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ON PROPERTIES OF DIFFERENTIAL INCLUSIONS WITH PROX-REGULAR SETS

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Abstract: In this paper, some regularity properties of solutions of the following differential inclusion

$$\begin{cases} \dot{x}(t) \in f\big(x(t)\big) - N_C\big(x(t)\big) \text{ a.e. } t \in [0, +\infty), \\ x(0) = x_0 \in C, \end{cases}$$

are analyzed where $f : H \to H$ is Lipschitz continuous and C is closed, uniformly prox-regular subset of a Hilbet space H. Here $N_C(\cdot)$ denotes the proximal normal cone of C. This work can be considered as an improvement of [9] since these properties are established without the additional tangential condition at each point in C.

Key words: *differential Inclusion, uniformly prox-regular set, normal cone*

Mathematics Subject Classification: *34A60, 49J52, 49J53*

1 Introduction

In the seventies, sweeping processes are introduced and deeply studied by J. J. Moreau through the series of papers [12–16] which plays an important role in elasto-plasticity, quasi-statics, dynamics, especially in mechanics [3, 17, 18]. Roughly speaking, a point is swept by a moving closed convex set $C(t)$ in a Hilbert space H and can be formulated in the form of differential inclusion as follows

$$\begin{cases} \dot{x}(t) \in -N_{C(t)}(x(t)) \text{ a.e. } t \in [0, T], \\ x(0) = x_0 \in C(0), \end{cases} \tag{1.1}$$

where $N_{C(t)}(\cdot)$ denotes the normal cone of $C(t)$ in the sense of convex analysis. When the systems are perturbed, it is natural to study the following variant

$$\begin{cases} \dot{x}(t) \in -N_{C(t)}(x(t)) + F(t, x(t)) \text{ a.e. } t \in [0, T], \\ x(0) = x_0 \in C(0), \end{cases} \tag{1.2}$$

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where $F : \mathbb{R}^+ \times H \to 2^H$ is a set-valued mapping with nonempty weakly compact convex values in H. For example, to study the planning procedures in mathematical economy, C. Henry [10] introduced and proved the existence of solutions in finite dimension of the system

$$\begin{cases} \dot{x}(t) \in P_{T_C\big(x(t)\big)}\big(F(x(t))\big) \text{ a.e. } t \in [0,T], \\ \\ x(0) = x_0 \in C, \end{cases} \tag{1.3}$$

where $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is upper semi-continuous with nonempty, convex, compact values and C is a closed, convex set in $\mathbb{R}^n$. Here T, P denote the tangent cone and projection operators, respectively. Later B. Cornet [6] extended the system (1.3) for the case $C \subset \mathbb{R}^n$ is Clarke tangentially regular and reduced to

$$\begin{cases} \dot{x}(t) \in F\big(x(t)\big) - N_C\big(x(t)\big) \text{ a.e. } t \in [0,T], \\ \\ x(0) = x_0 \in C. \end{cases} \tag{1.4}$$

There are numerous results for various variants of sweeping processes in literature but most of them are about the existence of solutions (see, e.g., [4, 7, 8, 22]). In this paper, we are interested in properties of solutions of the differential inclusion

$$\begin{cases} \dot{x}(t) \in f\big(x(t)\big) - N_C\big(x(t)\big) \text{ a.e. } t \in [0,+\infty), \\ \\ x(0) = x_0 \in C, \end{cases} \tag{1.5}$$

where $f : H \to H$ is Lipschitz continuous and C is closed, uniformly prox-regular subset of a Hilbet space H. It is known that (1.5) has a unique locally absolutely continuous solution $x(\cdot)$ on $[0, +\infty)$ (see [7] for example). However, it is also important to know more regularity properties of solutions, even the asymptotic behaviour, to understand better the systems. In [9], the authors considered this direction for the same problem. The main properties are the right differentiable of the solution and $\dot{x}^+(\cdot)$ is right continuous at each $t \geq 0$, which later play an important role in studying Lyapunov functions as well as asymptotic behaviour of solutions. However, these properties are obtained in [9] under the tangential condition: $f(x) \in T(C, x)$ for all $x \in C$. The condition is unnecessary since if C is closed, convex then $N_C(\cdot)$ is maximal monotone operator and thus we do not need such kind of condition [2]. It motivates us to establish the same properties but without the additional tangential condition.

