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A DUALITY APPROACH AND OPTIMALITY CONDITIONS FOR SIMPLE CONVEX BILEVEL PROGRAMMING PROBLEMS

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Abstract: The paper deals with a convex bilevel programming problem (S) which never satisfies the Slater's condition. Using ϵ-approximate solutions of the lower level problem, we consider a regularized bilevel problem (S_ϵ) of (S) investigated by Lignola and Morgan (1997) that satisfies this condition. As approximation results, they obtained that $\inf S_\epsilon \to \inf S$ when ϵ goes to zero and that any accumulation point of a sequence of regularized solutions solves the original problem (S). Via the Fenchel-Lagrange duality, we provide optimality conditions for the regularized problem. Then, necessary optimality conditions are established for a class of solutions of problem (S). Finally, sufficient optimality conditions are established for (S).

Key words: *bilevel optimization, convex analysis, conjugate duality, optimality conditions*

Mathematics Subject Classification: *91A65, 90C46, 47N10*

1 Introduction

We are concerned with the following bilevel programming problem

$$(S) \quad \begin{cases} \min\limits_{x\in\mathcal{M}} F(x) \\ \text{where } \mathcal{M} \text{ is the solution set of the lower level problem} \\ (\mathcal{P}) \ : \ \min\limits_{x\in X} f(x) \end{cases}$$

$F, f : \mathbb{R}^n \to \mathbb{R}$ are convex functions and X is a nonempty convex compact subset of $\mathbb{R}^n$, which is called in [10] a simple convex bilevel programming problem. This term will be adopted throughout the paper. This particular class of bilevel optimization problem has been first investigated by Solodov in [17].

The class of bilevel optimization problems of the form of (S) includes the large class of standard optimization problems of the form

$$\min_{x\in H} h(x)$$

where $H = \left\{x \in \hat{X} \,/\, g(x) \leq 0, Ax = a\right\}$, $h : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$ are differentiable convex functions, $A \in \mathbb{R}^{p\times n}$, $a \in \mathbb{R}^p$ and $\hat{X}$ is a closed convex subset of $\mathbb{R}^n$. In fact, in [17], Solodov showed that the constraint set H can be viewed as the solution set of the minimization problem

$$\min_{x\in\hat{X}} \left\{\|Ax-a\|^2 + \|\max\left\{0, g(x)\right\}\|^2\right\}$$

with the max is taken coordinate-wise. In [21], the authors showed that the problem (S) can also be viewed as a hierarchical equilibrium problem and variational inequality problem.

Let us summarize some interesting works on problem (S). In [9], Cabot gave a proximal point algorithm to solve (S). Under appropriate assumptions, he showed that the algorithm generates a sequence that converges to a solution of (S). In [10], for problem (S), Dempe et al. gave necessary and sufficient optimality conditions under convexity and differentiability assumptions and some qualification conditions. These qualification conditions use conjugacy and limiting normal cones. In [17], for problem (S) Solodov presented an explicit gradient descent method. Under convexity and regularity assumptions on the functions involved and some boundedness properties, he proved the convergence of an algorithm associated to this method. In [18], the same author provided an explicit bundle algorithm where the convergence is obtained under appropriate assumptions.

It is not difficult to see that under our data, the problem (S) admits at least one solution (the set $\mathcal{M}$ is compact and F is continuous on $\mathbb{R}^n$ as a finite convex function). Our aim in this paper is to give a new approach for problem (S) based on a regularization and the Fenchel-Lagrange duality. Via this approach, we obtain approximation results for (S), and optimality conditions for (S) and its regularized problem. The regularization is based on the use of ϵ-approximate solutions of the lower level problem $(\mathcal{P})$. It was principally used in the literature of two level optimization for pessimistic (or weak in the sense of [8]) bilevel programming problems. For papers using such a regularization see for example [1], [2], [12], [14]). On the other hand, we note that the Fenchel-Lagrange duality has been first introduced by Wanka and Boţ in [19] for ordinary convex programming problems. As an application of such a duality in two level optimization, it has been used in [3] to derive necessary and sufficient optimality conditions for a bilevel programming problem with extremal-value function. Furthermore, the corresponding Fenchel-Lagrange dual problem has been transformed to a one-level optimization problem. In this paper, for our investigation, as in [10], we will replace (S) by the following equivalent problem

