Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 13, No. 1, pp. 29–42, January 2017

ON UNCERTAIN CONICAL CONVEX OPTIMIZATION PROBLEM

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Abstract: In this paper, we give a necessary and sufficient condition for the equality between the worst value of an uncertain conical convex optimization problem and the value of its robust counterpart. We derive a sufficient condition for robust strong duality to hold.

Key words: *uncertain conical convex optimization problems, Fenchel conjugate, worst value, robust value, epigraphical duality, robust strong duality property*

Mathematics Subject Classification: *90C25, 90C46*

1 Introduction

In this work we deal with the uncertain conical convex optimization problem:

$$(P) \qquad \inf_x f(x) \quad s.t. \quad g_u(x) \in -S,$$

where u belongs to the set of uncertain parameters U, X and Y are two locally convex Hausdorff topological vector spaces, $f : X \to \mathbb{R} \cup \{+\infty\}$ a proper convex lower semi-continuous function, $S \subset Y$ a nonempty closed convex cone, and for each $u \in U$, the mapping $g_u : \mathrm{dom}\, g_u \subset X \to Y$ is S-level-closed convex or S-epi-closed convex.
To the uncertain problem (P) is associated its robust counterpart ([3],[4],[5]) which is the problem:

$$(RP) \qquad \inf_x f(x) \quad s.t. \quad g_u(x) \in -S\ ,\ \forall u \in U.$$

We call robust value, the value $\inf(RP)$ of the problem (RP).
Given $u \in U$, (P_u) is the corresponding instance of (P), namely:

$$(P_u) \qquad \inf_x f(x) \quad s.t. \quad g_u(x) \in -S.$$

Let us consider the problem of maximizing over U, the value of each problem (P_u):

$$(Q) \qquad \sup_u \inf_x \{f(x) \ : \ g_u(x) \in -S\} \quad s.t. \ u \in U,$$

and call worst value of the uncertain conical convex problem (P) the value of (Q) that is $\sup(Q)$. We note that $\sup(Q) \leq \inf(RP)$ and give an example in which this inequality is

*M. Barro has been supported by French embassy in Burkina, Scholarship SCAC-Ouagadougou.

strict (Proposition 3.1, Example 3.2). The purpose of this paper is to establish a necessary and sufficient condition for the equality between the worst value and the robust value of the uncertain problem (P), with attainment of the worst value. The outline of the paper is as follows: Section 2 presents elements of convex analysis needed in the sequel. Section 3 formulates a necessary and sufficient condition to obtain the equality between the worst value and the robust value with attainment for the worst value (Theorem 3.4, Corollary 3.5, Corollary 3.6, Corollary 3.7). Section 4 is concerned with the so-called optimistic dual problem (ODP) of (P). We note that $\sup(ODP) \leq \sup(Q)$ (Proposition 4.1) and give condition to obtain $\sup(ODP) = \sup(Q)$ (Proposition 4.6). In the case that robust strong duality holds, we get $\inf(RP) = \max(Q)$ (Proposition 4.3). Conversely, we establish robust strong duality (Theorem 4.8) from our previous results using a convex composite duality principle. Finally, we compare our Corollary 4.9 to [16] Corollary 3.1 (Remark 4.10, Proposition 4.11).

2 Preliminaries

Let X be a locally convex Hausdorff topological vector space with topological dual X^* and $\langle , \rangle$ the standard bilinear coupling function between X and X^*. Given a function $h : X \to \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, we note by $\operatorname{dom} h := \{x \in X \mid h(x) < +\infty\}$ the effective domain of h. One says that h is proper if $\operatorname{dom} h$ is non-empty and $-\infty \notin h(X)$. The epigraph of function h is the set $\operatorname{epi} h := \{(x,t) \in X \times \mathbb{R} \mid h(x) \leq t\}$. Recall that h is convex if and only if $\operatorname{epi} h$ is convex, h is lower semi-continuous if and only if $\operatorname{epi} h$ is closed. The set of all proper convex lower semi-continuous extended real-valued functions defined on X is denoted by $\Gamma(X)$.

The Legendre-Fenchel conjugate of $h : X \to \bar{\mathbb{R}}$ is the function

$$h^* : X^* \to \bar{\mathbb{R}} \quad , \quad h^*(x^*) := \sup_{x \in X} \{\langle x, x^* \rangle - h(x)\},$$

which is convex and weak* lower semi-continuous. The Legendre-Fenchel biconjugate of h is defined on X by

$$h^{**}(x) := \sup_{x^* \in X^*} \{\langle x, x^* \rangle - h^*(x^*)\}.$$

It holds that $h \geq h^{**}$, h^{**} being convex and lower semi-continuous.