The paper is organized as follow. In section 2, we recall some basic notations, definitions and results which are used throughout the paper. Some regularities properties of solutions are established without tangential condition in section 3. Some conclusions and perspectives end the paper in section 4.

2 Notations and Mathematical Background

Let us begin with some notations used in the paper. Let H be a Hilbert space. Denote by $\langle \cdot, \cdot \rangle$, $\| \cdot \|$ the scalar product and the corresponding norm in H. Denote by I the identity operator, by $\mathbb{B}$ the unit ball in H and $\mathbb{B}_r = r\mathbb{B}$, $\mathbb{B}_r(x) = x + r\mathbb{B}$. The distance from a point s to a closed set C is denoted by $d(s, C)$ or $d_C(s)$ and

$$d(s, C) = \inf_{x \in C} \|s - x\|.$$

Denote by C^0 the set of minimal norm elements of C, *i.e.*

$$C^0 = \{c \in C : \|c\| = \inf_{c' \in C} \|c'\|\}.$$

It is know that if C is closed and convex then C^0 contains exactly one element. The set of all points in C that are nearest to s is denoted by

$$\mathrm{Proj}(C, s) = \{x \in C : \|s - x\| = d(s, C)\}.$$

When $\mathrm{Proj}(C, s) = \{x\}$, we can write $x = \mathrm{proj}(C, s)$ to emphasize the single-valued property. Let $x \in \mathrm{Proj}(C, s)$ and $t \geq 0$, then the vector $t(s - x)$ is called *proximal normal* to C at x. The set of all such vectors is a cone, called *proximal normal cone* of C at x and denoted by $N^P(C, x)$. It is a known result [5, 20] that $\xi \in N^P(C, x)$ if and only if there exist some $\sigma > 0, \delta > 0$ such that

$$\langle \xi, y - x\rangle \leq \delta \|y - x\|^2 \text{ for all } y \in C \cap \mathbb{B}_\sigma(x).$$

The Fréchet normal cone $N^F(\cdot)$, the limiting normal cone $N^L(\cdot)$ and the Clarke normal cone $N^C(\cdot)$ are defined respectively as follows:

$$N^F(C, x) = \{\xi \in H : \forall \delta > 0, \exists \sigma > 0 \; s.\,t. \; \langle \xi, y - x\rangle \leq \delta \|y - x\| \text{ for all } y \in C \cap \mathbb{B}_\sigma(x)\}.$$

$$\begin{aligned} N^L(C, x) &= \{\xi \in H : \exists \, \xi_n \to \xi \text{ weakly and } \xi_n \in N^P(C, x_n), x_n \to x \text{ in } C\} \\ &= \{\xi \in H : \exists \, \xi_n \to \xi \text{ weakly and } \xi_n \in N^F(C, x_n), x_n \to x \text{ in } C\}. \end{aligned}$$

$$N^C(C, x) = \overline{\mathrm{co}} N^L(C, x).$$

If $x \notin C$, one has $N^P(C, x) = N^F(C, x) = N^L(C, x) = N^C(C, x) = \emptyset$ and for all $x \in C$:

$$N^P(C, x) \subset N^F(C, x) \subset N^L(C, x) \subset N^C(C, x).$$

If C is convex then these normal cones are coincide. It is in fact still true for prox-regular sets, which are defined as follows. Then we can write only $N(C, x)$ for simplicity.

Definition 2.1. The closed set C is called $r - prox - regular$ iff each point s in the r-enlargement of C

$$U_r(C) = \{w \in H : d(w, C) < r\},$$

has a unique nearest point $\mathrm{proj}(C, s)$ and the mapping $\mathrm{proj}(C, \cdot)$ is continuous in $U_r(C)$.