$$\min_{\substack{x\in X\\ f(x)\leq v}} F(x) \tag{1.1}$$

where v denotes the infimal value of $(\mathcal{P})$, i.e., $v = \inf_{x\in X} f(x)$. Unfortunately, such a problem does not satisfy the Slater's condition. Consequently, we cannot directly apply our duality to (S) under the formulation in (1.1). To avoid this situation, we will consider a regularized problem (S_ϵ) of (S) whose constraints are represented by the set of ϵ-approximate solutions of the lower level problem $(\mathcal{P})$, $\epsilon > 0$. Such a regularization for optimistic bilevel problems has been first investigated in [13]. Since the set of strict ϵ-approximate solutions is always nonempty, then, contrary to (S), this regularized problem will satisfy the Slater's condition. Moreover, such a regularized problem admits solutions under the assumptions quoted above. As approximation results, in [13] the authors have obtained that $\inf S_\epsilon$ converges to $\inf S$ when the parameter of regularization ϵ goes to zero, and that any accumulation point of a sequence of regularized solutions solves the original problem (S). Then, we provide necessary and sufficient optimality conditions for (S_ϵ) via the Fenchel-Lagrange duality. Under a qualification condition, necessary optimality conditions are established for some particular solutions of (S). These solutions are accumulation points of sequences of regularized solutions. Finally, sufficient optimality conditions are established for (S). The obtained optimality conditions are different from those given in the literature, and are expressed in terms of subdifferentials, normal cones (in the sense of convex analysis) and the conjugates of the functions involved.

The paper is organized as follows. Section 2 is devoted to the Fenchel-Lagrange duality for the regularized problem of (S). In Section 3, we provide necessary and sufficient optimality conditions for the regularized problem and its Fenchel-Lagrange dual. Then, we give necessary optimality conditions for a class of solutions of the original problem (S). Finally, sufficient optimality conditions are established for (S).

We recall that throughout the paper, we assume that F and f are convex functions and X is a nonempty convex compact subset of $\mathbb{R}^n$.

2 The Fenchel-Lagrange Duality Approach

Our aim in this section is to give a duality approach for problem (S). To begin our procedure, we consider its equivalent formulation

$$(S) \qquad \min_{\substack{x \in X \\ f(x) \leq v}} F(x)$$

where v is the infimal value of the lower level problem $(\mathcal{P})$. Such a formulation was first introduced in [10]. Unfortunately, the problem (S) under this formulation does not satisfy the Slater's constraint qualification which we will require for our procedure. To avoid this situation, we will replace (S) by a regularized problem which satisfies this condition. Then, for $\epsilon > 0$, we consider the following regularized problem of (S)

$$(S_\epsilon) \qquad \min_{\substack{x \in X \\ f(x) \leq v+\epsilon}} F(x).$$

That is the constraint set of (S_ϵ) is the set of ϵ-approximate solutions of the lower level problem $(\mathcal{P})$ which we denote by $\mathcal{M}_\epsilon$. Obviously, such a problem always satisfies the Slater's condition. This follows from the characterization of the infimal value v. Note that the regularization using ϵ-approximate solutions of the lower level problem has been first introduced by Loridan and Morgan (see [14] and [15]). As stability results, from [13], one obtains that $\inf S_\epsilon$ converges to $\inf S$ when the parameter of regularization ϵ goes to zero, and that any accumulation point of a sequence of regularized solutions is a solution of problem (S) (see Theorem 2.1 below).

More precisely, in this section, we will consider the Fenchel-Lagrange dual problem of (S_ϵ), which we denote by $(\mathcal{D}^*_\epsilon)$. Under the hypotheses quoted in the introduction, we will show that the problems (S_ϵ) and $(\mathcal{D}^*_\epsilon)$ are in strong duality. Such a duality will be used in the next section to establish optimality conditions.

In the sequel, for $\epsilon_k \searrow 0^+$, we denote $\mathcal{M}_{\epsilon_k}$ and (S_{ϵ_k}) by $\mathcal{M}_k$ and (S_k) respectively. Then, under the hypotheses quoted in the introduction we have the following fundamental results that we need in the sequel.

Theorem 2.1. i) *For every $\epsilon > 0$, the problem (S_ϵ) admits at least one solution,*

ii) $\lim_{\epsilon \to 0}(\inf S_\epsilon) = \inf S$,

iii) *Let $\epsilon_k \searrow 0^+$ and (x_k) be a sequence of solutions of the regularized problems (S_k). Then, any accumulation point of the sequence (x_k) solves the original problem (S).*

Proof. See Theorem 3.2 in [13]. □

Before starting the procedure of duality, let us recall the following definitions that we need in the sequel.

Let $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a function and A be a nonempty subset of $\mathbb{R}^n$.

- The effective domain of g denoted by dom$g$ is the set defined by

$$\text{dom}g = \Big\{x \in \mathbb{R}^n / \, g(x) < +\infty\Big\}.$$

We say that g is proper if $g(x) > -\infty$, for all $x \in \mathbb{R}^n$, and dom$g \neq \emptyset$.

- The conjugate function of g relative to the set A denoted by g_A^* is defined on $\mathbb{R}^n$ by (see [7])

$$g_A^*(p) = \sup_{x \in A} \big\{\langle p, x\rangle - g(x)\big\}.$$

When $A = \mathbb{R}^n$, we get the usual Legendre-Fenchel conjugate function of g, denoted by g^*. Note that we always have

$$g_A^*(p) + g(x) \geq \langle p, x\rangle \qquad \forall\, x \in A.$$

- The relative interior of A, denoted by ri$A$, is the interior of A relative to the smallest affine set containing A, equipped with the induced topology of $\mathbb{R}^n$.

