla G_i(\bar{z}) + \lambda_i^H \nabla H_i(\bar{z})], \tag{2.1}
$$

$$
\lambda_{I_g}^g \geq 0, \quad \lambda_\gamma^G = 0, \qquad \lambda_\alpha^H = 0, \tag{2.2}
$$

$$
\forall i \in \beta, \quad either \quad \lambda_i^G > 0, \quad \lambda_i^H > 0 \quad or \quad \lambda_i^G \lambda_i^H = 0.
$$

The following concept of Strong-stationary point was introduced in [20, 22].

Definition 2.7 (S-stationary point)**.** A feasible point $\bar{z}$ of MPEC is called strong stationary point if there exists $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{k+p+2l}$, such that (2.1), (2.2) and the following condition hold:

$$
\forall i \in \beta, \ \lambda_i^G \geq 0, \quad \lambda_i^H \geq 0.
$$

The following concept of C-stationary point was introduced by Scheel and Scholtes [22].

Definition 2.8 (C-stationary point)**.** A feasible point $\bar{z}$ of MPEC is called the Clarke stationary point if there exists $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{k+p+2l}$, such that (2.1), (2.2) and the following condition hold:

$$
\forall i \in \beta, \ \lambda_i^G \lambda_i^H \geq 0.
$$

The following concept of A-stationary point was introduced by Flegel and Kanzow [4].

Definition 2.9 (A-stationary point)**.** A feasible point $\bar{z}$ of MPEC is called alternatively stationary point if there exists $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{k+p+2l}$, such that (2.1), (2.2) and the following condition hold:

$$
\forall i \in \beta, \ \lambda_i^G \geq 0, \ or \ \lambda_i^H \geq 0.
$$

Remark 2.10 ([6])**.** The M-stationarity is the second strongest stationary condition after the S-stationarity. Also, the intersection of A-stationarity and C-stationarity give M-stationarity and strong stationarity implies M-, A- and C-stationarity.

The following definition of the no nonzero abnormal multiplier constraint qualification for MPEC is taken from Definition 2.10 in Ye [26].

Definition 2.11. Let $\bar{z}$ be a feasible point of MPEC, where all functions are continuously differentiable at $\bar{z}$. We say that the No Nonzero Abnormal Multiplier Constraint Qualification (NNAMCQ) is satisfied at $\bar{z}$ if there is no nonzero vector $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{k+p+2l}$, such that

$$0 = \sum_{i \in I_g} \lambda_i^g \nabla g_i(\bar{z}) + \sum_{i=1}^{p} \lambda_i^h \nabla h_i(\bar{z}) - \sum_{i=1}^{l} [\lambda_i^G \nabla G_i(\bar{z}) + \lambda_i^H \nabla H_i(\bar{z})],$$

$$\lambda_{I_g}^g \geq 0, \quad \lambda_\gamma^G = 0, \qquad \lambda_\alpha^H = 0,$$

$$\forall i \in \beta, \quad either \quad \lambda_i^G > 0, \quad \lambda_i^H > 0 \quad or \quad \lambda_i^G \lambda_i^H = 0.$$

The following definition of the generalized Mangasarian-Fromovitz constraint qualification for MPEC is taken from Definition 2.11 in Ye [26].

Definition 2.12. Let $\bar{z}$ be a feasible point of MPEC where all functions are continuously differentiable at $\bar{z}$. We say that MPEC generalized Mangasarian-Fromovitz constraint qualification (MPEC GMFCQ) is satisfied at $\bar{z}$ if
(i) for every partition of β into sets P, Q, R with $R \neq \phi$, there exist d, such that

$$\begin{aligned}
&\nabla g_i(\bar{z})^T d \leq 0, \ \forall i \in I_g, \\
&\nabla h_i(\bar{z})^T d = 0, \ \forall i = 1, 2, ..., p, \\
&\nabla G_i(\bar{z})^T d = 0, \ \forall i \in \alpha \cup Q, \\
&\nabla H_i(\bar{z})^T d = 0, \ \forall i \in \gamma \cup P, \\
&\nabla G_i(\bar{z})^T d \geq 0, \ \nabla H_i(\bar{z})^T d \geq 0, \ i \in R,
\end{aligned}$$

and for some $i \in R$ either $\nabla G_i(\bar{z})^T d > 0$ or $\nabla H_i(\bar{z})^T d > 0$;
(ii) for every partition of β into sets P, Q, the gradient vectors

$$\begin{aligned}
&\nabla h_i(\bar{z}), \ \forall i = 1, 2, ..., p, \\
&\nabla G_i(\bar{z}), \ \forall i \in \alpha \cup Q, \\
&\nabla H_i(\bar{z}), \ \forall i \in \gamma \cup P,
\end{aligned}$$

are linearly independent and there exists $d \in \mathbb{R}^n$, such that

$$\begin{aligned}
&\nabla g_i(\bar{z})^T d < 0, \ \forall i \in I_g, \\
&\nabla h_i(\bar{z})^T d = 0, \ \forall i = 1, 2, ..., p, \\
&\nabla G_i(\bar{z})^T d = 0, \ \forall i \in \alpha \cup Q, \\
&\nabla H_i(\bar{z})^T d = 0, \ \forall i \in \gamma \cup P.
\end{aligned}$$

The following proposition is Proposition 2.1 in [26], which proves that the NNAMCQ is equivalent to MPEC GMFCQ.

Proposition 2.13. NNAMCQ *is equivalent to* MPEC GMFCQ.

The following theorem is Theorem 2.1 in [26], which gives the Fritz-John type M-stationary condition for a feasible solution to be a local solution of the MPEC.

