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SECOND-ORDER RADIAL-ASYMPTOTIC DERIVATIVES AND APPLICATIONS IN SET-VALUED VECTOR OPTIMIZATION*

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Abstract: We propose the notion of second-order radial-asymptotic derivatives of a set-valued map, establish some simple calculus rules and apply directly them to obtain optimality conditions for some particular optimization problems. Then, we employ these derivatives together with second-order contingent derivatives to analyze sensitivity for nonsmooth set-valued vector optimization. Properties of second-order contingent derivatives of the proper perturbation maps of a parameterized optimization problem are obtained.

Key words: *second-order radial-asymptotic derivatives, calculus rules, set-valued vector optimization, second-order optimality conditions, proper perturbation map, second-order sensitivity analysis*

Mathematics Subject Classification: *90C46, 49J52, 46G05, 90C26, 90C29*

1 Introduction

In the last two decades, the first-order derivatives have been widely investigated to establish the first-order optimality conditions in optimization problems, analyze the sensitivity in parameterized optimization problems and apply to many aspects of variational analysis. For generalized derivatives and their applications, see the excellent books [18, 19, 21]. The first-order derivatives have the meaning in approximating the given mapping in the neighborhood of this point. To get better approximations, the higher-order derivatives were considered. Based on the different rates of change of the point under consideration in the domain space and the range space of a map, the mth-order variations of a map were proposed in [5] to obtain the open mapping principle. Another kind of the mth-order derivatives were presented in [23] and generalized to set-valued maps in [15, 24]. In [4], the notion of higher-order radial-contingent derivative of a set-valued map was proposed and employed together with higher-order contingent derivatives to analyze sensitivity for nonsmooth set-valued vector optimization. The higher-order contingent derivatives, defined in [3], have the graph coinciding with the higher-order contingent sets. Unlike the mth-order variations, the higher-order contingent derivatives are based on encounter information. Recently, the second-order contingent derivatives and some modified notions have been used to get the second-order optimality conditions and analyze the sensitivity in optimizations, see e.g. [7, 16, 29]. The asymptotic second-order tangent cones were presented in [20]. The second-order asymptotic derivatives, having the graph coinciding with the second-order asymptotic

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cones, were proposed in [8] and combined with the second-order contingent derivatives to get better optimality conditions. The second-order asymptotic derivatives were also used in [9] to establish optimality conditions in optimization problems. In [28], by combining the asymptotic notions and the variational sets, proposed in [10], we present second-order asymptotic variational sets and apply them to obtain some necessary optimality conditions.

For other theories of optimality conditions, an important point for a necessary optimality condition of this type is that the larger the separated sets, the stronger the result. In [10], variational sets were introduced to replace derivatives so that they are bigger than the sets defined by known derivatives and can be used in the mentioned separation to get stronger necessary optimality condition. Motivated by [4, 20, 10] and the above observations, we propose the second-order radial-asymptotic derivatives. The second-order radial-asymptotic derivatives contain both the second-order asymptotic derivatives and the second-order contingent derivatives. So, the stronger necessary optimality conditions are obtained when using the second-order radial-asymptotic derivatives. Moreover, the second-order radial-asymptotic derivatives also have similar properties of the TP-derivatives (see [22]). Hence, they can be used as a constraint qualification in sensitivity analysis in parameterized optimization problems.

The paper is organized as follows. In Section 2, some basic concepts and preliminary facts are recalled for our use in the sequel. We define the second-order radial-asymptotic derivatives, establish some calculus rules in Section 3. Section 4 consists of the optimality conditions for various kinds of solutions to some particular vector optimization problems by applying the second-order radial-asymptotic derivatives and their calculus rules. In Section 5, we discuss relations between the second-order contingent derivatives of a set-valued map and its profile map and relations between the second-order contingent derivatives of the proper perturbation map and the feasible-set map in the parameterized optimization problem.

2 Preliminaries

In the sequel, let X, Y and Z be normed spaces, $C \subseteq Y$ be a closed convex cone. B_X, B_Y stand for the closed unit balls in X, Y, respectively. $\mathcal{U}(x_0)$ is used to denote the set of all neighborhoods of $x_0 \in X$. For $A \subseteq X$, $\text{int}A$, $\text{cl}A$, ∂A denote its interior, closure and boundary, respectively. Furthermore, $\text{cone}A = \{\lambda a \mid \lambda \geq 0,\ a \in A\}$. A set $B \subset Y$ is called a base for C if $0 \notin \text{cl}B$ and $C = \{tb : t \in \mathbb{R}_+, b \in B\}$. If B is compact we say that C has a compact base B. The cone C has a compact base if and only if $C \cap \partial B$ is compact (see [22]). If Y is a finite dimensional space, then C has a compact base. For the set-valued map $H : X \rightrightarrows Y$, the domain, graph and epigraph of H are defined respectively by

$$\text{dom}H = \{x \in X : H(x) \neq \emptyset\},\ \text{gr}H = \{(x, y) \in X \times Y : y \in H(x)\},$$
$$\text{epi}H = \{(x, y) \in X \times Y : y \in H(x) + C\}.$$

The so-called profile mapping of H is $H+C$ defined by $(H+C)(x) = H(x)+C$. Throughout the rest of this section, let A be a nonempty subset of Y and $a_0 \in A$. One of the main concept in vector optimization is Pareto efficiency. Recall that a_0 is a local Pareto minimal point of A with respect to C ($a_0 \in \text{Min}_C A$)) if there exists $U \in \mathcal{U}(a_0)$ such that

$$(A \cap U - a_0) \cap (-C \setminus C) = \emptyset.$$

In this paper, we are concerned also with the following other concepts of efficiency.

(i) Supposing that $\text{int}C \neq \emptyset$, a_0 is a local weakly efficient point of A ($a_0 \in \text{WMin}_C A$) if there exists $U \in \mathcal{U}(a_0)$ such that

$$(A \cap U - a_0) \cap (-\mathrm{int}C) = \emptyset$$

(ii) Assuming that C is pointed, a_0 is termed a properly efficient point (Henig efficient point) of A, denoted by $a_0 \in \mathrm{PrMin}_C A$, if there exist a convex cone $K \subsetneqq Y$ with $C \setminus \{0\} \subseteq \mathrm{int}K$ and $U \in \mathcal{U}(a_0)$ such that

$$(A \cap U - a_0) \cap (-K) = \{0\}.$$

(iii) Let $Q \subseteq Y$ be an arbitrary nonempty open cone, different from Y. The point a_0 is called a local Q-minimal point (see [14]) of A ($a_0 \in \mathrm{QMin}_C A$) if there exists $U \in \mathcal{U}(a_0)$ such that

$$(A \cap U - a_0) \cap (-Q) = \emptyset.$$

If $U = Y$ the word "local" is omitted.