Now, we can start our duality approach. For $\epsilon > 0$, we consider the Fenchel-Lagrange dual problem (see [19])

$$(\mathcal{D}_\epsilon^*) \qquad \sup_{\substack{\alpha \in \mathbb{R}^+ \\ p \in \mathbb{R}^n}} \big\{ - F^*(p) - (\alpha f)_X^*(-p) - \alpha(v + \epsilon)\big\}$$

of the primal regularized problem (S_ϵ). Not that in [20], for a primal optimization problem with general constraints, other interesting dual problems using conjugacy are considered. Then, under appropriate assumptions, the authors have established equalities between the optimal values of these dual problems.

The following theorem establishes strong Fenchel-Lagrange duality for the pair (S_ϵ)-$(\mathcal{D}_\epsilon^*)$.

Theorem 2.2. *Let $\epsilon > 0$. Then, the problems (S_ϵ) and $(\mathcal{D}_\epsilon^*)$ are in strong Fenchel-Lagrange duality.*

Proof. See [7, Theorem 3.2.12]. □

3 Optimality Conditions

In this section, we first provide necessary and sufficient optimality conditions for the dual pair (S_ϵ)-$(\mathcal{D}_\epsilon^*)$, $\epsilon > 0$. As a consequence, a solution of the original problem can be deduced from the optimality conditions obtained for the dual-pair (S_{ϵ_k})-$(\mathcal{D}_{\epsilon_k}^*)$, with $\epsilon_k \searrow 0^+$ (Theorem 3.5). Then, we give necessary optimality conditions for such a solution (Theorem 3.7). Finally, sufficient optimality conditions are provided for (S).

Before going further, let us recall the following definitions related to convex analysis that we need in the sequel.

- Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function. Let $\bar{x} \in$ dom$g$. The subdifferential of g at $\bar{x}$ denoted by $\partial g(\bar{x})$ is the set defined by

$$\partial g(\bar{x}) = \Big\{x^* \in \mathbb{R}^n / \, g(x) \geq g(\bar{x}) + \langle x^*, x - \bar{x}\rangle, \, \forall x \in \mathbb{R}^n\Big\}.$$

- Let A be a nonempty convex subset of $\mathbb{R}^n$ and $\bar{x} \in A$. The normal cone to A at $\bar{x}$ denoted by $\mathcal{N}_A(\bar{x})$ is the set defined by

$$\mathcal{N}_A(\bar{x}) = \Big\{x^* \in \mathbb{R}^n / \langle x^*, x - \bar{x}\rangle \leq 0, \forall\, x \in A\Big\}.$$

It is not difficult to see that $\partial\psi_A(\bar{x}) = \mathcal{N}_A(\bar{x})$.

The proofs of the following two theorems (Theorems 3.1 and 3.3) can be found in [7, Theorem 3.3.22] in a more general case. So that, they are omitted.

Theorem 3.1 (Necessary optimality conditions)**.** *Let $\epsilon > 0$. Let x_ϵ be a solution of problem (S_ϵ). Then, there exists $(\alpha_\epsilon, p_\epsilon) \in \mathbb{R}^+ \times \mathbb{R}^n$, solution of $(\mathcal{D}^*_\epsilon)$, such that the following optimality conditions are satisfied*

i) $F(x_\epsilon) + F^*(p_\epsilon) = \langle p_\epsilon, x_\epsilon\rangle$,

ii) $(\alpha_\epsilon f)^*_X(-p_\epsilon) + \alpha_\epsilon f(x_\epsilon) = \langle -p_\epsilon, x_\epsilon\rangle$,

iii) $\alpha_\epsilon(-f(x_\epsilon) + v + \epsilon) = 0$.

Proof. See [7,Theorem 3.3.22]. □

Remark 3.2. 1) It is not difficult to see that the property ii) in Theorem 3.1 is equivalent to say that x_ϵ solves the problem

$$\min_{x\in X}(\alpha_\epsilon f(x) + \langle p_\epsilon, x\rangle).$$

2) In terms of subdifferentials and normal cones, the properties i) and ii) in Theorem 3.1 are respectively equivalent to

iv) $p_\epsilon \in \partial F(x_\epsilon)$,

v) $-p_\epsilon \in \partial(\alpha_\epsilon f)(x_\epsilon) + \mathcal{N}_X(x_\epsilon)$.