Theorem 2.14 (Fritz-John type M-stationary condition). *Let $\bar{z}$ be a local solution of* MPEC *where all functions are continuously differentiable at $\bar{z}$. Then, there exists $r \geq 0$, $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{k+p+2l}$, not all zero, such that*

$$0 = r\nabla f(\bar{z}) + \sum_{i \in I_g} \lambda_i^g \nabla g_i(\bar{z}) + \sum_{i=1}^{p} \lambda_i^h \nabla h_i(\bar{z}) - \sum_{i=1}^{l} [\lambda_i^G \nabla G_i(\bar{z}) + \lambda_i^H \nabla H_i(\bar{z})],$$

$$\lambda_{I_g}^g \geq 0, \quad \lambda_\gamma^G = 0, \qquad \lambda_\alpha^H = 0,$$

$$\forall i \in \beta, \quad \textit{either} \quad \lambda_i^G > 0, \quad \lambda_i^H > 0 \quad \textit{or} \quad \lambda_i^G \lambda_i^H = 0.$$

In the above Fritz-John type M-stationary condition if r is never zero, then, it can be taken as 1 and the following KKT type M-stationary condition follows immediately. The following theorem is Corollary 2.1 in [26].

Theorem 2.15 (Kuhn-Tucker type necessary M-stationary condition). *Let $\bar{z}$ be a locally optimal solution for* MPEC *where all the function are continuously differentiable at $\bar{z}$. Suppose* NNAMCQ *is satisfied at $\bar{z}$, then, $\bar{z}$ is M-stationary.*

In the following theorem taken from Theorem 2.3 in [26], we see that M-stationary condition turns into a sufficient optimality condition under certain MPEC generalized convexity condition.

Theorem 2.16 (Sufficient M-stationary condition). *Let $\bar{z}$ be a feasible point of* MPEC *and M-stationary condition holds at $\bar{z}$, i.e., there exists $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{k+p+2l}$, such that*

$$0 = \nabla f(\bar{z}) + \sum_{i \in I_g} \lambda_i^g \nabla g_i(\bar{z}) + \sum_{i=1}^{p} \lambda_i^h \nabla h_i(\bar{z}) - \sum_{i=1}^{l} [\lambda_i^G \nabla G_i(\bar{z}) + \lambda_i^H \nabla H_i(\bar{z})],$$

$$\lambda_{I_g}^g \geq 0, \quad \lambda_\gamma^G = 0, \qquad \lambda_\alpha^H = 0,$$

$$\forall i \in \beta, \quad \textit{either} \quad \lambda_i^G > 0, \quad \lambda_i^H > 0 \quad \textit{or} \quad \lambda_i^G \lambda_i^H = 0.$$

Let $J^+ := \{i : \lambda_i^h > 0\}, \quad J^- := \{i : \lambda_i^h < 0\}$,
$\beta^+ := \{i \in \beta : \lambda_i^G > 0, \lambda_i^H > 0\}$,
$\beta_G^+ := \{i \in \beta : \lambda_i^G = 0, \lambda_i^H > 0\}, \quad \beta_G^- := \{i \in \beta : \lambda_i^G = 0, \lambda_i^H < 0\}$,
$\beta_H^+ := \{i \in \beta : \lambda_i^H = 0, \lambda_i^G > 0\}, \quad \beta_H^- := \{i \in \beta : \lambda_i^H = 0, \lambda_i^G < 0\}$,
$\alpha^+ := \{i \in \alpha : \lambda_i^G > 0\}, \quad \alpha^- := \{i \in \alpha : \lambda_i^G < 0\}$,
$\gamma^+ := \{i \in \gamma : \lambda_i^H > 0\}, \quad \gamma^- := \{i \in \gamma : \lambda_i^H < 0\}$.
Further, suppose that f is pseudoconvex at $\bar{z}$, $g_i(i \in I_g), h_i(i \in J^+), -h_i(i \in J^-), G_i(i \in \alpha^- \cup \beta_H^-), -G_i(i \in \alpha^+ \cup \beta_H^+ \cup \beta^+), H_i(i \in \gamma^- \cup \beta_G^-), -H_i(i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$ are quasiconvex at $\bar{z}$. Then, in the case when $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \phi, \bar{z}$ is a global optimal solution for MPEC*; and when $\beta_G^- \cup \beta_H^- = \phi$ or when $\bar{z}$ is an interior point relative to the set*

$$S \cap \{z : G_i(z) = 0, H_i(z) = 0, i \in \beta_G^- \cup \beta_H^+\},$$

i.e., for every feasible point z which is close to $\bar{z}$, it holds that

$$G_i(z) = 0, \ H_i(z) = 0, \ \forall i \in \beta_G^- \cup \beta_H^-,$$

then, $\bar{z}$ is a locally optimal solution for MPEC*, where S denotes the set of feasible solutions of* MPEC.

3 Duality

In this section, we formulate a Wolfe type dual problem and a Mond-Weir type dual problem for the MPEC under convexity and generalized convexity assumptions.

$$WDMPEC(\bar{z}) \quad \max_{u,\lambda} \quad f(u) + \sum_{i \in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^{p} \lambda_i^h h_i(u) - \sum_{i=1}^{l} [\lambda_i^G G_i(u) + \lambda_i^H H_i(u)]$$

subject to:

$$0 = \nabla f(u) + \sum_{i \in I_g} \lambda_i^g \nabla g_i(u) + \sum_{i=1}^{p} \lambda_i^h \nabla h_i(u) - \sum_{i=1}^{l} [\lambda_i^G \nabla G_i(u) + \lambda_i^H \nabla H_i(u)], \tag{3.1}$$

$$\lambda_{I_g}^g \geq 0, \quad \lambda_\gamma^G = 0, \qquad \lambda_\alpha^H = 0,$$

$$\forall i \in \beta, \quad either \quad \lambda_i^G > 0, \quad \lambda_i^H > 0 \quad or \quad \lambda_i^G \lambda_i^H = 0,$$

where, $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{k+p+2l}$.

Remark 3.1. If $h(z) := 0, G(z) := 0, H(z) := 0$, then, Wolfe type dual problem WDMPEC $(\bar{z})$ for MPEC coincides with the classical Wolfe type dual problem for nonlinear programming given by Wolfe [23].