Definition 2.1. Let $F : X \rightrightarrows Y$ be a set-valued map,

(i) The contingent derivative (see [3]) of F at $(x_0, y_0) \in \mathrm{gr}F$ is

$$\begin{aligned} DF(x_0, y_0)(u) = \{v \in Y :\ &\exists t_n \to 0^+,\ \exists (u_n, v_n) \to (u, v),\ \forall n, \\ &y_0 + t_n v_n \in F(x_0 + t_n u_n)\}. \end{aligned}$$

(ii) The second-order contingent derivative (see [3]) of F at $(x_0, y_0) \in \mathrm{gr}F$ in the direction $(\overline{u}, \overline{v}) \in X \times Y$ is

$$\begin{aligned} D^2F(x_0, y_0, \overline{u}, \overline{v})(u) = \Big\{v \in Y :\ &\exists t_n \to 0^+,\ \exists (u_n, v_n) \to (u, v),\ \forall n, \\ &y_0 + t_n \overline{v} + \frac{1}{2} t_n^2 v_n \in F\Big(x_0 + t_n \overline{u} + \frac{1}{2} t_n^2 u_n\Big)\Big\}. \end{aligned}$$

(iii) The TP-derivative (see [22]) of F at $(x_0, y_0) \in \mathrm{gr}F$ is

$$\begin{aligned} D_S F(x_0, y_0)(u) = \{v \in Y :\ &\exists t_n > 0,\ \exists (u_n, v_n) \to (u, v),\ \forall n, \\ &y_0 + t_n v_n \in F(x_0 + t_n x_n),\ t_n u_n \to 0\}. \end{aligned}$$

(iv) The second-order asymptotic derivative (see [8]) of F at $(x_0, y_0) \in \mathrm{gr}F$ in the direction $(\overline{u}, \overline{v}) \in X \times Y$ is

$$\begin{aligned} D_A^2 F(x_0, y_0, \overline{u}, \overline{v})(u) = \Big\{v \in Y :\ &\exists (t_n, r_n) \to 0^+, \frac{t_n}{r_n} \to 0, \\ &\exists (u_n, v_n) \to (u, v),\ \forall n, y_0 + t_n \overline{v} \\ &\quad + \frac{1}{2} t_n r_n v_n \in F\Big(x_0 + t_n \overline{u} + \frac{1}{2} t_n r_n u_n\Big)\Big\}. \end{aligned}$$

Definition 2.2 (see [30])**.** Let $F : X \rightrightarrows Y$, $(x_0, y_0) \in \mathrm{gr}F$ and $(\overline{u}, \overline{v}) \in X \times Y$. The mapping F is said to be second-order directionally compact at (x_0, y_0) with respect to $(\overline{u}, \overline{v})$ in the direction $u \in X$ if for all sequences $t_n \to 0^+$ and $u_n \to u$, every sequences v_n with $y_0 + t_n \overline{v} + \frac{1}{2} t_n^2 v_n \in F(x_0 + t_n \overline{u} + \frac{1}{2} t_n^2 u_n)$ has a convergent subsequence.

$\boxed{3}$ Second-Order Radial-Asymptotic Derivatives

In this section, we propose the notion of second-order radial-asymptotic derivatives and establish some simple calculus rules.

Definition 3.1. Let $F : X \rightrightarrows Y$ be a set-valued map, the second-order radial-asymptotic derivative of F at $(x_0, y_0) \in \mathrm{gr}F$ in the direction $(\overline{u}, \overline{v}) \in X \times Y$ is

$$D_S^2 F(x_0, y_0, \overline{u}, \overline{v})(u) = \Big\{ v \in Y : \ \exists t_n \to 0^+ , \exists r_n > 0, \ \exists (u_n, v_n) \to (u, v), \ \forall n, \\ y_0 + t_n \overline{v} + \frac{1}{2} t_n r_n v_n \in F\Big(x_0 + t_n \overline{u} + \frac{1}{2} t_n r_n u_n\Big), t_n r_n u_n \to 0 \Big\}.$$

Remark 3.2. For all $(x_0, y_0) \in \mathrm{gr}F$ and for any $(\overline{u}, \overline{v}) \in X \times Y$,

(i) $D_S^2 F(x_0, y_0, \overline{u}, \overline{v})(u)$ is a cone for all $u \in X$,

(ii) $0 \in D_S^2 F(x_0, y_0, \overline{u}, \overline{v})(0)$.

Remark 3.3. For all $u \in X$,

(i) $D_S^2 F(x_0, y_0, 0, 0)(u) = D_S F(x_0, y_0)(u)$,

(ii) $D^2 F(x_0, y_0, \overline{u}, \overline{v})(u) \subseteq D_S^2 F(x_0, y_0, \overline{u}, \overline{v})(u)$,

(iii) $D_A^2 F(x_0, y_0, \overline{u}, \overline{v})(u) \subseteq D_S^2 F(x_0, y_0, \overline{u}, \overline{v})(u)$.

The inclusion may be strict as in the following example.

Example 3.4. Let $X = Y = \mathbb{R}$, $F : X \rightrightarrows Y$ be defined by

$$F(x) = \{ y \in \mathbb{R} \mid y \geq x^2 \text{ or } y = -x^2 \}.$$

Then, for $(x_0, y_0) = (0, 0) \in \mathrm{gr}F$, $(\overline{u}, \overline{v}) = (1, 0)$, one gets, for any $u \in \mathbb{R}$,

$$D^2 F(x_0, y_0, \overline{u}, \overline{v})(u) = \{ v \in \mathbb{R} \mid v \geq 2 \text{ or } v = -2 \},$$

$$D_A^2 F(x_0, y_0, \overline{u}, \overline{v})(u) = \{ v \in \mathbb{R} \mid v \geq 0 \}, \ D_S^2 F(x_0, y_0, \overline{u}, \overline{v})(u) = \mathbb{R}.$$

Moreover, our derivative is different from the second-order contingent-radial derivative (see [4]) of F at $(x_0, y_0) \in \mathrm{gr}F$, defined by

$$D_S^2 F(x_0, y_0)(u) = \{ v \in Y : \ \exists t_n > 0, \ \exists (u_n, v_n) \to (u, v), \ \forall n, \\ y_0 + t_n^2 v_n \in F(x_0 + t_n u_n), \ t_n u_n \to 0 \}.$$

The following example highlights detailed differences between the above-mentioned two derivatives.

Example 3.5. Let $X = Y = \mathbb{R}$, $F : X \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} \{0\}, & \text{if } x = 0 \\ \{1, -x^2\}, & \text{if } x \neq 0. \end{cases}$$

Then, for $(x_0, y_0) = (0, 0) \in \mathrm{gr}F$, $(\overline{u}, \overline{v}) = (1, 0)$, we have

$$D_S^2 F(x_0, y_0)(u) = \begin{cases} \mathbb{R}_+, & \text{if } u = 0, \\ \{-u^2\}, & \text{if } u \neq 0, \end{cases}$$

$$D_S^2 F(x_0, y_0, \overline{u}, \overline{v})(u) = \begin{cases} \mathbb{R}, & \text{if } u = 0, \\ \mathbb{R}_-, & \text{if } u \neq 0. \end{cases}$$

For some calculus rule of second order radial-asymptotic derivatives, we need the following notion.

Definition 3.6. Let $F : X \rightrightarrows Y$ be a set-valued map, $(x_0, y_0) \in \mathrm{gr}F$, $u \in X$ and vector $(\overline{u}, \overline{v}) \in X \times Y$. If

$$D_S^2F(x_0, y_0, \overline{u}, \overline{v})(u) = \Big\{v \in Y : \ \forall t_n \to 0^+ \ , \forall r_n > 0, \ \forall u_n \to u : t_n r_n u_n \to 0, \\ \exists v_n \to v, y_0 + t_n\overline{v} + \frac{1}{2}t_n r_n v_n \in F\Big(x_0 + t_n\overline{u} + \frac{1}{2}t_n r_n u_n\Big)\Big\},$$

and the set on the right side is nonempty, then $D_S^2F(x_0, y_0, \overline{u}, \overline{v})$ is called a second-order radial-semi-derivative of F at (x_0, y_0) with respect to $(\overline{u}, \overline{v})$ in direction u.

Note further that this property clearly holds if the left side of the equality in Definition 3.6 is a singleton.