Property 2.1. For each family of extended real-valued functions $(h_i)_{i \in I} \subset \bar{\mathbb{R}}^X$, one has:

$$\left(\inf_{i \in I} h_i\right)^* = \sup_{i \in I} h_i^*.$$

Given a subset A of X, we note by i_A the indicator function of A defined on X by $i_A(x) = 0$ if $x \in A$ and $i_A(x) = +\infty$ otherwise, $\sigma_A := i_A^*$ the support function of A , $\operatorname{co}(A)$ its convex hull, $\bar{A}$ its closure, $\overline{\operatorname{co}}(A)$ its closed convex hull. On the dual space X^* we only consider the weak* topology, and for any subset B of X^* we simply denote by $\bar{B}$ the weak* closure of B. Given A, B two subsets of X, we say that A is closed regarding B if $\bar{A} \cap B = A \cap B$ ([7]). Analogously, A is said to be closed convex regarding B if $\overline{\operatorname{co}}(A) \cap B = A \cap B$ ([12]).
Given $E \subset \bar{\mathbb{R}}$, we write $\min E$ (respectively $\max E$) instead of $\inf E$ (respectively $\sup E$) when the infimum (respectively supremum) of E is attained.

We recall below two classical properties.

Lemma 2.1 ([6] Theorem 2.1)**.** *For all* $h_1,\ h_2 \in \Gamma(X)$ *such that* $\mathrm{dom}\, h_1 \bigcap \mathrm{dom}\, h_2 \neq \emptyset$, *one has:*

$$\mathrm{epi}\,(h_1 + h_2)^* = \overline{\Big(\mathrm{epi}\, h_1^* + \mathrm{epi}\, h_2^*\Big)}. \tag{2.1}$$

Lemma 2.2 ([7])**.** *Let* $(h_i)_{i\in I} \subset \Gamma(X)$, *where* I *is an arbitrary nonempty index set. Assume that there exists* $\bar{x} \in X$ *such that* $\sup_{i\in I} h_i(\bar{x}) < +\infty$. *Then,*

$$\mathrm{epi}\,\Big(\sup_{i\in I} h_i\Big)^* = \overline{\mathrm{co}}\big(\bigcup_{i\in I} \mathrm{epi}\, h_i^*\big). \tag{2.2}$$

The closure in (2.1) and (2.2) is taken with respect to the product of the weak* topology on X^* and the natural topology on $\mathbb{R}$. Moreover, if h_1 is finite and continuous at some point of dom h_2 then, by Moreau-Rockafellar Theorem ([19] Theorem 3), the closure is unnecessary in (2.1).

Let Y be another locally convex Hausdorff topological vector space and $S \subset Y$ a nonempty closed convex cone. The S-epigraph of a mapping $g : \mathrm{dom}\, g \subset X \to Y$, is the set

$$\mathrm{epi}_S g := \{(x, y) \in \mathrm{dom}\, g \times Y \ : \ y - g(x) \in S\},$$

and the S-level set of g at level $y \in Y$ is defined as

$$\{x \in \mathrm{dom}\, g \ : \ g(x) \in y - S\}.$$

There are several notions about lower semi-continuity of mappings like g ([1],[10],[15], [17], [18]). Here, we only use the ones depending on the S-epigraph or the S-level sets of g. We shall say that g is S-epi-closed convex if $\mathrm{epi}_S g$ is closed and convex, and that g is S-level-closed convex if its S-level set at level y is closed and convex for each $y \in Y$. Of course any S-epi-closed convex function is also S-level-closed convex.

We denote by

$$S^+ := \{\lambda \in Y^* \ : \ \langle y, \lambda\rangle \geq 0,\ \forall y \in S\},$$

the positive polar cone of S.
Given $\lambda \in S^+$, we denote by λg the function defined on X by

$$\lambda g(x) = \begin{cases} \langle g(x), \lambda\rangle & \text{if } x \in \mathrm{dom}\, g \\ \\ +\infty & \text{otherwise.} \end{cases}$$

If g is defined on the whole space X then λg is the standard composition of λ by g usually denoted by $\lambda \circ g$.
The level set of g at level $y = 0_Y$ is

$$g^{-1}(-S) := \{x \in \mathrm{dom}\, g \ : \ g(x) \in -S\}.$$

Let us consider the set

$$K_g := \bigcup_{\lambda\in S^+} \mathrm{epi}\,(\lambda g)^*,$$

which can be viewed as the characteristic cone associated to the system of inequalities ([13]):

$$\Big\{x \in \mathrm{dom}\, g \ : \ \lambda g(x) \leq 0\ ,\ \forall \lambda \in S^+\Big\}.$$

Let us quote that, without any convexity assumption on g, K_g is always a convex cone. More precisely, one has:

Property 2.2. For each mapping $g : \text{dom}\, g \subset X \to Y$ such that $g^{-1}(-S) \neq \emptyset$, it holds:

i) $i_{g^{-1}(-S)} = \sup_{\lambda \in S^+} (\lambda g)$;

ii) K_g is a convex cone.