Proposition 2.2 ([19, 22])**.** *Let C be a closed set in H. The followings are equivalent:*

1) *C is $r - prox - regular$.*

2) *For all $x \in C$ and $\xi \in N^L(C, x)$ such that $\|\xi\| \leq r$, we have*

$$x = \mathrm{proj}(C, x + \xi). \tag{2.1}$$

3) *For all $x \in C$ and $\xi \in N^L(C, x)$, we have*

$$\langle \xi, y - x\rangle \leq \frac{\|\xi\|}{2r} \|y - x\|^2 \;\; \forall \, y \in C.$$

4) *(Hypo-monotonicity) For all* $x, x' \in C$, $\xi \in N^L(C, x)$, $\xi' \in N^L(C, x')$ *and* $\xi, \xi' \in \mathbb{B}_r$ *we have*

$$\langle \xi - \xi', x - x' \rangle \geq -\|x - x'\|^2.$$

If $r = +\infty$, then C is convex. Some examples of prox-regular sets [4]:

1. The finite union of disjoint intervals is non-convex but uniformly r-prox-regular and r depends on the distances between the intervals.
2. More generally, any finite union of disjoint convex subsets in H is non-convex but uniformly r-prox-regular and r depends on the distances between the sets.

We finish the section with a version of Gronwall's inequality (see, e.g., Lemma 4.1 in [21]).

Lemma 2.3. *Let* $T > 0$ *be given and* $a(\cdot), b(\cdot) \in L^1([t_0, t_0 + T]; \mathbb{R})$ *with* $b(t) \geq 0$ *for almost all* $t \in [t_0, t_0 + T]$. *Let the absolutely continuous function* $w : [t_0, t_0 + T] \to \mathbb{R}_+$ *satisfy:*

$$(1 - \alpha)w'(t) \leq a(t)w(t) + b(t)w^\alpha(t), \quad a.e. \ t \in [t_0, t_0 + T], \tag{2.2}$$

where $0 \leq \alpha < 1$. *Then for all* $t \in [t_0, t_0 + T]$*:*

$$w^{1-\alpha}(t) \leq w^{1-\alpha}(t_0)\exp\Big(\int_{t_0}^{t} a(\tau)d\tau\Big) + \int_{t_0}^{t} \exp\Big(\int_{s}^{t} a(\tau)d\tau\Big)b(s)ds. \tag{2.3}$$

3 Main Results

Let us first recall the existence and uniqueness result of (1.5) (see, e.g., [7]).

Theorem 3.1. *Let H be a Hilbert space and C be a closed, r-prox-regular set. Let $f : H \to H$ be a k-Lipschitz continuous function. Then for each $x_0 \in C$, the following differential inclusion*

$$\begin{cases} \dot{x}(t) \in f\big(x(t)\big) - N_C\big(x(t)\big) \text{ a.e. } t \in [0, +\infty), \\ \\ x(0) = x_0 \in C, \end{cases} \tag{3.1}$$

has a unique locally absolutely continuous solution $x(\cdot)$. *In addition, we have*

$$\|\dot{x}(t) - f\big(x(t)\big)\| \leq \|f\big(x(t)\big)\| \text{ for } a.e. \ t \geq 0. \tag{3.2}$$

Let $x(\cdot)$ be the unique solution of (1.5) satisfying $x(0) = x_0$. Define $v : \mathbb{R}_+ \to H$ by $v(t) := \Big(f\big(x(t)\big) - N\big(C, x(t)\big)\Big)^0$ and $v_0 := v(0) = \big(f(x_0) - N(C, x_0)\big)^0$. By using similar arguments as in Lemma 1.8 [11], we have the following lemma.