Proposition 3.7. *Let $F_1, F_2 : X \rightrightarrows Y$, $x_0 \in \mathrm{int}(\mathrm{dom}F_1) \cap \mathrm{dom}F_2$, $(x_0, y_i) \in \mathrm{gr}F_i$, $u \in X$ and $(\overline{u}, \overline{v}_i) \in X \times Y$ for $i = 1, 2$. Suppose that F_1 has a second-order radial-semi-derivative at (x_0, y_1) with respect to $(\overline{u}, \overline{v}_1)$ in direction u. Then,*

$$D_S^2F_1(x_0, y_1, \overline{u}, \overline{v}_1)(u) + D_S^2F_2(x_0, y_2, \overline{u}, \overline{v}_2)(u) \subseteq D_S^2(F_1 + F_2)(x_0, y_1 + y_2, \overline{u}, \overline{v}_1 + \overline{v}_2)(u).$$

Proof. Let $v_i \in D_S^2F_i(x_0, y_i, \overline{u}, \overline{v}_i)(u)$ for $i = 1, 2$. Because $v_2 \in D_S^2F_2(x_0, y_2, \overline{u}, \overline{v}_2)(u)$, there exist $t_n \to 0^+$, $r_n > 0$, $u_n \to u$, $v_n^2 \to v_2$ such that $t_n r_n u_n \to 0$ and

$$y_2 + t_n\overline{v}_2 + \frac{1}{2}t_n r_n v_n^2 \in F_2(x_0 + t_n\overline{u} + \frac{1}{2}t_n r_n u_n), \ \forall n.$$

Since $D_S^2F_1(x_0, y_1, \overline{u}, \overline{v}_1)$ is a second-order radial-semi-derivative of F at (x_0, y_1) with respect to $(\overline{u}, \overline{v}_1)$ in direction u, with t_n, r_n and u_n above, there exists $v_n^1 \to v_1$ such that $t_n r_n u_n \to 0$ and

$$y_1 + t_n\overline{v}_1 + \frac{1}{2}t_n r_n v_n^1 \in F_1\Big(x_0 + t_n\overline{u} + \frac{1}{2}t_n r_n u_n\Big).$$

Thus,

$$y_1 + y_2 + t_n(\overline{v}_1 + \overline{v}_2) + \frac{1}{2}t_n r_n(v_n^1 + v_n^2) \in (F_1 + F_2)\Big(x_0 + t_n\overline{u} + \frac{1}{2}t_n r_n u_n\Big).$$

The conclusion is obtained. □

The following example shows that the assumption about the existence of the second-order radial-semi-derivative of F_1 in Proposition 3.7 cannot be dropped.

Example 3.8. Let $X = Y = \mathbb{R}$ and $F_1, F_2 : X \rightrightarrows Y$ be given by

$$F_1(x) = \begin{cases} \{1\}, & \text{if } x = \frac{1}{n}, n \in \mathbb{N}, \\ \{0\}, & \text{if } x = 0, \end{cases} \qquad F_2(x) = \begin{cases} \{0\}, & \text{if } x = \frac{1}{n}, n \in \mathbb{N}, \\ \{1\}, & \text{if } x = 0. \end{cases}$$

Let $(x_0, y_1) = (0, 0) \in \mathrm{gr}F_1$, $(x_0, y_2) = (0, 1) \in \mathrm{gr}F_2$, $(\overline{u}, \overline{v}_1) = (1, 0)$, $(\overline{u}, \overline{v}_2) = (1, 0)$ and $u = 0$. It is easy to see that for $i = 1, 2$, F_i has not a second-order radial-semi-derivative at (x_0, y_i) with respect to $(\overline{u}, \overline{v}_i)$ in direction u. We have

$$D_S^2F_1(0, 0, 1, 0)(0) = \mathbb{R}_+, D_S^2F_2(0, 1, 1, 0)(0) = \mathbb{R}_-.$$

On the other hand,

$$(F_1+F_2)(x)=\begin{cases}\{1\}, & \text{if } x=\frac{1}{n}, n\in\mathbb{N},\\ \{1\}, & \text{if } x=0.\end{cases}$$

Direct calculations yield

$$D_S^2(F_1+F_2)(0,1,1,0)(0)=\{0\}.$$

Hence,

$$D_S^2F_1(0,0,1,0)(0)+D_S^2F_2(0,1,1,0)(0)\not\subseteq D_S^2(F_1+F_2)(0,1,1,0)(0).$$

Proposition 3.9. *Let $F: X \rightrightarrows Y, G: Y \rightrightarrows Z$ with $\mathrm{Im}F \subseteq \mathrm{dom}G$, $(x_0,y_0)\in \mathrm{gr}F$, $(y_0,z_0)\in \mathrm{gr}G$, $(\overline{u},\overline{v},\overline{w})\in X\times Y\times Z$. Suppose that G has a second-order radial-semi-derivative at (y_0,z_0) with respect $(\overline{v},\overline{w})$ in any direction in $D_S^2F(x_0,y_0,\overline{u},\overline{v})(u)$. Then, for all $u\in X$,*

$$D_S^2G(y_0,z_0,\overline{v},\overline{w})[D_S^2F(x_0,y_0,\overline{u},\overline{v})(u)]\subseteq D_S^2(G\circ F)(x_0,z_0,\overline{u},\overline{w})(u).$$

Proof. Let $w\in D_S^2G(y_0,z_0,\overline{v},\overline{w})[D_S^2F(x_0,y_0,\overline{u},\overline{v})(u)]$. It follows that there exists $v\in D_S^2F(x_0,y_0,\overline{u},\overline{v})(u)$ such that $w\in D_S^2G(y_0,z_0,\overline{v},\overline{w})(v)$. Therefore, there exist $t_n\to 0^+$, $r_n>0$ and $u_n\to u$ such that $y_0+t_n\overline{v}+\frac{1}{2}t_nr_nv_n\in F(x_0+t_n\overline{u}+\frac{1}{2}t_nr_nu_n)$ and $t_nr_nu_n\to 0$. Since G has a second-order radial-semi-derivative at (y_0,z_0) with respect to $(\overline{v},\overline{w})$, with t_n, r_n and u_n above, there exists $w_n\to w$ such that

$$\begin{aligned} z_0+t_n\overline{w}+\frac{1}{2}t_nr_nw_n &\in G\Big(y_0+t_n\overline{v}+\frac{1}{2}t_nr_nv_n\Big)\\ &\subseteq G\Big[F\Big(x_0+t_n\overline{u}+\frac{1}{2}t_nr_nu_n\Big)\Big]\\ &= (G\circ F)\Big(x_0+t_n\overline{u}+\frac{1}{2}t_nr_nu_n\Big). \end{aligned}$$

Hence, $w\in D_S^2(G\circ F)(x_0,z_0,\overline{u},\overline{w})(u)$. □

The following properties are immediate from the definition.

Proposition 3.10. *Let $F: X\rightrightarrows Y$, $u\in X$, $(x_0,y_0)\in \mathrm{gr}F$ and $(\overline{u},\overline{v})\in X\times Y$. Then for all $u\in X$,*

(i) $D_S^2(\lambda F)(x_0,\lambda y_0,\overline{u},\lambda\overline{v})(u)=\lambda D_S^2F(x_0,y_0,\overline{u},\overline{v})(u)$, *for all* $\lambda\in\mathbb{R}$,

(ii) $D_S^2F(x_0,y_0,\lambda\overline{u},\lambda\overline{v})(\lambda u)=\lambda D_S^2F(x_0,y_0,\overline{u},\overline{v})(u)$, *for all* $\lambda>0$.

4 Second-Order Optimality Conditions

In this section, we apply the second-order asymptotic-contingent derivatives to establish the necessary optimality conditions for Q-minimal solutions of some kinds of the unconstrained set-valued vector optimization problems.

Consider the following unconstrained set-valued vector optimization problem, for $F: X\rightrightarrows Y$,

(P) $$F(x),\ x\in X.$$

For a set-valued vector optimization problem, from the concepts of efficiency recalled at the Section 2, we define in the usual and natural way, the corresponding solution notions. For instance, $(x_0,y_0)\in \mathrm{gr}F$ is called a local Q-minimal solution of (P) if there exists $U\in\mathcal{U}(x_0)$ such that $(F(U)-y_0)\cap(-Q)=\emptyset$.