Then, iv) and v) imply that

$$0 \in \partial F(x_\epsilon) + \partial(\alpha_\epsilon f)(x_\epsilon) + \mathcal{N}_X(x_\epsilon).$$

That is x_ϵ solves the problem

$$\min_{x\in X}(F + \alpha_\epsilon f)(x).$$

Theorem 3.3 (Sufficient optimality conditions)**.** *Let $\epsilon > 0$. Let x_ϵ and $(\alpha_\epsilon, p_\epsilon) \in \mathbb{R}^+ \times \mathbb{R}^n$ be feasible points of problems (S_ϵ) and $(\mathcal{D}^*_\epsilon)$ respectively. Assume that they satisfy the conditions i) to iii) in Theorem 3.1. Then, x_ϵ and $(\alpha_\epsilon, p_\epsilon)$ solve respectively the problems (S_ϵ) and $(\mathcal{D}^*_\epsilon)$. Moreover, the problems (S_ϵ) and $(\mathcal{D}^*_\epsilon)$ are in strong duality.*

Proof. See [7,Theorem 3.3.22]. □

As a consequence, we obtain the following necessary and sufficient optimality conditions for the dual pair (S_ϵ)-$(\mathcal{D}^*_\epsilon)$.

Corollary 3.4. *(necessary and sufficient optimality conditions) Let* $\epsilon > 0$. *Let* x_ϵ *and* $(\alpha_\epsilon, p_\epsilon) \in \mathbb{R}^+ \times \mathbb{R}^n$ *be feasible points of problems* (S_ϵ) *and* $(\mathcal{D}^*_\epsilon)$ *respectively. Then,* x_ϵ *and* $(\alpha_\epsilon, p_\epsilon) \in \mathbb{R}^+ \times \mathbb{R}^n$ *solve* (S_ϵ) *and* $(\mathcal{D}^*_\epsilon)$ *respectively if and only if they satisfy the conditions i) to iii) in Theorem 3.1.*

Proof. Apply Theorems 3.1 and 3.3. □

For $\epsilon_k \searrow 0^+$, denote $(\mathcal{D}^*_{\epsilon_k})$ by $(\mathcal{D}^*_k)$. The following result shows that we can obtain a solution of the original problem (S) from the optimality conditions given for the pair (S_k)-$(\mathcal{D}^*_k)$, $k \in \mathbb{N}$.

Theorem 3.5. *Let* $\epsilon_k \searrow 0^+$. *Let* x_k *be a feasible point of problem* (S_k) *that satisfies together with a certain feasible point* (α_k, p_k) *of the dual problem* $(\mathcal{D}^*_k)$ *the conditions i) to iii) in Theorem 3.1. Let* $\bar{x}$ *be an accumulation point of the sequence* (x_k). *Then,* $\bar{x}$ *solves the problem* (S).

Proof. Apply Theorems 3.3 and 2.1. □

In order to give necessary optimality conditions for a class of solutions of problem (S), we need the following additional assumptions:

$$(1)\quad \begin{cases} \inf_{x\in\mathbb{R}^n} F(x) < \inf_{x\in X} F(x), \\ \\ \inf_{x\in\mathbb{R}^n} f(x) < \inf_{x\in X} f(x), \end{cases}$$

(2) For any $\epsilon > 0$ sufficiently small, there exists $x_\epsilon \in \text{int}X$ such that $f(x_\epsilon) \le v + \epsilon$.

Remark 3.6. Assumption (1) implies that for any $x \in X$, we have $0 \notin \partial F(x) \cup \partial f(x)$.

The following result gives necessary optimality conditions for the class of solutions of (S) which are accumulation points of sequences of regularized solutions.

Theorem 3.7. *Let* $\epsilon_k \searrow 0^+$. *Assume that assumptions (1) and (2) are satisfied. For* ϵ_k, *let* x_k *be the feasible point of problem* (S_k) *given by assumption (2). Moreover, assume that* x_k *satisfies together with a certain feasible point* (α_k, p_k), $\alpha_k > 0$, *of the dual problem* $(\mathcal{D}^*_k)$ *the conditions i) to iii) in Theorem 3.1. Let* $\bar{x}$ *be an accumulation point of the sequence* (x_k). *Then,* $\bar{x}$ *solves the problem* (S) *and there exists* $(\bar{\alpha}, \bar{p}) \in \mathbb{R}^*_+ \times \mathbb{R}^n$ *such that*

a) $0 \in \partial f(\bar{x}) + \mathcal{N}_X(\bar{x})$,

b) $\bar{p} \in \partial F(\bar{x})$,

c) $-\bar{p} \in \bar{\alpha}\partial f(\bar{x})$.