Lemma 3.2. *We have*

$$\|v_0\| \leq \liminf_{t \to 0^+} \|v(t)\|. \tag{3.3}$$

Proof. If $\liminf_{t \to 0^+} \|v(t)\| = +\infty$ then the conclusion holds. If $\liminf_{t \to 0^+} \|v(t)\| = \gamma < +\infty$, then there exists a sequence $(t_n)_{n \geq 1}$ such that $t_n \to 0^+$ and $\lim_{n \to +\infty} \|v(t_n)\| = \gamma$. In particular, the sequence $\big(v(t_n)\big)_{n \geq 1}$ is bounded hence there exist a subsequence $\big(v(t_{n_k})\big)_{k \geq 1}$ and $\xi \in H$ such that $\big(v(t_{n_k})\big)_{k \geq 1}$ converges weakly to ξ. Recall that

$$v(t_{n_k}) = \Big(f\big(x(t_{n_k})\big) - N\big(C; x(t_{n_k})\big)\Big)^0 \in f\big(x(t_{n_k})\big) - N\big(C; x(t_{n_k})\big).$$

Hence $f\big(x(t_{n_k})\big) - v(t_{n_k}) \in N\big(C; x(t_{n_k})\big)$. We can find some $\beta > 0$ such that $\|f\big(x(t_{n_k})\big) - v(t_{n_k})\| \leq \beta$ for all $k \geq 1$. Using the prox-regularity of C, one has

$$\langle f\big(x\big(t_{n_k}\big)\big) - v(t_{n_k}), c - x(t_{n_k})\rangle \leq \frac{\beta}{2r}\|c - x(t_{n_k})\|^2 \text{ for all } c \in C,\ k \geq 1. \tag{3.4}$$

Let $k \to +\infty$, we get

$$\langle f(x_0) - \xi, c - x_0\rangle \leq \frac{\beta}{2r}\|c - x_0\|^2 \text{ for all } c \in C. \tag{3.5}$$

Thus $f(x_0) - \xi \in N(C; x_0)$ or equivalently $\xi \in f(x_0) - N(C; x_0)$. Then

$$\|\xi\| \leq \liminf_{k\to+\infty} \|v(t_{n_k})\| = \liminf_{n\to+\infty} \|v(t_n)\| = \gamma, \tag{3.6}$$

due to the weak lower semicontinuity of the norm and the conclusion follows. □

Lemma 3.3. *Let $x(\cdot)$ be the unique solution of* (1.5) *satisfying $x(0) = x_0$. Then one has*

$$\limsup_{t\to 0^+} \|\frac{x(t) - x_0}{t}\| \leq \|v_0\|, \tag{3.7}$$

where $v_0 = \big(f(x_0) - N(C, x_0)\big)^0 = f(x_0) - \text{proj}\big(f(x_0), N_C(x_0)\big)$.

Proof. We have

$$\begin{cases} \dot{x}(t) - f\big(x(t)\big) \in -N_C\big(x(t)\big) \text{ a.e. } t \in [0, +\infty), \\ v_0 - f(x_0) \in -N_C(x_0), \end{cases} \tag{3.8}$$

and $\|\dot{x}(t) - f\big(x(t)\big)\| \leq \|f\big(x(t)\big)\|$ for *a.e.* $t \geq 0$. Using the prox-regularity of C and Proposition 2.2, one has

$$\langle \dot{x}(t) - f\big(x(t)\big) - v_0 + f(x_0), x(t) - x_0\rangle \leq \frac{1}{r}\big(\|f\big(x(t)\big)\| + \|v_0 - f(x_0)\|\big)\|x(t) - x_0\|^2. \tag{3.9}$$

Combining with the k-Lipschitz continuity of $f(\cdot)$, one deduces that

$$\frac{1}{2}\frac{d}{dt}\|x(t) - x_0\|^2 \leq \|v_0\|\|x(t) - x_0\| + a(t)\|x(t) - x_0\|^2, \tag{3.10}$$

where $a(t) = k + \frac{1}{r}\big(\|f\big(x(t)\big)\| + \|v_0 - f(x_0)\|\big)$. Using Gronwall's inequality (Lemma 2.3), one obtains for all $t \geq 0$ that

$$\|x(t) - x_0\| \leq \|v_0\| \int_0^t \exp\Big(\int_s^t a(\tau)d\tau\Big)ds. \tag{3.11}$$