Proposition 4.1. *Let $(x_0, y_0) \in \mathrm{gr}F$ be local Q-minimal solution of* (P)*, the nonempty open cone Q satisfy $Q + C \subset Q$ and $(\overline{u}, \overline{v}) \in X \times (-C)$. Then, for all $u \in X$,*

$$D^2_S F(x_0, y_0, \overline{u}, \overline{v})(u) \cap (-Q) = \emptyset.$$

Proof. Suppose there exist $u \in X$ and $v \in D^2_S F(x_0, y_0, \overline{u}, \overline{v})(u) \cap (-Q)$. Then, there exist sequences $t_n \to 0^+$, $r_n > 0$ and $(u_n, v_n) \to (u, v)$ such that $t_n r_n u_n \to 0$ and $y_0 + t_n \overline{v} + \frac{1}{2} t_n r_n v_n \in F(x_0 + t_n \overline{u} + \frac{1}{2} t_n r_n u_n)$. Since the cone Q is open, $t_n r_n v_n \in -Q$ for large n. Hence, for such n,

$$t_n \overline{v} + \frac{1}{2} t_n r_n v_n \in -C - Q \subset -Q.$$

Therefore, $t_n \overline{v} + \frac{1}{2} t_n r_n v_n \in (F(x_0 + t_n \overline{u} + \frac{1}{2} t_n r_n u_n) - y_0) \cap (-Q)$, a contradiction. $\square$

Observe that $y \notin -Q = -\mathrm{int}Q$ is equivalent to the existence of $c^* \in Q^*$ with $\langle c^*, y \rangle \geq 0$. Hence, we can formulate dual forms of the Proposition 4.1 as follows. The proofs are straightforward.

Corollary 4.2. *Let $(x_0, y_0) \in \mathrm{gr}F$ be local Q-minimal solution of* (P)*, the nonempty open cone Q satisfy $Q + C \subset Q$ and $(\overline{u}, \overline{v}) \in X \times (-C)$. Then, for all $u \in X$ and $v \in D^2_S F(x_0, y_0, \overline{u}, \overline{v})(u)$, there exists $c^* \in Q^*$ such that*

$$\langle c^*, v \rangle \geq 0.$$

Next two examples explain advantages of Proposition 4.1 over recent existing results.

Example 4.3. Let $X = Y = \mathbb{R}, C = \mathbb{R}_+, Q = \mathrm{int}C$, $F : X \rightrightarrows Y$ be defined by

$$F(x) = \{y \in \mathbb{R} \mid y \geq x^2 \text{ or } y = -x^2\}.$$

Then, for $(x_0, y_0) = (0, 0) \in \mathrm{gr}F$, one gets, for any $u \in \mathbb{R}$,

$$DF(x_0, y_0)(u) = D_S F(x_0, y_0)(u) = \mathbb{R}_+.$$

Since $DF(x_0, y_0)(u) \cap -\mathrm{int}C = D_S F(x_0, y_0)(u) \cap -\mathrm{int}C = \emptyset$, for all $u \in X$, we cannot use Theorem 2.1 in [20] and Theorem 3.1 in [25] to reject (x_0, y_0) for a local weakly efficient solution. But with $(\overline{u}, \overline{v}) = (1, 0)$, $\forall u \in X$,

$$D^2_S F(x_0, y_0, \overline{u}, \overline{v})(u) = \mathbb{R},$$

i.e. $D^2_S F(x_0, y_0, \overline{u}, \overline{v})(u) \cap -\mathrm{int}C \neq \emptyset, \forall u \in X$. So, Proposition 4.1 rejects (x_0, y_0).

Example 4.4. Let $X = Y = \mathbb{R}, C = \mathbb{R}_+, Q = \mathrm{int}C$, $F : X \rightrightarrows Y$ be defined by

$$F(x) = \{y \in \mathbb{R} \mid y \geq -x^3\}.$$

Then, for all $x \in X$, $F_+(x) = F(x) + C = F(x)$. For $(x_0, y_0) = (0, 0) \in \mathrm{gr}F$, one gets

$$DF_+(x_0, y_0)(u) = \mathbb{R}_+, \forall u \in X.$$

Hence $DF_+(x_0, y_0)(u) \cap -\partial C = \{0\}$ and $S_0 := \mathrm{dom}DF_+(x_0, y_0) = \mathbb{R}$. Therefore, for every $\overline{v} \in DF_+(x_0, y_0)(u) \cap -\partial C$, i.e. $\overline{v} = 0$, and for any $\overline{u} \in S_0$, one gets

$$D^2 F_+(x_0, y_0, \overline{u}, \overline{v})(u) = D^2_A F_+(x_0, y_0, \overline{u}, \overline{v})(u) = \mathbb{R}_+,$$

$$\mathrm{dom}D^2F_+(x_0, y_0, \overline{u}, \overline{v}) = \mathrm{dom}D^2_AF_+(x_0, y_0, \overline{u}, \overline{v}) = \mathbb{R}.$$

Consequently, for all $u \in \mathbb{R}$ and $\overline{u} \in S_0$,

$$D^2F_+(x_0, y_0, \overline{u}, \overline{v})(u) \cap (-\mathrm{int}C - \{\overline{v}\}) = \emptyset,$$

$$D^2_AF_+(x_0, y_0, \overline{u}, \overline{v})(u) \cap (-\mathrm{int}C - \{\overline{v}\}) = \emptyset.$$

Therefore, Theorem 3.1 in [7] and Theorem 3.1 in [8] cannot be used to reject (x_0, y_0) for a local weakly solution. Moreover, for any $u \in \mathrm{dom}DF(x_0, y_0) = \mathbb{R}, \overline{v} \in DF(x_0, y_0)(u) \cap -C$, i.e. $\overline{v} = 0$, one has

$$\mathrm{IT}(-C, \overline{v}) = \mathrm{intcone}(-C - \overline{v}) = \mathrm{intcone}(-C) = \mathrm{int}(-C) = -\mathrm{int}C,$$

and epi(F) is asymptotically derivable at $(x_0, y_0, \overline{u}, \overline{v})$. Hence,

$$D^2F_+(x_0, y_0, \overline{u}, \overline{v})[\mathrm{IT}(-C, \overline{v})]^- = \{u \in X | D^2F_+(x_0, y_0, \overline{u}, \overline{v})(u) \cap (\mathrm{IT}(-C, \overline{v})) \neq \emptyset\} = \emptyset,$$

$$D^2_AF_+(x_0, y_0, \overline{u}, \overline{v})[\mathrm{IT}(-C, \overline{v})]^- = \{u \in X | D^2_AF_+(x_0, y_0, \overline{u}, \overline{v})(u) \cap (\mathrm{IT}(-C, \overline{v})) \neq \emptyset\} = \emptyset.$$

Hence, Theorem 4.2 in [9] cannot also be used to reject (x_0, y_0) for a local weakly solution.

But with $(\overline{u}, \overline{v}) = (1, 0)$, $\forall u \in X$,

$$D^2_SF(x_0, y_0, \overline{u}, \overline{v})(u) = \mathbb{R},$$

i.e. $D^2_SF(x_0, y_0, \overline{u}, \overline{v})(u) \cap -\mathrm{int}C \neq \emptyset, \forall u \in X$. So, Proposition 4.1 can be used to reject (x_0, y_0).

Proposition 4.5. *Assume that X is a finite dimensional space, C has a compact base B, $F : X \rightrightarrows Y$, $(x_0, y_0) \in \mathrm{gr}F$. If for any $(\overline{u}, \overline{v}) \in X \times (-C)$,*

(i) $D^2_SF(x_0, y_0, \overline{u}, \overline{v})(0) \cap (-C) = \{0\}$;

(ii) $D^2_SF(x_0, y_0, \overline{u}, \overline{v})(u) \cap (-C) = \emptyset$ *for all nonzero* $u \in X$,

then (x_0, y_0) is a local Pareto minimal solution of (P).

Proof. Note that (i) and (ii) are required to be satisfied also $(\overline{u}, \overline{v}) = (0, 0)$. So, from the Remark 3.2, (i) and (ii) imply

(i') $D_SF(x_0, y_0)(0) \cap (-C) = \{0\}$;

(ii') $D_SF(x_0, y_0)(u) \cap (-C) = \emptyset$ for all nonzero $u \in X$.