Theorem 3.2 (Weak Duality). *Let $\bar{z}$ be feasible for* MPEC, (u, λ) *feasible for* WDMPEC $(\bar{z})$ *and index sets $I_g, \alpha, \beta, \gamma$ defined accordingly. Suppose that $f, g_i(i \in I_g), h_i(i \in J^+), -h_i(i \in J^-), G_i(i \in \alpha^- \cup \beta_H^-), -G_i(i \in \alpha^+ \cup \beta_H^+ \cup \beta^+), H_i(i \in \gamma^- \cup \beta_G^-), -H_i(i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$ are convex functions at u. If $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \phi$. Then, for any z feasible for the* MPEC, *we have*

$$f(z) \geq f(u) + \sum_{i \in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^{p} \lambda_i^h h_i(u) - \sum_{i=1}^{l} [\lambda_i^G G_i(u) + \lambda_i^H H_i(u)].$$

Proof. Let z be any feasible point for MPEC. Then, we have

$$g_i(z) \leq 0, \quad \forall i \in I_g$$

and

$$h_i(z) = 0, \quad i = 1, 2..., p.$$

Since f is convex at u, then,

$$f(z) - f(u) \geq \langle \nabla f(u), z - u \rangle . \tag{3.2}$$

Similarly, we have

$$g_i(z) - g_i(u) \geq \langle \nabla g_i(u), z - u \rangle , \quad \forall i \in I_g, \tag{3.3}$$

$$h_i(z) - h_i(u) \geq \langle \nabla h_i(u), z - u \rangle , \quad \forall i \in J^+, \tag{3.4}$$

$$-h_i(z) + h_i(u) \geq - \langle \nabla h_i(u), z - u \rangle , \quad \forall i \in J^-, \tag{3.5}$$

$$-G_i(z) + G_i(u) \geq - \langle \nabla G_i(u), z - u \rangle , \quad \forall i \in \alpha^+ \cup \beta_H^+ \cup \beta^+, \tag{3.6}$$

$$-H_i(z) + H_i(u) \geq - \langle \nabla H_i(u), z - u \rangle , \quad \forall i \in \gamma^+ \cup \beta_G^+ \cup \beta^+. \tag{3.7}$$

If $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \phi$, multiplying $(3.3) - (3.7)$ by $\lambda_i^g \geq 0 (i \in I_g), \lambda_i^h > 0 (i \in J^+), -\lambda_i^h > 0 (i \in J^-), \lambda_i^G > 0 (i \in \alpha^+ \cup \beta_H^+ \cup \beta^+), \lambda_i^H > 0 (i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$, respectively and adding $(3.2) - (3.7)$, we get

$$\begin{aligned} & f(z) - f(u) + \sum_{i \in I_g} \lambda_i^g g_i(z) - \sum_{i \in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^{p} \lambda_i^h h_i(z) - \sum_{i=1}^{p} \lambda_i^h h_i(u) - \sum_{i=1}^{l} \lambda_i^G G_i(z) \\ & + \sum_{i=1}^{l} \lambda_i^G G_i(u) - \sum_{i=1}^{l} \lambda_i^H H_i(z) + \sum_{i=1}^{l} \lambda_i^H H_i(u) \\ & \geq \left\langle \nabla f(u) + \sum_{i \in I_g} \lambda_i^g \nabla g_i(u) + \sum_{i=1}^{p} \lambda_i^h \nabla h_i(u) - \sum_{i=1}^{l} [\lambda_i^G \nabla G_i(u) + \lambda_i^H \nabla H_i(u)], z - u \right\rangle . \end{aligned}$$

Using condition (3.1), we have

$$\begin{aligned} & f(z) - f(u) + \sum_{i \in I_g} \lambda_i^g g_i(z) - \sum_{i \in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^{p} \lambda_i^h h_i(z) - \sum_{i=1}^{p} \lambda_i^h h_i(u) \\ & - \sum_{i=1}^{l} \lambda_i^G G_i(z) + \sum_{i=1}^{l} \lambda_i^G G_i(u) - \sum_{i=1}^{l} \lambda_i^H H_i(z) + \sum_{i=1}^{l} \lambda_i^H H_i(u) \geq 0. \end{aligned}$$

Now, using the feasibility of z for MPEC, that is, $g_i(z) \leq 0, h_i(z) = 0, G_i(z) \geq 0, H_i(z) \geq 0$, we get

$$f(z) - f(u) - \sum_{i \in I_g} \lambda_i^g g_i(u) - \sum_{i=1}^{p} \lambda_i^h h_i(u) + \sum_{i=1}^{l} \lambda_i^G G_i(u) + \sum_{i=1}^{l} \lambda_i^H H_i(u) \geq 0.$$

Hence,

$$f(z) \geq f(u) + \sum_{i \in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^{p} \lambda_i^h h_i(u) - \sum_{i=1}^{l} [\lambda_i^G G_i(u) + \sum_{i=1}^{l} \lambda_i^H H_i(u)].$$

This completes the proof. □

Remark 3.3. If $h(z) := 0, G(z) := 0, H(z) := 0$, then, Theorem 3.2 coincides with Theorem 8.1.3 in [14].

The following corollary is a direct consequence of Theorem 3.2.