Furthermore, $\bar{x}$ *solves the unconstrained minimization problem*

$$\min_{x\in\mathbb{R}^n} (F + \bar{\alpha} f)(x).$$

Proof. Let $\mathcal{N}$ be an infinite subset of $\mathbb{N}$ such that $x_k \to \bar{x}$, as $k \to +\infty$, $k \in \mathcal{N}$. From Theorem 3.5, the accumulation point $\bar{x}$ solves the original problem (S). Moreover, for every $k \in \mathcal{N}$, we have

i) $F(x_k) + F^*(p_k) = \langle p_k, x_k \rangle$

ii) $(\alpha_k f)^*_X(-p_k) + \alpha_k f(x_k) = \langle -p_k, x_k \rangle$

iii) $\alpha_k(-f(x_k) + v + \epsilon_k) = 0.$

Since $\alpha_k \neq 0$, then after simplification, the third equation *iii*) gives

$$-f(x_k) + v + \epsilon_k = 0.$$

Letting $k \to +\infty$, $k \in \mathcal{N}$, we get $f(\bar{x}) = v$, which is equivalent to

$$0 \in \partial f(\bar{x}) + \mathcal{N}_X(\bar{x}).$$

Furthermore, the first equation *i*) is equivalent to $p_k \in \partial F(x_k)$. We have $\partial F(x_k) \subset \bigcup_{x \in X} \partial F(x)$. Since X is compact, it follows that the set (see [16,Theorem 24.7])

$$\partial F(X) = \bigcup_{x \in X} \partial F(x)$$

is compact. Then, there exists an infinite subset $\mathcal{N}'$ of $\mathcal{N}$ such that the sequence $(p_k)_{k \in \mathcal{N}'}$ converges to $\bar{p}$. Since, for any $k \in \mathcal{N}'$, we have

$$F(y) \geq F(x_k) + \langle p_k, y - x_k \rangle \qquad \forall\, y \in \mathbb{R}^n$$

then, passing to the limit as $k \to +\infty$, $k \in \mathcal{N}'$, we get

$$F(y) \geq F(\bar{x}) + \langle \bar{p}, y - \bar{x} \rangle \qquad \forall\, y \in \mathbb{R}^n.$$

That is

$$\bar{p} \in \partial F(\bar{x}). \tag{3.1}$$

On the other hand, for every $k \in \mathcal{N}'$, the equation ii) gives

$$\langle -p_k, x_k \rangle - \alpha_k f(x_k) \geq \langle -p_k, x \rangle - \alpha_k f(x) \qquad \forall\, x \in X.$$

So that

$$\alpha_k f(x) + \langle p_k, x \rangle \geq \alpha_k f(x_k) + \langle p_k, x_k \rangle \qquad \forall\, x \in X.$$

That is for all $k \in \mathcal{N}'$, x_k is a solution of the problem

$$\min_{x \in X} (\alpha_k f(x) + \langle p_k, x \rangle).$$

Therefore

$$0 \in \alpha_k \partial f(x_k) + p_k + \mathcal{N}_X(x_k).$$

Since $x_k \in \text{int}X$ (assumption (2)), it follows that (see for example [11])

$$\mathcal{N}_X(x_k) = \big\{0\big\}.$$

We deduce that

$$-\frac{p_k}{\alpha_k} \in \partial f(x_k) \qquad \forall\, k \in \mathcal{N}.$$

Using the fact that

$$\partial f(x_k) \subset \bigcup_{x \in X} \partial f(x)$$

and that $\bigcup_{x\in X}\partial f(x)$ is compact, we deduce that there exists an infinite subset $\mathcal{N}''$ of $\mathcal{N}'$ such that $-\frac{p_k}{\alpha_k}\to\bar{q}$, as $k\to+\infty$, $k\in\mathcal{N}''$. On the other hand, for $k\in\mathcal{N}''$, we have

$$f(y)\geq f(x_k)+\left\langle -\frac{p_k}{\alpha_k}, y-x_k\right\rangle \qquad \forall\, y\in\mathbb{R}^n.$$

Then, passing to the limit as $k\to+\infty$, $k\in\mathcal{N}''$, we get

$$f(y)\geq f(\bar{x})+\langle\bar{q}, y-\bar{x}\rangle \qquad \forall\, y\in\mathbb{R}^n.$$

That is

$$\bar{q}\in\partial f(\bar{x}).$$

Assumption (1) implies that $\bar{q}\neq 0$ and $\bar{p}\neq 0$ (see Remark 3.6). Set $q_k=-\frac{p_k}{\alpha_k}$, $k\in\mathcal{N}''$. Since

$$\|q_k\|\to\|\bar{q}\|>0 \qquad \text{as } k\to+\infty, k\in\mathcal{N}''$$

it follows that there exists $k_0\in\mathcal{N}''$ such that

$$\|q_k\|>0 \quad \forall k\geq k_0, k\in\mathcal{N}''.$$

Then, for $k\in\mathcal{N}''$, $k\geq k_0$, we have $\alpha_k=\frac{\|p_k\|}{\|q_k\|}$. So that, the sequence $(\alpha_k)_{k\in\mathcal{N}''}$ converges to $\bar{\alpha}=\frac{\|\bar{p}\|}{\|\bar{q}\|}$, and $\bar{\alpha}\neq 0$. Furthermore, the sequence $\left(\frac{p_k}{\alpha_k}\right)_{k\in\mathcal{N}''}$ converges to $\dfrac{\bar{p}}{\bar{\alpha}}=-\bar{q}$. Hence

$$-\bar{p}\in\bar{\alpha}\partial f(\bar{x}). \tag{3.2}$$

Finally, using (3.1) and (3.2), we deduce that $0\in\partial(F+\bar{\alpha}f)(\bar{x})$. That is $\bar{x}$ solves the problem

$$\min_{x\in\mathbb{R}^n}(F+\bar{\alpha}f)(x).$$

□

The following result gives sufficient optimality conditions for problem (S).