Applying Theorem 4.1 in [25], one gets conclusion. □

Now, we apply calculus rules to obtain necessary conditions for weakly efficient solutions of several particular optimization problems. Firstly, applying the above chain rule, we easily establish necessary optimality condition for local Q-minimal solutions of the following problem

$$(\mathrm{P}_1) \qquad \min F(x') \text{ subject to } x' \in G(x) \text{ and } x \in X,$$

where $F : X \rightrightarrows Y$ and $G : X \rightrightarrows X$. This problem can be restated as the unconstrained problem $\min(F \circ G)(x)$ s.t. $x \in X$.

Proposition 4.6. *Let* $\mathrm{Im}G \subseteq \mathrm{dom}F$*,* $(x_0, z_0) \in \mathrm{gr}G$*,* $(z_0, y_0) \in \mathrm{gr}F$*,* $u \in X$*, and* $(\overline{u}, \overline{v}, \overline{w}) \in X \times X \times Y$*. Assume that* (x_0, y_0) *is a local* Q*-minimal solution of* (P_1)*. If* F *has a second-order radial-semi-derivative at* (z_0, y_0) *with respect to* $(\overline{w}, \overline{v})$ *for any direction in* $D^2_S G(x_0, z_0, \overline{u}, \overline{w})(u)$*, then*

$$D^2_S F(z_0, y_0, \overline{w}, \overline{v})[D^2_S G(x_0, z_0, \overline{u}, \overline{w})(u)] \cap (-Q) = \emptyset.$$

Proof. By Proposition 4.1, for $u \in X$ we have $D^2_S(F \circ G)(x_0, y_0, \overline{u}, \overline{v})(u) \cap (-Q) = \emptyset$. Proposition 3.9 implies that $D^2_S F(z_0, y_0, \overline{w}, \overline{v})[D^2_S G(x_0, z_0, \overline{u}, \overline{w})(u)] \cap (-Q) = \emptyset$. ☐

To compare Proposition 4.5 with a result in [6], we recall the definition of contingent epiderivative. A single-valued map $EDF : X \to Y$ satisfying $\mathrm{epi}(EDF(x_0, y_0)) = T_{\mathrm{epi}F}(x_0, y_0)$ is said to be the contingent epiderivative of F at $(x_0, y_0) \in \mathrm{gr}F$.

Example 4.7. Let $X = Y = \mathbb{R}$, $Q = \mathrm{int}\mathbb{R}_+, C = \mathbb{R}_+$, $G(x) = \{-|x|\}$, and

$$F(x) = \begin{cases} \mathbb{R}_-, & \text{if } x \leq 0, \\ \emptyset, & \text{if } x > 0. \end{cases}$$

Since G is single-valued, we try to make use of Proposition 5.2 of [6]. We can check that $DG(0, G(0))(u) = \{-|u|\}$ for all $u \in X$, and $T_{\mathrm{epi}F}(G(0), 0) = \mathbb{R}_- \times \mathbb{R}$. Hence, the contingent epiderivative $EDF(G(0), 0)(u)$ does not exist for any $u \in X$ and the mentioned Proposition 5.2 of [6] cannot be applied. However, F has a second-order radial-semi-derivative at $(G(0), 0)$ with respect to $(-1, 0)$ in all directions in $D^2_S G(0, G(0), 0, 0)(0) = \{0\}$, $D^2_S F(G(0), 0, -1, 0)[D^2_S G(0, G(0), 0, 0)(0)] = \mathbb{R}_-$, which meets $-\mathrm{int}C$. Therefore, Proposition 4.6 above rejects the candidate $(0, 0)$.

To illustrate the sum rule, we consider the following problem

$$(\mathrm{P}_2) \qquad \min F(x) \text{ subject to } g(x) \in -C,$$

where $g : X \to Y$. Define $M := \{x \in X \mid g(x) \in -C\}$ (the feasible set) and $G : X \rightrightarrows Y$ by

$$G(x) = \begin{cases} \{0\}, & \text{if } x \in M, \\ \{g(x)\}, & \text{otherwise.} \end{cases}$$

Consider the following unconstrained optimization problem, for an arbitrary positive s,

$$(\mathrm{P}_C) \qquad \min(F + sG)(x).$$

In the particular case, where $Y = \mathbb{R}$ and F is single-valued, (P_C) is used to approximate (P_2) in penalty methods (see [21]). Then, usually s is large or tends to infinity. Think of a simple one dimensional case: $f(x) = x$ and $g(x) = -x + 2$. Then, $x^* = 2$ is a solution of (P_2) and also of (P_C) for large s, e.g., $s = 1000$. But, for $s = 1/2$, the solution of (P_C) is not close to 2. (P_C) has also been studied independently from (P_2). Optimality conditions for this general problem (P_C) were obtained in [6] by using sum rules and scalar product rules for contingent epiderivatives. Now, we apply Propositions 4.1, 3.7, and 3.10 for second-order radial-asymptotic derivatives to get the following necessary condition for local Q-minimal solutions of (P_C). Here, s can be any positive number.

Proposition 4.8. *Let* $\mathrm{dom}F \subseteq \mathrm{dom}G$*,* $x_0 \in M, y_0 \in F(x_0)$*,* $(\overline{u}, \overline{v}) \in X \times Y$*,* $u \in X$*, and either* F *has a second-order radial-semi-derivative at* (x_0, y_0) *with respect to* $(\overline{u}, \overline{v})$ *in direction* u *or* G *has a second-order radial-semi-derivative at* $(x_0, 0)$ *with respect to* $(\overline{u}, 0)$ *in direction* u*. If* (x_0, y_0) *is a local* Q*-minimal solution of* (P_C)*, then,*

$$(D^2_S F(x_0, y_0, \overline{u}, \overline{v})(u) + sD^2_S G(x_0, 0, \overline{u}, 0)(u)) \cap (-Q) = \emptyset.$$

Proof. By Proposition 4.1, one gets $D^2_S(F+sG)(x_0,y_0,\overline{u},\overline{v})(u)\cap(-\mathrm{Q})=\emptyset$. According to Proposition 3.10, $sD^2_SG(x_0,0,\overline{u},0)(u)=D^2_S(sG)(x_0,0,\overline{v},0)(u)$. Then, Proposition 3.7 completes the proof:

$$D^2_SF(x_0,y_0,\overline{u},\overline{v})(u)+sD^2_SG(x_0,0,\overline{u},0)(u)\subseteq D^2_S(F+sG)(x_0,y_0,\overline{u},\overline{v})(u).$$

□

Next example illustrates a case, where Proposition 4.8 is more advantageous than the earlier existing results.

Example 4.9. Let $X=Y=\mathbb{R}$, $Q=\mathrm{int}\mathbb{R}_+, C=\mathbb{R}_+$, $g(x)=x^4-2x^3$, and

$$F(x)=\begin{cases}\mathbb{R}_-, & \text{if } x\leq 0,\\ \emptyset, & \text{if } x>0.\end{cases}$$

Then, $M=[0,2]$ and $G(x)=\{\max\{0,x^4-2x^3\}\}$. Since $T_{\mathrm{epi}F}(0,0)=\mathbb{R}_-\times\mathbb{R}$ and $T_{\mathrm{epi}G}(0,0)=\{(x,y)|\ y\geq 0\}$, the contingent epiderivative $EDF(0,0)(u)$ does not exist for any $u\in X$. Hence, Proposition 5.1 in [6] cannot be employed. But, F has a second-order radial-semi-derivative of at (0,0) with respect to $(-1,0)$ in any direction and we can check that $D^2_SF(0,0,-1,0)(0)=\mathbb{R}_-$, and $D^2_SG(0,0,0,0)(0)=\{0\}$. Therefore, $(D^2_SF(0,0,-1,0)(0)+sD^2_SG(0,0,0,0)(0))\cap(-\mathrm{int}C)\neq\emptyset$. In view of Proposition 4.8, (x_0,y_0) is not a local weakly solution of (P_C). This fact can be checked directly too.