Hence

$$\limsup_{t\to 0^+} \|\frac{x(t) - x_0}{t}\| \leq \|v_0\| \limsup_{t\to 0^+} \frac{1}{t}\int_0^t \exp\Big(\int_s^t a(\tau)d\tau\Big)ds = \|v_0\|. \tag{3.12}$$

□

Lemma 3.4. *Let $x(\cdot), y(\cdot)$ be the unique solution of* (1.5) *satisfying initial conditions* $x(0) = x_0, y(0) = y_0$ *respectively. Then for all* $t \geq 0$:

$$\|x(t) - y(t)\| \leq \|x(0) - y(0)\| \exp\Big(\int_0^t b(s)ds \Big) \quad t \geq 0, \tag{3.13}$$

where $b(t) = k + \frac{1}{r}(\|f(x(t))\| + \|f(y(t))\|)$. *In particular, for a.e.* $t \geq 0$, *one has*

$$\|\dot{x}(t)\| \leq \|v_0\| \exp\Big(\int_0^t (k + \frac{2\|f(x(s))\|}{r})ds \Big), \tag{3.14}$$

where v_0 *is defined in Lemma 3.3.*

Proof. Using the prox-regularity of C and Lipschitz continuity of $f(\cdot)$ similarly as above, we have

$$\frac{1}{2}\frac{d}{dt}\|x(t) - y(t)\|^2 \leq b(t)\|x(t) - y(t)\|^2 \quad a.e.\ t \geq 0, \tag{3.15}$$

where $b(t) = k + \frac{1}{r}\big(\|f\big(x(t)\big)\| + \|f\big(y(t)\big)\|\big)$. Then the Gronwall's inequality (Lemma 2.3) implies (3.13). Given some $h > 0$, and we take $y(0) = x(h)$ then $y(t) = x(t+h)$ for all $t \geq 0$. From (3.13), we deduce that

$$\Big\|\frac{x(t+h) - x(t)}{h}\Big\| \leq \Big\|\frac{x(h) - x(0)}{h}\Big\| \exp\Big(\int_0^t (k + \frac{\|f(x(s))\| + \|f(x(s+h))\|}{r})ds \Big) \text{ for all } t \geq 0. \tag{3.16}$$

Fixed some $t_0 \geq 0$ such that $\dot{x}(t_0)$ exists. Taking the limsup of both sides of (3.16) as $h \to 0^+$ and using Lemma 3.3, one gets

$$\|\dot{x}(t_0)\| \leq \|v_0\| \exp\Big(\int_0^{t_0} (k + \frac{2\|f(x(s))\|}{r})ds \Big).$$

Thus (3.14) follows. □

Now, we are ready for the main result which states that the solution is right differentiable and $\dot{x}^+(\cdot)$ is right continuous at each $t \geq 0$. We also recall an important property (Theorem 3.5-i) acquired in Proposition 2.6 [9] by using a different approach.

Theorem 3.5. *Let $x(\cdot)$ be the unique solution of the system satisfying $x(0) = x_0$. Then we have:*

(*i*) $\dot{x}(t) = v(t) = \Big(f\big(x(t)\big) - N\big(C, x(t)\big)\Big)^0$ *for almost every* $t \in [0, +\infty)$.