Theorem 3.8 (Sufficient optimality conditions)**.** *Let $\bar{x}\in\mathbb{R}^n$. Assume that there exists $(\bar{\alpha},\bar{p})\in\mathbb{R}^*_+\times\mathbb{R}^n$ such that*

i) $0\in\partial f(\bar{x})+\mathcal{N}_X(\bar{x})$,

ii) $\bar{p}\in\partial F(\bar{x})$,

iii) $-\bar{p}\in\bar{\alpha}\partial f(\bar{x})$.

Then, $\bar{x}$ solves the problem (S).

Proof. Feasibility: The property *i)* implies that $\bar{x}$ solves the problem $(\mathcal{P})$. That is $\bar{x}\in\mathcal{M}$.

Optimality: Let x be a feasible point of (S), i.e., $x\in\mathcal{M}$. The property *ii)* implies that

$$F(y)\geq F(\bar{x})+\langle\bar{p}, y-\bar{x}\rangle \qquad \forall\, y\in\mathbb{R}^n. \tag{3.3}$$

On the other hand, property iii) says that

$$\bar{\alpha}f(y)\geq\bar{\alpha}f(\bar{x})+\langle-\bar{p}, y-\bar{x}\rangle \qquad \forall\, y\in\mathbb{R}^n. \tag{3.4}$$

In particular, for $y = x$ in (3.4), we obtain

$$\langle \bar{p}, x - \bar{x} \rangle \geq \bar{\alpha}(f(\bar{x}) - f(x)) = 0$$

where the equality follows from the fact that $x, \bar{x} \in \mathcal{M}$. Then, in (3.3) we obtain

$$F(x) \geq F(\bar{x}).$$

Therefore, $\bar{x}$ solves (S). □

For illustration of our method, we give the following example. Then, in order to see the advantages of this method, we will give a comparison with other classical ones.

Example 3.9. Let F and f be the functions defined on $\mathbb{R}^2$ by

$$F(x_1, x_2) = x_1^2 + x_2^2 \qquad\qquad f(x_1, x_2) = -x_1 - x_2$$

and $X = \{(x_1, x_2) \in \mathbb{R}^2 / g(x_1, x_2) \leq 0\}$, with $g = (g_1, ..., g_4)^t$,

$$\begin{cases} g_1(x_1, x_2) = -x_1 & g_2(x_1, x_2) = -x_2 \\ g_3(x_1, x_2) = x_1 + x_2 - 1 & g_4(x_1, x_2) = (x_1 - \frac{1}{2})^2 - x_2. \end{cases}$$

Then, the functions F and f are convex and X is a convex compact set. It is not difficult to check that

$$\inf \mathcal{P} = v = -1 \quad \text{and} \quad \mathcal{M} = \text{conv}\Big\{(0,1)^t, \Big(\frac{\sqrt{3}}{2}, \frac{2-\sqrt{3}}{2}\Big)^t\Big\}$$

where "conv" stands for the convex hull.

Let $\epsilon > 0$ sufficiently small. We have

$$\mathcal{M}_\epsilon = \Big\{(x_1, x_2) \in X / -x_1 - x_2 \leq -1 + \epsilon\Big\}, \quad (S_\epsilon) : \min_{(x_1,x_2) \in \mathcal{M}_\epsilon} (x_1^2 + x_2^2).$$

Remark that (S_ϵ) has a strict convex objective function constrained by a nonempty convex compact set. So that, it admits a unique solution.

1) Resolution via the Fenchel-Lagrange duality

Let $x_\epsilon = (x_1^\epsilon, x_2^\epsilon)^t$ and $(\alpha_\epsilon, p_\epsilon)$ be feasible points of problems (S_ϵ) and $(\mathcal{D}_\epsilon^*)$ respectively, which are in strong Fenchel-Lagrange duality (Theorem 2.2). According to Corollary 3.4, x_ϵ and $(\alpha_\epsilon, p_\epsilon)$ solve (S_ϵ) and $(\mathcal{D}_\epsilon^*)$ respectively if and only if they satisfy the following conditions [see 2) of Remark 3.2]

i) $p_\epsilon \in \partial F(x_\epsilon)$, $p_\epsilon = (p_1^\epsilon, p_2^\epsilon)^t$,

ii) $-p_\epsilon \in \partial(\alpha_\epsilon f)(x_\epsilon) + \mathcal{N}_X(x_\epsilon)$,

iii) $\alpha_\epsilon(x_1^\epsilon + x_2^\epsilon - 1 + \epsilon) = 0$ (complementary slackness).