Remark 4.10. As the remark of an Anonymous Referee, the results in the Sect. 4, shown for Q-minimal solutions of (P), can be developed analogously to the set-valued optimization problems. Much more interesting is the well-known solution concept by Kuroiwa for set-valued optimization problems (see [13]). The author thinks that second-order necessary optimality conditions for solutions given by Kuroiwa's concept can be derived by using the second-order derivatives based on the set of continuous selections of set-valued map as the approach of Alonso and Rodriguez-Marin in [1,2].

$\boxed{5}$ Sensitivity Analysis in Nonconvex Set-Valued Optimization

The first-order sensitivity analysis using the first-order contingent derivative was investigated by Tanino in [26, 27]. See also [11, 12, 22] and references therein. In this section, we discuss the relations between the second-order contingent-type derivatives of a set-valued map and those of its profile map. Such relations for proper efficient points of these derivatives are also investigated. We first give a simple sufficient condition ensuring the compactness in Definition **??** in the following.

Proposition 5.1. *Let $F:X\rightrightarrows Y$, $(x_0,y_0)\in\mathrm{gr}F$, $(\overline{u},\overline{v})\in X\times Y$ and Y be finite dimensional. If $D^2_SF(x_0,y_0,\overline{u},\overline{v})(0)=\{0\}$, then F is second-order directionally compact at (x_0,y_0) with respect to $(\overline{u},\overline{v})$ in all directions $u\in X$.*

Proof. Let $t_n\to 0^+$ and $u_n\to u$, and v_n with $y_0+t_n\overline{v}+\frac{1}{2}t_n^2v_n\in F(x_0+t_n\overline{u}+\frac{1}{2}t_n^2u_n)$. It is sufficient to show the boundedness of $\{v_n\}$. Indeed, suppose to contrary that $\lim_{n\to\infty}\|v_n\|=+\infty$. Setting $\widetilde{v}_n:=v_n/\|v_n\|$, $\widetilde{u}_n:=u_n/\|v_n\|$ and $r_n:=t_n\|v_n\|$. Then, $r_n>0$ and $\widetilde{u}_n\to 0$. Hence, there exist $t_n\to 0^+, r_n>0$ and $(\widetilde{u}_n,\widetilde{v}_n)\to(0,\widetilde{v})$ with $\|\widetilde{v}\|=1$ such that $y_0+t_n\overline{v}+\frac{1}{2}t_nr_n\widetilde{v}_n\in F(x_0+t_n\overline{u}+\frac{1}{2}t_nr_n\widetilde{u}_n)$ and $t_nr_n\widetilde{u}_n\to 0$. This leads $\widetilde{v}\in D^2_SF(x_0,y_0,\overline{u},\overline{v})(0)$, a contradiction. □

In the following, the relation between the second-order contingent-type derivatives of a set-valued map and those of its profile map is investigated.

Proposition 5.2. *Let $F : X \rightrightarrows Y$, $(x_0, y_0) \in \mathrm{gr}F$, $(\overline{u}, \overline{v}) \in X \times Y$.*

(i) *Suppose that C has a compact base and $D^2_S F(x_0, y_0, \overline{u}, \overline{v})(0) \cap (-C) = \{0\}$. Then, for all $u \in X$,*

$$D^2F(x_0, y_0, \overline{u}, \overline{v})(u) + C = D^2(F + C)(x_0, y_0, \overline{u}, \overline{v})(u). \tag{5.1}$$

(ii) *If F is second-order directionally compact at (x_0, y_0) with respect to $(\overline{u}, \overline{v})$ in any direction $u \in X$ then (5.1) holds.*

Proof. (i) Let $v \in D^2F(x_0, y_0, \overline{u}, \overline{v})(u)$ and $c \in C$. Then, there exist $t_n \to 0^+$, $(u_n, v_n) \to (u, v)$ such that $y_0 + t_n\overline{v} + \frac{1}{2}t_n^2 v_n \in F(x_0 + t_n\overline{u} + \frac{1}{2}t_n^2 u_n)$, for all n. Setting $\widetilde{v}_n := v_n + c$, for all n. Then, $\widetilde{v}_n \to v + c$ and

$$y_0 + t_n\overline{v} + \frac{1}{2}t_n^2\widetilde{v}_n = y_0 + t_n\overline{v} + \frac{1}{2}t_n^2 v_n + \frac{1}{2}t_n^2 c \in F(x_0 + t_n\overline{u} + \frac{1}{2}t_n^2 u_n) + C, \forall n.$$

Hence, $v + c \in D^2(F + C)(x_0, y_0, \overline{u}, \overline{v})(u)$. Therefore,

$$D^2F(x_0, y_0, \overline{u}, \overline{v})(u) + C \subseteq D^2(F + C)(x_0, y_0, \overline{u}, \overline{v})(u).$$

For the reverse inclusion, let $u \in X$ and $v \in D^2(F + C)(x_0, y_0, \overline{u}, \overline{v})(u)$ be arbitrary. Then, there exist $t_n \to 0^+$, $(u_n, v_n) \to (u, v)$, and $c_n \in C$ such that

$$y_0 + t_n\overline{v} + \frac{1}{2}t_n^2(v_n - c_n) \in F(x_0 + t_n\overline{u} + \frac{1}{2}t_n^2 u_n), \forall n.$$

If there exists n_0 such that $c_n = 0, \forall n \geq n_0$, then

$$v \in D^2F(x_0, y_0, \overline{u}, \overline{v})(u) + 0 \subseteq D^2F(x_0, y_0, \overline{u}, \overline{v})(u) + C.$$

Now assume that $c_n \neq 0$ and $c_n/\|c_n\| \to \overline{c}$ for some $\overline{c} \in C$ with norm one. Setting $r_n := t_n\|c_n\|$. There is two cases for $\|c_n\|$.
Case 1: $\|c_n\| \to +\infty$. Then, $\widetilde{u}_n := u_n/\|c_n\| \to 0$. Since

$$y_0 + t_n\overline{v} + \frac{1}{2}t_n r_n(v_n/\|c_n\| - c_n/\|c_n\|) \in F(x_0 + t_n\overline{u} + \frac{1}{2}t_n r_n\widetilde{u}_n),$$

$v_n/\|c_n\| - c_n/\|c_n\| \to -\overline{c}$ and $t_n r_n\widetilde{u}_n \to 0$, one has $-\overline{c} \in D^2_S F(x_0, y_0, \overline{u}, \overline{v})(0) \cap (-C)$, an impossibility.
Case 2: $\|c_n\|$ is bounded and assume $\|c_n\| \to \alpha \geq 0$. Hence, $c_n = \|c_n\|(c_n/\|c_n\|) \to \alpha\overline{c}$. Then, since

$$y_0 + t_n\overline{v} + \frac{1}{2}t_n^2(v_n - c_n) \in F(x_0 + t_n\overline{u} + \frac{1}{2}t_n^2 u_n),$$

$v_n - c_n \to v - \alpha\overline{c}$ and $u_n \to u$, one has $v - \alpha\overline{c} \in D^2F(x_0, y_0, \overline{u}, \overline{v})(u)$. Therefore, $v \in D^2F(x_0, y_0, \overline{u}, \overline{v})(u) + C$.
(ii) See Proposition 5.1 in [16]. □

The following example shows that the conditions in Proposition **??** are essential.