(*ii*) *For all* $t^* \in [0, +\infty)$, *the right derivative* $\dot{x}^+(t^*)$ *exists and*

$$\dot{x}^+(t^*) = \Big(f\big(x(t^*)\big) - N_C\big(x(t^*)\big)\Big)^0.$$

Furthermore $\dot{x}^+(\cdot)$ *is continuous on the right.*

Proof. Let $E = \{t \in [0, +\infty) : \dot{x}(t) \text{ exists}\}$. It is clear that the Lebesgue measure of $[0, +\infty) \setminus E$ is zero.
(*i*) Fixed $t_0 \in E$. Let $y(\cdot)$ be the unique solution of the system with initial condition $y(0) = x(t_0)$. Then $y(t) = x(t + t_0)$ for all $t \geq 0$. Applying Lemma 3.3, we get

$$\limsup_{t \to 0^+} \Big\|\frac{y(t) - y(0)}{t}\Big\| \leq \Big\|\Big(f\big(y(0)\big) - N\big(C, y(0)\big)\Big)^0\Big\|, \tag{3.17}$$

or equivalently

$$\limsup_{t\to 0^+}\Big\|\frac{x(t+t_0)-x(t_0)}{t}\Big\| \le \Big\|\Big(f\big(x(t_0)\big)-N\big(C,x(t_0)\big)\Big)^0\Big\|. \tag{3.18}$$

Hence

$$\|\dot{x}(t_0)\| \le \|\Big(f\big(x(t_0)\big)-N(C,x(t_0))\Big)^0\|. \tag{3.19}$$

On the other hand $\dot{x}(t_0)\in f\big(x(t_0)\big)-N\big(C,x(t_0)\big)$, thus $\dot{x}(t_0)=\Big(f\big(x(t_0)\big)-N\big(C,x(t_0)\big)\Big)^0$.
(ii) Due to the property of semi-group, it is sufficient to prove for $t^*=0$. Using (i) and (3.14) of Lemma 3.4, for all $t\in E$, we have

$$\|v(t)\| \le \|v_0\|\exp\Big(\int_0^t \big(k+\frac{2\|f(x(s))\|}{r}\big)ds\Big), \tag{3.20}$$

where $v(t)=\Big(f\big(x(t)\big)-N\big(C,x(t)\big)\Big)^0$. It implies that

$$\limsup_{t\to 0^+,t\in E}\|v(t)\| \le \|v_0\|. \tag{3.21}$$

On the other hand, Lemma 3.2 deduces that

$$\|v_0\| \le \liminf_{t\to 0^+}\|v(t)\| \le \liminf_{t\to 0^+,t\in E}\|v(t)\|. \tag{3.22}$$

From (3.21) and (3.22), we obtain

$$\lim_{t\to 0^+,t\in E}\|v(t)\| = \|v_0\|. \tag{3.23}$$

Thus for any sequence $(t_n)_{n\ge 1}\subset E$ and $t_n\to 0$, we have

$$\|v(t_n)\|\to\|v_0\| \text{ as } n\to+\infty. \tag{3.24}$$

Then $\big(v(t_n)\big)_{n\ge 1}$ is bounded and therefore there exists some $v^*\in H$ such that a subsequence $(v(t_{n_k}))_{k\ge 1}$ converges weakly to v^* when $k\to+\infty$. Similarly as in Lemma 3.2, we can prove that $v^*\in f(x_0)-N(C;x_0)$. On the other hand

$$\|v^*\| \le \liminf_{k\to+\infty}\|v(t_{n_k})\| = \lim_{k\to+\infty}\|v(t_n)\| = \|v_0\|, \tag{3.25}$$

due to (3.24). Thus, we must have $v^*=v_0$ and the set of weak cluster point of $\big(v(t_n)\big)_{n\ge 1}$ contains only v_0. It implies that $v(t_n)$ converges weakly to v_0. Combining with (3.24), one deduces that $v(t_n)$ converges strongly to v_0. In conclusion