Assume that $\alpha_\epsilon > 0$. The complementary slackness condition gives

$$x_1^\epsilon + x_2^\epsilon - 1 + \epsilon = 0. \tag{3.5}$$

It follows that $g_3(x_1^\epsilon, x_2^\epsilon) = x_1^\epsilon + x_2^\epsilon - 1 < 0$. Moreover, since ϵ is sufficiently small, we also deduce from (3.5) that $(x_1^\epsilon, x_2^\epsilon) \neq (0,0)$. Assume that the constraint g_4 is not active at $(x_1^\epsilon, x_2^\epsilon)$, i.e., $(x_1^\epsilon - \frac{1}{2})^2 - x_2^\epsilon < 0$. Adding i) to ii) we obtain

$$0 \in \partial F(x_\epsilon) + \partial(\alpha_\epsilon f)(x_\epsilon) + \mathcal{N}_X(x_\epsilon). \tag{3.6}$$

Then, by using (3.6), it is not difficult to verify that $x_1^\epsilon \neq 0$ and $x_2^\epsilon \neq 0$. Hence, $x_\epsilon \in \text{int}X$. It follows that $\mathcal{N}_X(x_\epsilon) = \{0\}$ and property (3.6) becomes $0 \in \partial F(x_\epsilon) + \alpha_\epsilon \partial f(x_\epsilon)$. So that $\binom{-\alpha_\epsilon + 2x_1^\epsilon}{-\alpha_\epsilon + 2x_2^\epsilon} = \binom{0}{0}$. Using (3.5), we obtain $x_\epsilon = \frac{1}{2}(1-\epsilon, 1-\epsilon)^t$ and $\alpha_\epsilon = 1-\epsilon$. Moreover, from i) we obtain $p_\epsilon = \binom{2x_1^\epsilon}{2x_2^\epsilon} = \binom{1-\epsilon}{1-\epsilon}$. We have $x_\epsilon \to \bar{x} = \left(\frac{1}{2}, \frac{1}{2}\right)^t$, as $\epsilon \to 0^+$. Then, according to Theorem 2.1, the point $\bar{x}$ solves the original bilevel problem (S). Note that $\bar{x}$ is in the boundary of X, since the constraint g_3 is active at $\bar{x}$.

2) Resolution via the classical Lagrangian duality

Let $(\mathcal{D}_\epsilon)$ denote the Lagrangian dual problem of (S_ϵ), i.e.

$$(\mathcal{D}_\epsilon) \qquad \sup_{\alpha \in \mathbb{R}_+} \inf_{x \in X} \left\{ F(x) + \alpha(f(x) - v - \epsilon) \right\}.$$

In the following, we summarize the main steps of the procedure. From Theorem 2.1, we have $\inf S_\epsilon > -\infty$. Moreover, (S_ϵ) satisfies the Slater's constraint qualification. So that, under the data, strong Lagrangian duality holds between (S_ϵ) and its dual $(\mathcal{D}_\epsilon)$ (see for example [5]). Therefore, $\inf S_\epsilon = \sup \mathcal{D}_\epsilon$, and the problem $(\mathcal{D}_\epsilon)$ admits a solution $\bar{\alpha}_\epsilon$. Let $\bar{x}_\epsilon = (\bar{x}_1^\epsilon, \bar{x}_2^\epsilon)^t$ denote the unique solution of problem (S_ϵ) (the objective function F is strictly convex). Denote by θ_ϵ the Lagrangian dual function, i.e.,

$$\theta_\epsilon(\alpha) = \inf_{(x_1,x_2) \in X} \mathcal{L}_\epsilon((x_1, x_2), \alpha)$$

where $\mathcal{L}_\epsilon((x_1, x_2), \alpha) = x_1^2 + x_2^2 + \alpha(-x_1 - x_2 + 1 - \epsilon)$ is the Lagrangian function associated to (S_ϵ). Then, we have first to determine the explicit expression of the function θ_ϵ by solving the problem

$$(\mathcal{D}_\epsilon) \qquad \min_{(x_1,x_2) \in X} \mathcal{L}_\epsilon((x_1, x_2), \alpha). \tag{3.7}$$

The second step is to find $\bar{\alpha}_\epsilon$ solution of the dual problem

$$\max_{\alpha \in \mathbb{R}_+} \theta_\epsilon(\alpha). \tag{3.8}$$

Finally, the solution $\bar{x}_\epsilon$ is given by the resolution of the problem

$$\min_{(x_1,x_2) \in X} \mathcal{L}_\epsilon((x_1, x_2), \bar{\alpha}_\epsilon). \tag{3.9}$$

Therefore, the determination of the solution $\bar{x}_\epsilon$ of problem (S_ϵ) needs the resolution of problems (3.7)-(3.9). Then, via optimality conditions applied to these problems, we find the unique solution $\bar{x}_\epsilon = \frac{1}{2}(1-\epsilon, 1-\epsilon)^t$ of problem (S_ϵ). Letting $\epsilon \to 0^+$, we get the solution $\left(\frac{1}{2}, \frac{1}{2}\right)^t$ of problem (S) (Theorem 2.1).