Example 5.3. Let $X = \mathbb{R}^2, Y = \mathbb{R}, C = \mathbb{R}_+$ and $F : X \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} \{x_1^2 + x_1, -1\}, & \text{if } x_2 = 0, x_1 \geq 0, \\ \emptyset, & \text{otherwise,} \end{cases}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$. Let $(x_0, y_0) = ((0,0), 0) \in \mathrm{gr}F$ and $(\overline{u}, \overline{v}) = ((1,0), 1)$. Since $D_S^2 F(x_0, y_0, \overline{u}, \overline{v})(0,0) = \mathbb{R} \neq \{0\}$, the condition in (i) in Proposition **??** is not satisfied. Moreover, for the direction $u = (1,0)$, for every $t_n \to 0^+$, for $u_n = (u_n^1, 0) \to u$, the sequence $\{v_n\} \subseteq \mathbb{R}$ with

$$y_0 + t_n \overline{v} + \frac{1}{2} t_n^2 v_n = -1 \in F(x_0 + t_n \overline{u} + \frac{1}{2} t_n^2 u_n),$$

i.e. $v_n = -\frac{2}{t_n} - \frac{2}{t_n^2}$, has no convergent subsequence. Hence, the condition in (ii) in Proposition 5.2 is not also satisfied. We have, for all $u = (u_1, u_2) \in X$,

$$D^2 F(x_0, y_0, \overline{u}, \overline{v})(u) = \{2 + u_1\}, D^2 F_+(x_0, y_0, \overline{u}, \overline{v})(u) = \mathbb{R},$$

$$\mathrm{dom} D^2 F(x_0, y_0, \overline{u}, \overline{v}) = \mathrm{dom} D^2 F_+(x_0, y_0, \overline{u}, \overline{v}) = \mathbb{R} \times \{0\}.$$

Thus, for all $u \in X$,

$$D^2 F(x_0, y_0, \overline{u}, \overline{v})(u) + C \neq D^2 F_+(x_0, y_0, \overline{u}, \overline{v})(u).$$

Proposition 5.4. *Let $F : X \rightrightarrows Y$, $(x_0, y_0) \in \mathrm{gr}F$, $(\overline{u}, \overline{v}) \in X \times Y$.*

(i) *Assume that C has a compact base and $D_S^2 F(x_0, y_0, \overline{u}, \overline{v})(0) \cap (-C) = \{0\}$. Then, for all $u \in X$,*

$$\mathrm{PrMin}_C D^2 F(x_0, y_0, \overline{u}, \overline{v})(u) = \mathrm{PrMin}_C D^2 (F + C)(x_0, y_0, \overline{u}, \overline{v})(u). \tag{5.2}$$

(ii) *If F is second-order directionally compact at (x_0, y_0) with respect to $(\overline{u}, \overline{v})$ in any direction $u \in X$ then (5.2) holds.*

Proof. Since the similarity, only (i) is proven. We first prove the inclusion

$$\mathrm{PrMin}_C D^2 F(x_0, y_0, \overline{u}, \overline{v})(u) \subseteq \mathrm{PrMin}_C D^2 (F + C)(x_0, y_0, \overline{u}, \overline{v})(u). \tag{5.3}$$

Let an arbitrary $v \in \mathrm{PrMin}_C D^2 F(x_0, y_0, \overline{u}, \overline{v})(u)$. Then, there exists a convex cone K with $C \setminus \{0\} \subseteq \mathrm{int}K$ such that $v \in \mathrm{Min}_K D_C^2 F(x_0, y_0, \overline{u}, \overline{v})(u)$. We show that v belongs to the right-hand side of (5.3) relative to the same cone K. Suppose to contrary that there exists $w \in D^2(F + C)(x_0, y_0, \overline{u}, \overline{v})(u)$ with $v - w := \overline{k} \in K \setminus (-K)$. By Proposition 5.2 (i), there exists $w' \in D^2 F(x_0, y_0, \overline{u}, \overline{v})(u)$ such that $w - w' := c' \in C$. This leads a contradiction:

$$v - w' = \overline{k} + c' \in K \setminus (-K) + C \subseteq K \setminus (-K).$$

To prove the inclusion reverse to (5.3), let $v \in \mathrm{PrMin}_C D^2(F + C)(x_0, y_0, \overline{u}, \overline{v})(u)$ relative to K. Then, $v \in D^2(F + C)(x_0, y_0, \overline{u}, \overline{v})(u)$. According to Proposition 5.2 (i), there exists $v' \in D^2 F(x_0, y_0, \overline{u}, \overline{v})(u) \subseteq D^2(F + C)(x_0, y_0, \overline{u}, \overline{v})(u)$ such that $v - v' := c' \in C$. If $c' \in C \setminus \{0\} \subseteq \mathrm{int}K$, then $v \notin \mathrm{PrMin}_C D^2(F + C)(x_0, y_0, \overline{u}, \overline{v})(u)$. Hence, $c' = 0$ and $v \in D^2 F(x_0, y_0, \overline{u}, \overline{v})(u)$. Therefore,

$$v \in \mathrm{PrMin}_C D^2(F + C)(x_0, y_0, \overline{u}, \overline{v})(u) \cap D^2 F(x_0, y_0, \overline{u}, \overline{v})(u).$$

So, with the same cone K, one gets $v \in \mathrm{PrMin}_C D^2 F(x_0, y_0, \overline{u}, \overline{v})(u)$. □

Now, we consider the following parameterized vector optimization problem:

$$\min_K f(x,u) = (f_1(x,u), f_2(x,u), ..., f_q(x,u)), \text{ s.t. } x \in X(u) \subseteq \mathbb{R}^l,$$

where x is a l-dimensional decision variable, u is a p-dimensional parameter, f_i is a real valued objective function on $\mathbb{R}^l \times \mathbb{R}^p$ for $i = 1, 2, ..., q$, X is a set-valued map from $\mathbb{R}^p$ to $\mathbb{R}^l$, which defines a feasible decision set, and K is a nonempty pointed closed convex ordering cone in $\mathbb{R}^q$. Let $F(u)$ be the value at u of the feasible set map in the objective space, i.e.,

$$F(u) := \{y \in \mathbb{R}^q \mid y = f(x,u) \text{ for some } x \in X(u)\}.$$

We consider the proper perturbation map of the considered problem

$$P(u) := \text{PrMin}_K F(u).$$

Definition 5.5. For $u_0 \in \mathbb{R}^p$, F is said to be K-dominated by P near u_0 if for all u in some $U \in \mathcal{U}(u_0)$,

$$F(u) \subseteq P(u) + K.$$

Remark 5.6. Since $P(u) \subseteq F(u)$, the K-dominatedness of F by P near u_0 (relative to U) implies that, for all $u \in U$, $F(u) + K = P(u) + K$. Hence, for any $y_0 \in P(u_0)$, $u_0 \in U$, $(\overline{u}, \overline{v}) \in \mathbb{R}^p \times \mathbb{R}^q$, and $u \in \mathbb{R}^p$,

$$D^2(P+K)(u_0, y_0, \overline{u}, \overline{v})(u) = D^2(F+K)(u_0, y_0, \overline{u}, \overline{v})(u).$$

Proposition 5.7. *Let $(\overline{u}, \overline{v}) \in \mathbb{R}^p \times \mathbb{R}^q$, $u_0 \in \mathbb{R}^p$ and u near u_0. Suppose that F is K-dominated by P near u_0.*

(i) *If $D^2_S F(u_0, y_0, \overline{u}, \overline{v})(0) \cap (-K) = \{0\}$, then*

$$\text{PrMin}_K D^2 F(u_0, y_0, \overline{u}, \overline{v})(u) \subseteq D^2 P(u_0, y_0, \overline{u}, \overline{v})(u). \tag{5.4}$$

(ii) *Assume that F is second-order directionally compact at (u_0, y_0) with respect to $(\overline{u}, \overline{v})$ in direction u. Then, assertions (5.4) holds.*

Proof. We prove only (i). The other can be proved similarly. Observe that, being a pointed closed convex cone in $\mathbb{R}^q$, K clearly has a compact base. Furthermore, since $P(u) \subseteq F(u)$, for any $u \in \mathbb{R}^p$, $D^2_S P(u_0, y_0, \overline{u}, \overline{v})(0) \cap (-K) = \{0\}$ and P is second-order directionally compact at (u_0, y_0) with respect to $(\overline{u}, \overline{v})$ in direction u. Therefore, we have

$$\begin{aligned}
\text{PrMin}_K D^2 F(u_0, y_0, \overline{u}, \overline{v})(u) &= \text{PrMin}_K D^2(F+K)(u_0, y_0, \overline{u}, \overline{v})(u) \\
&= \text{PrMin}_K D^2(P+K)(u_0, y_0, \overline{u}, \overline{v})(u) \\
&= \text{PrMin}_K D^2 P(u_0, y_0, \overline{u}, \overline{v})(u) \\
&\subseteq D^2 P(u_0, y_0, \overline{u}, \overline{v})(u).
\end{aligned}$$

Here the first and third equalities are due to Proposition 5.4, and the second one follows from Remark 5.6. □

The essentialness of the K-dominatedness by P near u_0 is justified as follows.