Corollary 3.4. *Let $\bar{z}$ be feasible for* MPEC *where all constraint functions g_i, h_i, G_i, H_i are affine and index sets $I_g, \alpha, \beta, \gamma$ defined accordingly. Then, for any z feasible for the* MPEC *and (u, λ) feasible for* WDMPEC $(\bar{z})$, *we have*

$$f(z) \geq f(u) + \sum_{i \in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^{p} \lambda_i^h h_i(u) - \sum_{i=1}^{l} [\lambda_i^G G_i(u) + \lambda_i^H H_i(u)].$$

Theorem 3.5 (Strong Duality). *If $\bar{z}$ is a global optimal solution of* MPEC*, such that* NNAMCQ *is satisfied at $\bar{z}$ and index sets $I_g, \alpha, \beta, \gamma$ defined accordingly. Let $f, g_i(i \in I_g), h_i(i \in J^+), -h_i(i \in J^-), G_i(i \in \alpha^- \cup \beta_H^-), -G_i(i \in \alpha^+ \cup \beta_H^+ \cup \beta^+), H_i(i \in \gamma^- \cup \beta_G^-), -H_i(i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$ satisfy the assumption of the Theorem 3.2. Then, there exists $\bar{\lambda}$, such that $(\bar{z}, \bar{\lambda})$ is a global optimal solution of* WDMPEC $(\bar{z})$ *and respective objective values are equal.*

Proof. Since, $\bar{z}$ is a global optimal solution of MPEC and the NNAMCQ is satisfied at $\bar{z}$, hence, $\exists\, \bar{\lambda} = (\bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \mathbb{R}^{k+p+2l}$, such that the M-stationarity conditions for MPEC are satisfied, that is,

$$0 = \nabla f(\bar{z}) + \sum_{i\in I_g} \bar{\lambda}_i^g \nabla g_i(\bar{z}) + \sum_{i=1}^{p} \bar{\lambda}_i^h \nabla h_i(\bar{z}) - \sum_{i=1}^{l} [\bar{\lambda}_i^G \nabla G_i(\bar{z}) + \bar{\lambda}_i^H \nabla H_i(\bar{z})], \tag{3.8}$$

$$\bar{\lambda}_{I_g}^g \geq 0, \quad \bar{\lambda}_\gamma^G = 0, \qquad \bar{\lambda}_\alpha^H = 0,$$

$$\forall i \in \beta, \quad \textit{either} \quad \bar{\lambda}_i^G > 0, \quad \bar{\lambda}_i^H > 0 \quad \textit{or} \quad \bar{\lambda}_i^G \bar{\lambda}_i^H = 0.$$

Therefore, $(\bar{z}, \bar{\lambda})$ is feasible for WDMPEC $(\bar{z})$. By Theorem 3.2, we have

$$f(\bar{z}) \geq f(u) + \sum_{i\in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^{p} \lambda_i^h h_i(u) - \sum_{i=1}^{l} [\lambda_i^G G_i(u) + \lambda_i^H H_i(u)], \tag{3.9}$$

for any feasible solution (u, λ) for WDMPEC $(\bar{z})$. Also, from the feasibility condition of MPEC and WDMPEC $(\bar{z})$, that is, for $i \in I_g(\bar{z}), g_i(\bar{z}) = 0$, also $h_i(\bar{z}) = 0, G_i(\bar{z}) = 0, \forall i \in \alpha \cup \beta$ and $H_i(\bar{z}) = 0, \forall i \in \beta \cup \gamma$, then, we have

$$f(\bar{z}) = f(\bar{z}) + \sum_{i\in I_g} \bar{\lambda}_i^g g_i(\bar{z}) + \sum_{i=1}^{p} \bar{\lambda}_i^h h_i(\bar{z}) - \sum_{i=1}^{l} [\bar{\lambda}_i^G G_i(\bar{z}) + \bar{\lambda}_i^H H_i(\bar{z})]. \tag{3.10}$$

Using (3.9) and (3.10), we have

$$\begin{aligned} &f(\bar{z}) + \sum_{i\in I_g} \bar{\lambda}_i^g g_i(\bar{z}) + \sum_{i=1}^{p} \bar{\lambda}_i^h h_i(\bar{z}) - \sum_{i=1}^{l} [\bar{\lambda}_i^G G_i(\bar{z}) + \bar{\lambda}_i^H H_i(\bar{z})] \\ &\geq f(u) + \sum_{i\in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^{p} \lambda_i^h h_i(u) - \sum_{i=1}^{l} [\lambda_i^G G_i(u) + \lambda_i^H H_i(u)]. \end{aligned}$$

Hence,$(\bar{z}, \bar{\lambda})$ is a global optimal solution for WDMPEC $(\bar{z})$ and the respective objective values are equal. □

Remark 3.6. If $h(z) := 0, G(z) := 0, H(z) := 0$, then, Theorem 3.5 coincides with Theorem 8.1.4 in [14].

The following corollary is a consequence of Theorem 3.5.

Corollary 3.7. *If $\bar{z}$ is a optimal solution of* MPEC*, where all constraint functions g_i, h_i, G_i, H_i are affine. Then, there exists $\bar{\lambda}$, such that $(\bar{z}, \bar{\lambda})$ is a optimal solution of* WDMPEC $(\bar{z})$ *and respective objective values are equal.*

Proof. Let $\bar{z}$ is a global optimal solution of MPEC. If all constraint functions g_i, h_i, G_i, H_i are affine, then, from Theorem 3.6, Theorem 4.3 and Proposition 4.2 in [25], the M-stationary condition holds for MPEC. The remaining part of the proof is same as in the proof of Theorem 3.5. □

Example 3.8. Consider the following MPEC in $\mathbb{R}^2$:

$$\begin{array}{lll} \text{MPEC(1)} & \min & z_1^2 + z_2^2 \\ & \text{subject to :} & z_1 \geq 0, \\ & & z_2 \geq 0, \\ & & z_1 z_2 = 0. \end{array}$$

Let $\Xi := \{(z_1, z_2) \mid z_1 \geq 0, z_2 \geq 0, z_1 z_2 = 0\}$ be feasible region of MPEC(1). Now, we formulate Wolfe type dual problem WDMPEC $(\bar{z})$ for MPEC(1):

$$\max_{u,\lambda} u_1^2 + u_2^2 - [\lambda^G u_1 + \lambda^H u_2]$$

subject to:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2u_1 \\ 2u_2 \end{pmatrix} - \lambda^G \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \lambda^H \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

If β is non-empty, then, either

$$\lambda^G > 0, \quad \lambda^H > 0, \quad \text{or} \quad \lambda^G \lambda^H = 0.$$

If we take point $\bar{z} = (0, 0)$ from feasible region Ξ, then, index sets $\alpha\,(0, 0)$ and $\gamma\,(0, 0)$ are empty sets, but $\beta := \beta\,(0, 0)$ is non-empty. Also, from solving constraint equation in feasible region of WDMPEC$(0, 0)$, we get $\lambda^G = 2u_1$ and $\lambda^H = 2u_2$. Since β is non-empty, we consider a $\beta^+, \beta_G^+, \beta_H^+$ to decide the feasible region of WDMPEC$(0, 0)$. It is clear that assumptions of Theorem 3.2 are satisfied, so, Theorem 3.2 holds between MPEC(1) and WDMPEC$(0, 0)$.