$$\lim_{t\to 0^+,t\in E} v(t) = v_0. \tag{3.26}$$

Due to the absolute continuity of $x(\cdot)$ and (i), for all $h>0$, we have

$$x(h)-x_0 = \int_0^h \dot{x}(s)ds = \int_0^h v(s)ds, \tag{3.27}$$

where $v(\cdot)$ is locally integrable and satisfying (3.26). Now we prove that

$$\lim_{h\to 0^+}\frac{1}{h}\int_0^h v(s)ds = v_0. \tag{3.28}$$

Indeed, given $\epsilon > 0$. From (3.26), there exists $\delta > 0$ such that for all $s \in E, s \le \delta$ then $\|v(s) - v_0\| \le \epsilon$. Hence for all $h \le \delta$:

$$\Big\|\frac{1}{h}\int_0^h v(s)ds - v_0\Big\| \le \frac{1}{h}\int_0^h \|v(s)-v_0\|ds = \frac{1}{h}\int_{[0,h]\cap E}\|v(s)-v_0\|ds \le \frac{\epsilon}{h}\int_{[0,h]\cap E} ds = \epsilon.$$

So we have (3.28) and thus from (3.27), the right derivative $\dot{x}^+(0)$ exists and

$$\dot{x}^+(0) = v_0 = \big(f(x_0) - N(C, x_0)\big)^0. \tag{3.29}$$

It implies for all $t \ge 0$ that

$$\dot{x}^+(t) = v(t) = \Big(f\big(x(t)\big) - N\big(C, x(t)\big)\Big)^0. \tag{3.30}$$

Then taking the limit both sides of (3.16), we deduce for all $t \ge 0$ that

$$\|\dot{x}^+(t)\| \le \|\dot{x}^+(0)\|\exp\Big(\int_0^t \big(k + \frac{2\|f(x(s))\|}{r}\big)ds\Big),$$

or equivalently

$$\|v(t)\| \le \|v_0\|\exp\Big(\int_0^t \big(k + \frac{2\|f(x(s))\|}{r}\big)ds\Big).$$

Therefore

$$\limsup_{t\to 0^+}\|v(t)\| \le \|v_0\|.$$

Combining with (3.22), we obtain $\lim_{t\to 0^+}\|v(t)\| = \|v_0\|$. Similar as (3.26), we can prove that $\lim_{t\to 0^+} v(t) = v_0$. It means that $\dot{x}^+(\cdot)$ is right continuous at 0 and due to the property of semi-group, it is right continuous at any $t \ge 0$. $\square$

Now we consider the case $f(\cdot) = -\nabla V(\cdot)$ where V is $C^{1,+}$ function (i.e., V is differentiable and ∇V is Lipschitz continuous) and study some asymptotic properties of the solutions. The system then can be considered as an extension of "gradient equation" [1].

Proposition 3.6. *Let $V : H \to \mathbb{R}$ be a $C^{1,+}$ function. Let $x(\cdot)$ be the solution of the system*

$$\begin{cases} \dot{x}(t) \in -\nabla V\big(x(t)\big) - N\big(C, x(t)\big) \text{ a.e. } t \in [0, +\infty), \\ x(0) = x_0 \in C. \end{cases} \tag{3.31}$$

Then we have

$$\frac{d}{dt}V\big(x(t)\big) + \|\dot{x}(t)\|^2 = 0, \text{ for } a.e.\ t \ge 0. \tag{3.32}$$

In particular, V is a Lyapunov function of the system. Furthermore *(i) if V is coercive, i.e.,*

$$V(x) \to +\infty \text{ as } \|x\| \to +\infty,$$

then $x(\cdot)$ is bounded on $\mathbb{R}_+$.

(ii) if V is bounded from below on C then $\lim_{t\to+\infty} V\big(x(t)\big) = V_\infty$ exists and $\dot{x} \in L^2([0,+\infty); H)$ with $\int_0^{+\infty} \|\dot{x}(s)\|^2 ds = V(x_0) - V_\infty$.

(iii) If V is convex and bounded from below on C then $V_\infty = \inf_{y\in C} V(y)$.