Example 5.8. Let $p = 1$, $q = 2$, $K = \mathbb{R}^2_+$ and $F : \mathbb{R} \to \mathbb{R}^2$ be given by

$$F(x) = \Big\{(y_1, y_2) \in \mathbb{R}^2 \ : \ \frac{1}{2}x^2 \le y_1 \le x^2, 0 \le y_2 \le \frac{1}{2}x^2, (y_1 - x^2)^2 + \Big(y_2 - \frac{1}{2}x^2\Big)^2 \le \frac{1}{4}x^4\Big\}$$
$$\cup\Big\{(y_1, y_2) \in \mathbb{R}^2 \ : \ 0 < y_1 \le \frac{1}{2}x^2, y_2 = \frac{1}{2}x^2\Big\}.$$

Then, for any $x \in X$,

$$P(x) = \begin{cases} \{(y_1, y_2) \in \mathbb{R}^2 \ : \ \frac{1}{2}x^2 < y_1 < x^2, 0 < y_2 < \frac{1}{2}x^2, & \\ \qquad (y_1 - x^2)^2 + (y_2 - \frac{1}{2}x^2)^2 = \frac{1}{4}x^4\}, & \text{if } x \ne 0 \\ \{(0, 0)\}, & \text{if } x = 0. \end{cases}$$

Hence F is not K-dominated by P near $u_0 = 0$. Let $(u_0, y_0) = (0, (0, 0)) \in \text{gr}P$ and $(\overline{u}, \overline{v}) = (1, (0, 0))$. For any $u \in \mathbb{R}$,

$$D^2F(u_0, y_0, \overline{u}, \overline{v})(u) = \{(y_1, y_2) : 1 \le y_1 \le 2, 0 \le y_2 \le 1, (y_1 - 2)^2 + (y_2 - 1)^2 \le 1\}$$
$$\cup\{(y_1, y_2) : 0 \le y_1 \le 1, y_2 = 1\},$$
$$D^2P(u_0, y_0, \overline{u}, \overline{v})(u) = \{(y_1, y_2) : 1 \le y_1 \le 2, 0 \le y_2 \le 1, (y_1 - 2)^2 + (y_2 - 1)^2 = 1\},$$
$$D^2_S F(x_0, y_0, \overline{u}, \overline{v})(0) = \mathbb{R}^2_+.$$

Therefore, the condition $D^2_S F(u_0, y_0, \overline{u}, \overline{v})(0) \cap (-K) = \{(0, 0)\}$ is satisfied. Moreover, F is obviously second-order directionally compact at (u_0, y_0) with respect to $(\overline{u}, \overline{v})$ in direction $u \in \mathbb{R}$. Hence, the condition (ii) in Proposition **??** is also satisfied. We can check that for all $u \in \mathbb{R}$,

$$\text{PrMin}D^2F(u_0, y_0, \overline{u}, \overline{v})(u)$$
$$= \{(y_1, y_2) : 1 < y_1 < 2, 0 < y_2 < 1, (y_1 - 2)^2 + (y_2 - 1)^2 = 1\} \cup \{(0, 1)\}.$$

So, $\forall u \in \mathbb{R}$,

$$\text{PrMin}D^2F(u_0, y_0, \overline{u}, \overline{v})(u) \not\subseteq D^2P(u_0, y_0, \overline{u}, \overline{v})(u).$$

The similar result of Proposition **??** has been obtained in Theorem 5.2 in [29]. But our conditions are different from that of [29]. The following example indicates that our result has advantage in some cases.

Example 5.9. Let $p = 1, q = 2$, $K = \mathbb{R}^2_+$ and $F : \mathbb{R} \to \mathbb{R}^2$ be given by

$$F(x) = \begin{cases} \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \ge 0, y_2 \ge 0\}, & \text{if } x = 0, \\ \{(y_1, y_2) \in \mathbb{R}^2 : (y_1)^{2/3} + (y_2)^{2/3} \ge x^{4/3}, y_1 \ge 0, y_2 \ge 0\}, & \text{if } x \ne 0. \end{cases}$$

Then,

$$P(x) = \begin{cases} \{(0, 0)\}, & \text{if } x = 0, \\ \{(y_1, y_2) \in \mathbb{R}^2 : (y_1)^{2/3} + (y_2)^{2/3} = x^{4/3}, y_1 > 0, y_2 > 0\}, & \text{if } x \ne 0. \end{cases}$$

Let $(u_0, y_0) = (0, (0, 0)) \in \text{gr}P$ and $(\overline{u}, \overline{v}) = (1, (0, 0))$. Then, for all $u \in \mathbb{R}$,

$$D^2F(u_0, y_0, \overline{u}, \overline{v})(u) = \{(y_1, y_2) \in \mathbb{R}^2 : (y_1)^{2/3} + (y_2)^{2/3} \ge 2^{2/3}, y_1 \ge 0, y_2 \ge 0\},$$
$$D^2P(u_0, y_0, \overline{u}, \overline{v})(u) = \{(y_1, y_2) \in \mathbb{R}^2 : (y_1)^{2/3} + (y_2)^{2/3} = 2^{2/3}, y_1 \ge 0, y_2 \ge 0\}.$$

Hence,

$$\text{PrMin}_K D^2F(u_0, y_0, \overline{u}, \overline{v})(u) = \{(y_1, y_2) \in \mathbb{R}^2 : (y_1)^{2/3} + (y_2)^{2/3} = 2^{2/3}, y_1 > 0, y_2 > 0\}.$$

Since for all $x \in X$, $F_+(x) = F(x)$, we have $D^2F_+(u_0, y_0, \overline{u}, \overline{v})(u) = D^2F(u_0, y_0, \overline{u}, \overline{v})(u)$, $\forall u \in \mathbb{R}$ and $\text{PrMin}_K D^2F_+(u_0, y_0, \overline{u}, \overline{v})(u) = \text{PrMin}_K D^2F(u_0, y_0, \overline{u}, \overline{v})(u)$ for all $u \in \mathbb{R}$. It implies that

$$D^2F_+(u_0, y_0, \overline{u}, \overline{v})(u) \not\subseteq \text{PrMin}_K D^2F_+(u_0, y_0, \overline{u}, \overline{v})(u) + K,$$

i.e. the condition $P(x) := D^2F_+(u_0, y_0, \overline{u}, \overline{v})(u)$ satisfying the proper K-domination property in Theorem 5.2 in [29] is not fulfilled.
However, since $D^2_S F(u_0, y_0, \overline{u}, \overline{v})(0) = \mathbb{R}^2_+$, $D^2_S F(u_0, y_0, \overline{u}, \overline{v})(0) \cap (-K) = \{(0, 0)\}$. It follows that our condition is fulfilled. It is clear that

$$\text{PrMin}_K D^2F(u_0, y_0, \overline{u}, \overline{v})(u) \subseteq D^2P(u_0, y_0, \overline{u}, \overline{v})(u).$$

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