3) Resolution via the conditions of Kuhn-Tucker

Let $(\tilde{x}_1^\epsilon, \tilde{x}_2^\epsilon)$ be a feasible point of the convex problem (S_ϵ). Under the data, the point $(\tilde{x}_1^\epsilon, \tilde{x}_2^\epsilon)$ solves (S_ϵ) if and only if there exists $\lambda_\epsilon = (\lambda_1^\epsilon, ..., \lambda_4^\epsilon) \in \mathbb{R}_+^4$ such that the following conditions are satisfied :

a) $\nabla F(\tilde{x}_1^\epsilon, \tilde{x}_2^\epsilon) + \sum_{i=1}^{4} \lambda_i^\epsilon \nabla g_i((\tilde{x}_1^\epsilon, \tilde{x}_2^\epsilon) = 0_{\mathbb{R}^2}$

b) $\lambda_i^\epsilon g_i(\tilde{x}_1^\epsilon, \tilde{x}_2^\epsilon) = 0, i = 1, ..., 4.$

So that, we are led to solve a system of six nonlinear real equations. Then, after a laborious calculation, we find the unique solution $(\tilde{x}_1^\epsilon, \tilde{x}_2^\epsilon) = \left(\frac{1-\epsilon}{2}, \frac{1-\epsilon}{2}\right)^t$. Letting $\epsilon \to 0^+$ and using Theorem 2.1, we deduce the solution of problem (S).

Therefore, making a comparison, we see that by adopting the Fenchel-Lagrange duality, we obtain directly the solution of (S) via the conditions i)-iii). However, the resolution via the Lagrangian duality requires several steeps to get this solution, and the resolution via the application of the Kuhn-Tucker conditions requires the resolution of six nonlinear real equations. So that, the first method seems to be more adequate. The reader can also verify that our method is more simple to use than other primal methods as the descent methods.

Remark 3.10. Return to our Fenchel-Lagrange duality approach in Example 3.9 and remark that assumptions (1) and (2) are also satisfied. In fact, we have $x_\epsilon \in \mathcal{M}_\epsilon \cap \text{int}X$,

$$\inf_{(x_1,x_2)\in\mathbb{R}^2} F(x_1, x_2) = 0 < \inf_{(x_1,x_2)\in X} F(x_1, x_2)$$

(the strict inequality follows from the fact that $0 \notin X$) and

$$\inf_{(x_1,x_2)\in\mathbb{R}^2} f(x_1, x_2) = -\infty < \inf_{(x_1,x_2)\in X} f(x_1, x_2) = -1.$$

So that, we can also apply the result of Theorem 3.7.

4 Conclusions

In order to investigate the simple convex bilevel programming problem (S), we have considered a new duality approach using a regularization and the Fenchel-Lagrange duality. In a first time, we have considered an equivalent problem of (S) whose constraints are expressed in terms of the value of the lower level problem $(\mathcal{P})$. Unfortunately, such a problem does not satisfy the Slater's constraint qualification. For this reason, we have introduced a regularized problem (S_ϵ), $\epsilon > 0$, which satisfies this condition. The regularization is based on the use of ϵ-approximate solutions of the lower level problem $(\mathcal{P})$. As approximation results, from [13], one obtains under appropriate assumptions that any accumulation point of a sequence of regularized solutions solves the original problem (S), and that $\inf S_\epsilon$ converges to $\inf S$ when the parameter of regularization ϵ goes to zero. Via the Fenchel-Lagrange duality, we have established necessary and sufficient optimality conditions for (S_ϵ) and its dual. Furthermore, we have shown that a solution of the original problem (S) can be obtained from the optimality conditions established for the regularized problems (S_{ϵ_k}), $\epsilon_k \searrow 0^+$. Then, necessary optimality conditions are given for the solutions of (S) which are accumulation points of regularized solutions. Finally, sufficient optimality conditions are provided for (S). These optimality conditions are different from those existing in the literature. Therefore, they possibly generate new algorithms for the resolution of problem (S).

As quoted in the introduction the same composed duality was used in [3] for another class of bilevel programming problems, the so-called bilevel programming problems with extremal-value function. But in [3], no regularization was required since under appropriate assumptions, the original bilevel programming problem satisfies the Slater's constraint qualification. Then, similar optimality conditions are obtained in [3]. The composed duality which is the combination of the Fenchel duality to the Lagrangian duality allows us to express optimality conditions in terms of subdifferentials, normal cones and conjugate functions. Hence, the differentiability of the functions involved is not necessary. Note that other dualities can also be applied to problem (S), as Fenchel and Fenchel-Rockafellar dualities. We refer to [20] where relationships between the optimal values of the Lagrange dual, the Fenchel dual and the Fenchel-Lagrange dual problems are investigated in a general case.

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