Proof. Fixed some $t \geq 0$ such that (i) of Theorem 3.5 holds, i.e., $\dot{x}(t) = \big(-\nabla V\big(x(t)\big) - N_C\big(x(t)\big)\big)^0 = -\nabla V\big(x(t)\big) - \text{proj}\big(N\big(C, x(t)\big); -\nabla V\big(x(t)\big)\big)$. Then

$$\begin{aligned} & \big\langle \dot{x}(t) + \nabla V\big(x(t)\big), \dot{x}(t) \big\rangle \\ = & \Big\langle -\text{proj}\Big(N\big(C, x(t)\big); -\nabla V\big(x(t)\big)\Big), \nabla V(x(t)) - \text{proj}\Big(N\big(C, x(t)\big); -\nabla V\big(x(t)\big)\Big)\Big\rangle \\ = & \ 0. \end{aligned}$$

Note that $\frac{d}{dt}V\big(x(t)\big) = \big\langle \nabla V\big(x(t)\big), \dot{x}(t)\big\rangle$ and (3.32) follows. In particular, we have $\frac{d}{dt}V\big(x(t)\big) \leq 0$ for a.e. $t \geq 0$. It means that V is a Lyapunov function of the system. Then (i) and (ii) follow classically.
(iii) Fix some $y \in C$ and consider the function $\varphi(t) = \frac{1}{2}\|x(t) - y\|^2$. Due to the r-prox-regularity of C and the fact that $\dot{x}(t) + \nabla V\big(x(t)\big) \in -N\big(C, x(t)\big)$ a.e. $t \in [0, +\infty)$, one has

$$\langle \dot{x}(t) + \nabla V\big(x(t)\big), x(t) - y\rangle \leq \frac{\|\nabla V\big(x(t)\big)\|}{r}\|x(t) - y\|^2.$$

Thus

$$\begin{aligned} \dot{\varphi}(t) = \langle \dot{x}(t), x(t) - y\rangle &\leq \frac{2\|\nabla V\big(x(t)\big)\|}{r}\varphi(t) + \langle \nabla V\big(x(t)\big), y - x(t)\rangle \\ &\leq \frac{2\|\nabla V\big(x(t)\big)\|}{r}\varphi(t) + V(y) - V\big(x(t)\big), \end{aligned}$$

due to the convexity of V. Using Gronwall's inequality (Lemma 2.3), for all $t \geq 0$ one obtains

$$\begin{aligned} 0 \leq \varphi(t) &\leq \varphi(0)\exp\Big(\int_0^t \frac{2\|\nabla V(x(\tau))\|}{r}d\tau\Big) \\ &\quad + \int_0^t \exp\Big(\int_s^t \frac{2\|\nabla V\big(x(\tau)\big)\|}{r}d\tau\Big)[V(y) - V\big(x(s)\big)]ds \\ &\leq \exp\Big(\int_0^t \frac{2\|\nabla V(x(\tau))\|}{r}d\tau\Big)\big[\varphi(0) + t\Big(V(y) - V\big(x(t)\big)\Big)\big], \end{aligned}$$

since $V\big(x(s)\big) \geq V\big(x(t)\big)$ for all $s \in [0, t]$. It implies that

$$V\big(x(t)\big) \leq V(y) + \frac{\varphi(0)}{t}.$$

Let $t \to +\infty$, one gets $V_\infty \leq V(y)$. Since y is arbitrary in C, it deduces that $V_\infty \leq \inf_{y\in C} V(y)$. On the other hand $V\big(x(t)\big) \geq \inf_{y\in C} V(y)$ since $x(t) \in C$ for all $t \geq 0$. Hence $V_\infty \geq \inf_{y\in C} V(y)$. Therefore $V_\infty = \inf_{y\in C} V(y)$, it means the trajectory is minimizing for V on C. □

4 Conclusion

In this paper, we have established some important regularity properties for a class of differential inclusions involving normal cone operator of prox-regular sets without tangential assumption. Some asymptotic behaviours of the solutions are also studied. It is interesting to consider properties of solutions of sweeping process with prox-regular sets, where C can depend on time and even the state. It is out of scope of the current work and will be considered in the future.

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