Also, if we put value of λ^G and λ^H in the objective function of dual problem, then, $u_1^2 + u_2^2 - [\lambda^G u_1 + \lambda^H u_2] = -u_1^2 - u_2^2$. Further, it is clear that for any $z \in \Xi, f\,(z_1, z_2) = 0$ or z_1^2 or z_2^2 which is greater than equal to $-u_1^2 - u_2^2$, for any feasible u. Hence, Theorem 3.2 is verified.

If we take feasible point $\bar{z} = (1, 0)$, then, index sets $\alpha\,(1, 0)$ and $\beta\,(1, 0)$ are empty sets but $\gamma\,(1, 0)$ is non-empty set. Therefore, we find the feasible region of WDMPEC$(1, 0)$ according to these index sets, Theorem 3.2 holds between MPEC(1) and WDMPEC$(1, 0)$. Similarly, if we take feasible point $\bar{z} = (0, 1)$, then, the only non-empty set is $\gamma\,(0, 1)$. Then, we find the feasible region of WDMPEC$(0, 1)$ according to this index set, Theorem 3.2. holds between MPEC(1) and WDMPEC$(0, 1)$.

It is clear that $\bar{z} = (0, 0)$ is the optimal solution of MPEC(1). Also, $\nabla G(\bar{z})$ and $\nabla H(\bar{z})$ are linearly independent, so, we say that MPEC linear independence constraint qualification (MPEC LICQ) is satisfied at $\bar{z}$. Then, by Definition 2.8, Theorem 3.2 in [26], NNAMCQ is satisfied at $\bar{z}$. Hence, the assumptions of the Theorem 3.5 are satisfied. Then, by Theorem 3.5, there exists $\bar{\lambda}$ such that $(\bar{z}, \bar{\lambda})$ is an optimal solution of WDMPEC$(0, 0)$ and respective values are equal. Also, it is easy to see that

$$0 = f(\bar{z}) = f(\bar{z}) - [\bar{\lambda}^G G(\bar{z}) + \bar{\lambda}^H H(\bar{z})].$$

We now prove that duality relation between the mathematical programming problem with equilibrium constraints (MPEC) and the following Mond-Weir type dual problem

$$MWDMPEC(\bar{z}) \quad \max_{u,\lambda} \; f(u)$$

subject to:

$$0 = \nabla f(u) + \sum_{i\in I_g} \lambda_i^g \nabla g_i(u) + \sum_{i=1}^{p} \lambda_i^h \nabla h_i(u) - \sum_{i=1}^{l} [\lambda_i^G \nabla G_i(u) + \lambda_i^H \nabla H_i(u)], \tag{3.11}$$

$$\sum_{i\in I_g} \lambda_i^g g_i(u) \geq 0, \quad \sum_{i=1}^{p} \lambda_i^h h_i(u) \geq 0,$$

$$\sum_{i=1}^{l} \lambda_i^G G_i(u) \leq 0, \quad \sum_{i=1}^{l} \lambda_i^H H_i(u) \leq 0,$$

$$\lambda_{I_g}^g \geq 0, \quad \lambda_\gamma^G = 0, \qquad \lambda_\alpha^H = 0,$$

$$\forall i \in \beta, \quad \textit{either} \quad \lambda_i^G > 0, \quad \lambda_i^H > 0 \quad \textit{or} \quad \lambda_i^G \lambda_i^H = 0,$$

where, $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{k+p+2l}$.

Remark 3.9. If $h(z) := 0, G(z) := 0, H(z) := 0$, then, Mond-Weir type dual problem MWDMPEC $(\bar{z})$ for MPEC coincides with the Mond-Weir type dual problem for nonlinear programming given in [16].

Theorem 3.10 (Weak Duality)**.** *Let $\bar{z}$ be feasible for* MPEC*, (u, λ) be feasible for* MWDMPEC $(\bar{z})$ *and the index sets $I_g, \alpha, \beta, \gamma$ defined accordingly. Suppose that $f, g_i(i \in I_g), h_i(i \in J^+), -h_i(i \in J^-), G_i(i \in \alpha^- \cup \beta_H^-), -G_i(i \in \alpha^+ \cup \beta_H^+ \cup \beta^+), H_i(i \in \gamma^- \cup \beta_G^-), -H_i(i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$ are convex functions at u. If $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \phi$, then, for any z feasible for the* MPEC*, we have*

$$f(z) \geq f(u).$$

Proof. Let z be any feasible point for MPEC. Then, we have

$$g_i(z) \leq 0, \quad \forall i \in I_g$$

and

$$h_i(z) = 0, \quad i = 1, 2..., p.$$

Since, f is convex at u, we have

$$f(z) - f(u) \geq \langle \nabla f(u), z - u \rangle . \tag{3.12}$$

Similarly, we have

$$g_i(z) - g_i(u) \geq \langle \nabla g_i(u), z - u \rangle , \quad \forall i \in I_g, \tag{3.13}$$

$$h_i(z) - h_i(u) \geq \langle \nabla h_i(u), z - u \rangle , \quad \forall i \in J^+, \tag{3.14}$$

$$-h_i(z) + h_i(u) \geq - \langle \nabla h_i(u), z - u \rangle , \quad \forall i \in J^-, \tag{3.15}$$

$$-G_i(z) + G_i(u) \geq - \langle \nabla G_i(u), z - u \rangle , \quad \forall i \in \alpha^+ \cup \beta_H^+ \cup \beta^+, \tag{3.16}$$

$$-H_i(z) + H_i(u) \geq - \langle \nabla H_i(u), z - u \rangle , \quad \forall i \in \gamma^+ \cup \beta_G^+ \cup \beta^+. \tag{3.17}$$

If $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \phi$, multiplying (3.13) − (3.17) by $\lambda_i^g \geq 0 (i \in I_g), \lambda_i^h > 0 (i \in J^+), -\lambda_i^h > 0 (i \in J^-), \lambda_i^G > 0 (i \in \alpha^+ \cup \beta_H^+ \cup \beta^+), \lambda_i^H > 0 (i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$, respectively and adding (3.12) − (3.17), we get

$$\begin{aligned} & f(z) - f(u) + \sum_{i\in I_g} \lambda_i^g g_i(z) - \sum_{i\in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^{p} \lambda_i^h h_i(z) - \sum_{i=1}^{p} \lambda_i^h h_i(u) - \sum_{i=1}^{l} \lambda_i^G G_i(z) \\ & + \sum_{i=1}^{l} \lambda_i^G G_i(u) - \sum_{i=1}^{l} \lambda_i^H H_i(z) + \sum_{i=1}^{l} \lambda_i^H H_i(u) \\ & \geq \left\langle \nabla f(u) + \sum_{i\in I_g} \lambda_i^g \nabla g_i(u) + \sum_{i=1}^{p} \lambda_i^h \nabla h_i(u) - \sum_{i=1}^{l} [\lambda_i^G \nabla G_i(u) + \lambda_i^H \nabla H_i(u)], z - u \right\rangle. \end{aligned}$$

Using condition (3.11), the above inequality gives

$$\begin{aligned} & f(z) - f(u) + \sum_{i\in I_g} \lambda_i^g g_i(z) - \sum_{i\in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^{p} \lambda_i^h h_i(z) - \sum_{i=1}^{p} \lambda_i^h h_i(u) \\ & - \sum_{i=1}^{l} \lambda_i^G G_i(z) + \sum_{i=1}^{l} \lambda_i^G G_i(u) - \sum_{i=1}^{l} \lambda_i^H H_i(z) + \sum_{i=1}^{l} \lambda_i^H H_i(u) \geq 0. \end{aligned}$$

Now, using the feasibilty of z and u for MPEC and MWDMPEC($\bar{z}$), respectively, we get

$$f(z) - f(u) \geq 0$$

or

$$f(z) \geq f(u).$$

This completes the proof. □

Remark 3.11. If $h(z) := 0, G(z) := 0, H(z) := 0$, then, weak duality Theorem 3.10 coincides with the weak duality theorem for Mond-Weir type dual for standard nonlinear programming in [16].

The following corollary is a consequence of Theorem 3.10.

Corollary 3.12. *Let $\bar{z}$ be feasible for* MPEC *where all constraint functions g_i, h_i, G_i, H_i are affine and index sets $I_g, \alpha, \beta, \gamma$ defined accordingly. Then, for any z feasible for the* MPEC *and (u, λ) feasible for* MWDMPEC ($\bar{z}$), *we have*

$$f(z) \geq f(u).$$

Theorem 3.13 (Strong Duality)**.** *If $\bar{z}$ is a global optimal solution of* MPEC, *such that the* NNAMCQ *is satisfied at $\bar{z}$ and index sets $I_g, \alpha, \beta, \gamma$ defined accordingly. Let $f, g_i (i \in I_g), h_i (i \in J^+), -h_i (i \in J^-), G_i (i \in \alpha^- \cup \beta_H^-), -G_i (i \in \alpha^+ \cup \beta_H^+ \cup \beta^+), H_i (i \in \gamma^- \cup \beta_G^-), -H_i (i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$ satisfy the assumptions of the Theorem 3.10. Then, there exists $\bar{\lambda}$, such that $(\bar{z}, \bar{\lambda})$ is a global optimal solution of* MWDMPEC($\bar{z}$) *and respective objective values are equal.*

Proof. The proof follows the lines of the proof of Theorem 3.5 and invoking Theorem 3.10. □

Remark 3.14. If $h(z) := 0, G(z) := 0, H(z) := 0$, then, strong duality Theorem 3.13 coincides with the strong duality theorem for Mond-Weir type dual for standard nonlinear programming in [16].

The following corollary is a consequence of Theorem 3.13.

Corollary 3.15. *If $\bar{z}$ is a optimal solution of* MPEC *where all constraint functions g_i, h_i, G_i, H_i are affine. Then, there exists $\bar{\lambda}$ such that $(\bar{z}, \bar{\lambda})$ is a optimal solution of* MWDMPEC $(\bar{z})$ *and respective objective values are equal.*

Proof. If $\bar{z}$ is an optimal solution of MPEC where all constraint functions g_i, h_i, G_i, H_i are affine. Then, by Theorems 3.6, 4.3 and Proposition 4.2 in [25], the M-stationary condition holds for MPEC. Hence, by the proof of Theorem 3.13, there exists $\bar{\lambda}$, such that $(\bar{z}, \bar{\lambda})$ is an optimal solution of MWDMPEC$(\bar{z})$ and respective objective values are equal. $\square$

Example 3.16. Consider the following MPEC problem in $\mathbb{R}^2$:

$$\begin{array}{lrl} \text{MPEC(2)} & \min & z_1 + z_2 \\ & \text{subject to :} & z_1 + z_2 \geq 0, \\ & & z_2 - z_1 \geq 0, \\ & & (z_1 + z_2)(z_2 - z_1) = 0. \end{array}$$

The Mond-Weir type dual problem MWDMPEC$(\bar{z})$ for the MPEC(2) is:

$$\max_{u,\lambda} u_1 + u_2$$

subject to:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \lambda^G \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \lambda^H \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \tag{3.18}$$

$$\lambda^G (u_1 + u_2) \leq 0, \tag{3.19}$$

$$\lambda^H (u_2 - u_1) \leq 0, \tag{3.20}$$

if β is non-empty, then, either

$$\lambda^G > 0, \quad \lambda^H > 0, \quad \text{or} \quad \lambda^G \lambda^H = 0.$$

From (3.18), $\lambda^G - \lambda^H = 1$ and $\lambda^G + \lambda^H = 1$, then, we get $\lambda^G = 1$ and $\lambda^H = 0$. If $\bar{z} = (0, 0)$, then, index sets $\alpha(0, 0)$ and $\gamma(0, 0)$ are empty sets, but $\beta(0, 0)$ is non-empty. Also, by definition of sets $\beta^+, \beta^+_G, \beta^+_H$, we can see that only β^+_H is non-empty. It is clear that the assumptions of Theorem 3.10 are satisfied. So, Theorem 3.10 holds between MPEC(2) and MWDMPEC$(0, 0)$. Now, if we put $\lambda^G = 1$ in (3.19), we get $u_1 + u_2 \leq 0$. But, for any feasible z for MPEC(2), $f(z) = 0$ or $2z_1$, $z_1 > 0$, so, $f(z) \geq f(u)$, for any feasible u for MWDMPEC$(0, 0)$.

Similarly, If $\bar{z} = (1, 1)$, then, index sets $\alpha(1, 1)$ and $\beta(1, 1)$ are empty sets, but $\gamma(1, 1)$ is non-empty. On the other hand, if $\bar{z} = (-1, 1)$, then, the only non-empty set is $\alpha(-1, 1)$. In all the cases, assumptions of the Theorem 3.10 are satisfied, therefore, Theorem 3.10 holds between MPEC(2) and its corresponding Mond-Weir dual problem MWDMPEC$(\bar{z})$.

Again, $\bar{z} = (0,0)$ is the optimal solution of MPEC(2). Also, $\nabla G(\bar{z}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\nabla H(\bar{z}) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are linearly independent, so, MPEC linear independence constraint qualification (MPEC LICQ) is satisfied at $\bar{z}$. Then, by Definition 2.8 and Theorem 3.2 in [26], NNAMCQ is satisfied at $\bar{z}$. Then, by Theorem 3.13 there exists $\bar{\lambda}$ such that $(\bar{z}, \bar{\lambda})$ is an optimal solution of MWDMPEC$(0,0)$ and optimal values are equal.

Now, we establish weak and strong duality theorems for the MPEC and its Mond-Weir type dual problem under generalized convexity assumptions.

Theorem 3.17 (Weak Duality). *Let $\bar{z}$ be feasible for the* MPEC, *(u,λ) be feasible for* MWDMPEC $(\bar{z})$ *and index sets $I_g, \alpha, \beta, \gamma$ defined accordingly. Also, suppose that f is pseudoconvex at u, and the functions*

$$\sum_{i\in I_g} \lambda_i^g g_i(.)(\lambda_i^g \geq 0),$$

$$\sum_{i=1}^{p} \lambda_i^h h_i(.)(\lambda_i^h > 0, \forall i \in J^+, \lambda_i^h < 0, \forall i \in J^-),$$

$$\sum_{i\in \alpha^+\cup\beta_G^+\cup\beta^+} \lambda_i^H(-H_i)(.)(\lambda_i^H > 0)$$

and

$$\sum_{i\in \alpha^+\cup\beta_H^+\cup\beta^+} \lambda_i^G(-G_i)(.)(\lambda_i^G > 0)$$

are quasiconvex at u. If $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \phi$, then, for any z feasible for MPEC, *we have*

$$f(z) \geq f(u).$$

Proof. By feasibility conditions of MPEC, MWDMPEC$(\bar{z})$ and quasiconvexity of $\sum_{i\in I_g} \lambda_i^g g_i(.)$, we have

$$\sum_{i\in I_g} \lambda_i^g g_i(z) - \sum_{i\in I_g} \lambda_i^g g_i(u) \leq 0 \Rightarrow \left\langle \sum_{i\in I_g} \lambda_i^g \nabla g_i(u), z-u \right\rangle \leq 0. \tag{3.21}$$

Similarly, by quasiconvexity of $\sum_{i=1}^{p} \lambda_i^h h_i(.)$, we have

$$\sum_{i=1}^{p} \lambda_i^h h_i(z) - \sum_{i=1}^{p} \lambda_i^h h_i(u) \leq 0 \Rightarrow \left\langle \sum_{i=1}^{p} \lambda_i^h \nabla h_i(u), z-u \right\rangle \leq 0. \tag{3.22}$$

For $i \in \alpha^+ \cup \beta_H^+ \cup \beta^+$, we get

$$\sum_{i} \lambda_i^G(-G_i)(z) - \sum_{i} \lambda_i^G(-G_i)(u) \leq 0 \Rightarrow \left\langle \sum_{i} \lambda_i^G(-\nabla G_i)(u), z-u \right\rangle \leq 0. \tag{3.23}$$

Similarly, for $i \in \alpha^+ \cup \beta_G^+ \cup \beta^+$, we have

$$\sum_{i} \lambda_i^H(-H_i)(z) - \sum_{i} \lambda_i^H(-H_i)(u) \leq 0 \Rightarrow \left\langle \sum_{i} \lambda_i^H(-\nabla H_i)(u), z-u \right\rangle \leq 0. \tag{3.24}$$

Adding $(3.21), (3.22), (3.23)$ and (3.24), we get

$$\left\langle \sum_{i\in I_g} \lambda_i^g \nabla g_i(u) + \sum_{i=1}^{p} \lambda_i^h \nabla h_i(u) - \left[\sum_{i}^{l} \lambda_i^G \nabla G_i(u) + \sum_{i}^{l} \lambda_i^H \nabla H_i(u)\right], z-u \right\rangle \leq 0.$$

From, (3.11) and pseudoconvexity of f, we have

$$\langle \nabla f(u), z-u\rangle \geq 0 \Rightarrow f(z) \geq f(u).$$

This completes the proof. □

Remark 3.18. If $h(z) := 0, G(z) := 0, H(z) := 0$, then, weak duality Theorem 3.17 coincides with the weak duality theorem for Mond-Weir type dual for standard nonlinear programming in [16].

Theorem 3.19 (Strong Duality)**.** *If $\bar{z}$ is a global optimal solution of* MPEC*, such that the* NNAMCQ *is satisfied at $\bar{z}$ and index sets $I_g, \alpha, \beta, \gamma$ defined accordingly. Let $f, g_i(i \in I_g), h_i(i \in J^+), -h_i(i \in J^-), G_i(i \in \alpha^- \cup \beta_H^-), -G_i(i \in \alpha^+ \cup \beta_H^+ \cup \beta^+), H_i(i \in \gamma^- \cup \beta_G^-)$, and $-H_i(i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$ satisfy the assumptions of the Theorem 3.17. Then, there exists $\bar{\lambda}$, such that $(\bar{z}, \bar{\lambda})$ is a global optimal solution of* MWDMPEC $(\bar{z})$ *and respective objective values are equal.*

Proof. As $\bar{z}$ is a global optimal solution for MPEC and the NNAMCQ is satisfied at $\bar{z}$, hence, there exists $\bar{\lambda} \in \mathbb{R}^{k+p+l}$ such that the M-stationarity conditions for MPEC are satisfied, that is

$$0 = \nabla f(\bar{z}) + \sum_{i\in I_g} \bar{\lambda}_i^g \nabla g_i(\bar{z}) + \sum_{i=1}^{p} \bar{\lambda}_i^h \nabla h_i(\bar{z}) - \sum_{i=1}^{l} [\bar{\lambda}_i^G \nabla G_i(\bar{z}) + \bar{\lambda}_i^H \nabla H_i(\bar{z})],$$

$$\bar{\lambda}_{I_g}^g \geq 0, \quad \bar{\lambda}_\gamma^G = 0, \qquad \bar{\lambda}_\alpha^H = 0,$$

$$\forall i \in \beta, \quad \textit{either} \quad \bar{\lambda}_i^G > 0, \quad \bar{\lambda}_i^H > 0 \quad \textit{or} \quad \bar{\lambda}_i^G \bar{\lambda}_i^H = 0.$$

Since, $\bar{z}$ is a optimal solution for MPEC, we have

$$\sum_{i\in I_g} \bar{\lambda}_i^g g_i(\bar{z}) = 0, \sum_{i=1}^{p} \bar{\lambda}_i^h h_i(\bar{z}) = 0, \sum_{i=1}^{l} \bar{\lambda}_i^G G_i(\bar{z}) = 0, \sum_{i=1}^{l} \bar{\lambda}_i^H H_i(\bar{z}) = 0.$$

Therefore, $(\bar{z}, \bar{\lambda})$ is feasible for MWDMPEC$(\bar{z})$. Also, by Theorem 3.17, for any feasible (u, λ), we have

$$f(\bar{z}) \geq f(u).$$

Thus, $(\bar{z}, \bar{\lambda})$ is an optimal solution for MWDMPEC $(\bar{z})$ and the respective objective values are equal. This completes the proof. □

Remark 3.20. If $h(z) := 0, G(z) := 0, H(z) := 0$, then, strong duality Theorem 3.19 coincides with the strong duality theorem for Mond-Weir type dual for standard nonlinear programming in [16].

Remark 3.21. The sum of quasiconvex functions is not in general quasiconvex. For example, the functions $f(x) = x^3, g(x) = -3x$ are quasiconvex in $\mathbb{R}$ but their sum $h(x) = f(x) + g(x) = x^3 - 3x$ is not quasiconvex (see, e.g. [2]). So, the conditions taken in Theorem 3.17 and Theorem 3.19 are weaker than conditions in which $g_i(i \in I_g), h_i(i \in J^+), -h_i(i \in$

J^-), $-G_i(i \in \alpha^+ \cup \beta_H^+ \cup \beta^+)$ and $-H_i(i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$ are quasiconvex functions. To check the sum functions are quasiconvex at feasible point of dual problem MWDMPEC ($\bar{z}$), first we have to find feasible region according to given point $\bar{z}$ as in Example 3.8 and Example 3.16 and then used the Definition 2.4 and various characterization of quasiconvexity at point given in [2].

The following corollary is a consequence of Theorem 3.17 and 3.19.

Corollary 3.22. *If $\bar{z}$ is an optimal solution of MPEC where all constraint functions g_i, h_i, G_i, H_i are affine. Then, there exists $\bar{\lambda}$, such that $(\bar{z}, \bar{\lambda})$ is an optimal solution of MWDMPEC($\bar{z}$).*

4 Conclusions

We have studied mathematical programs with equilibrium constraints (MPECs) and introduced the Wolfe type dual WDMPEC($\bar{z}$) and Mond-Weir type dual MWDMPEC($\bar{z}$) for the MPEC. We have established weak and strong duality theorems relating the MPEC and the two duals WDMPEC($\bar{z}$) and MWDMPEC($\bar{z}$). Moreover, we have discussed the cases when all the constraint functions are affine. Suitable examples have been given to illustrate the significance of the results. The results presented in this paper can be extended for more generalized convex functions as well as for nonsmooth functions on the lines of Ye and Zhang [27]. Further, the results presented in this paper can be extended for the second order duality for tighter lower bound on the primal problem, following the work of Guo *et al.* [9].

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