Proof. i). Let $x \in X$. If $x \in g^{-1}(-S)$ then $\langle g(x), \lambda \rangle \leq 0$ for each $\lambda \in S^+$, thus

$$\sup_{\lambda \in S^+} (\lambda g)(x) = 0.$$

If $x \notin g^{-1}(-S)$ then $g(x) \notin -S$ which is closed convex. By Hahn-Banach separation Theorem, there exists $(y^*, r) \in Y^* \times \mathbb{R}$ such that $\langle y, y^* \rangle < r < \langle g(x), y^* \rangle$ for all $y \in -S$. Since $0_Y \in S$, it then follows that $r > 0$ and thus $y^* \in S^+$. Therefore $\sup_{\lambda \in S^+} (\lambda g)(x) \geq \langle g(x), ny^* \rangle > 0$ for all $n \geq 1$. Letting $n \to +\infty$, one gets $\sup_{\lambda \in S^+} (\lambda g)(x) = +\infty$, and i) holds.
ii). We first prove that K_g is a cone. Let $(x^*, r) \in K_g$ and $t > 0$. There exists $\lambda \in S^+$ such that $(\lambda g)^*(x^*) \leq r$, and we have

$$(t\lambda g)^*(tx^*) = \sup_{x \in X} \{\langle x, tx^* \rangle - t\lambda g(x)\} = t \sup_{x \in X} \{\langle x, x^* \rangle - \lambda g(x)\} = t(\lambda g)^*(x^*) \leq tr.$$

Therefore, $t(x^*, r) \in \text{epi}\,(t\lambda g)^* \subset K_g$.
We now prove that K_g is convex. Let $(x_i^*, r_i) \in K_g$ for $i = 1, 2$. One has to check that $(x_1^* + x_2^*\ ,\ r_1 + r_2) \in K_g$. There exists $\lambda_i \in S^+$ such that $(x_i^*, r_i) \in \text{epi}\,(\lambda_i g)^*$. For all $x \in \text{dom}\, g$, we have

$$\begin{aligned} \langle x, x_1^* + x_2^* \rangle - \langle g(x), \lambda_1 + \lambda_2 \rangle =& \langle x, x_1^* \rangle - \langle g(x), \lambda_1 \rangle + \langle x, x_2^* \rangle - \langle g(x), \lambda_2 \rangle \\ \leq& (\lambda_1 g)^*(x_1^*) + (\lambda_2 g)^*(x_2^*) \\ \leq& \ r_1 + r_2. \end{aligned}$$

Taking the supremum over $x \in \text{dom}\, g$ we get:

$$(x_1^* + x_2^*\ ,\ r_1 + r_2) \in \text{epi}\,((\lambda_1 + \lambda_2)g)^* \subset K_g.$$

□

In the presence of convexity, we have:

Proposition 2.3. *For any S-epi-closed convex mapping $g : \text{dom}\, g \subset X \to Y$, such that $g^{-1}(-S) \neq \emptyset$, we have:*

$$\text{epi}\, \sigma_{g^{-1}(-S)} = \overline{K_g}. \tag{2.3}$$

Proof. Let us defined the function $H : X \times Y \to \mathbb{R} \cup \{+\infty\}$ by:

$$H(x, y) = i_{\text{epi}_S g}(x, -y).$$

It is easy to check that

$$H^*(x^*, \lambda) = \begin{cases} (\lambda g)^*(x^*) & \text{if } \lambda \in S^+ \\ \\ +\infty & \text{else.} \end{cases}$$

Since g is S-epi-closed convex and $g^{-1}(-S) \neq \emptyset$, we have $H \in \Gamma(X \times Y)$. In particular for each $x \in X$, one has:

$$\begin{aligned} H(x, 0_Y) =& H^{**}(x, 0_Y) \\ &= \sup_{\substack{x^* \in X^* \\ \lambda \in Y^*}} \{\langle x, x^* \rangle - \langle 0_Y, \lambda \rangle - H^*(x^*, \lambda)\} \\ &= \sup_{\substack{x^* \in X^* \\ \lambda \in S^+}} \{\langle x, x^* \rangle - (\lambda g)^*(x^*)\} \\ &= \sup_{\lambda \in S^+} \left\{ \sup_{x^* \in X^*} \{\langle x, x^* \rangle - (\lambda g)^*(x^*)\} \right\} \\ &= \sup_{\lambda \in S^+} (\lambda g)^{**}(x). \end{aligned}$$

On the other hand, for each $x \in X$, one has:

$$H(x, 0_Y) = i_{\text{epi}_S g}(x, 0_Y) = i_{g^{-1}(-S)}(x).$$

Therefore $i_{g^{-1}(-S)} = \sup_{\lambda \in S^+} (\lambda g)^{**}$ and since $g^{-1}(-S) \neq \emptyset$, Lemma 2.2 says that

$$\text{epi } \sigma_{g^{-1}(-S)} = \overline{\text{co}} \left(\bigcup_{\lambda \in S^+} \text{epi}\,(\lambda g)^{***} \right) = \overline{\text{co}} \left(\bigcup_{\lambda \in S^+} \text{epi}\,(\lambda g)^{*} \right) = \overline{K_g},$$

where the last equality is due to Property 2.2.ii) and the definition of K_g. □

Remark 2.4. Despite the fact that

$$i_{g^{-1}(-S)} = \sup_{\lambda \in S^+} (\lambda g),$$

one cannot directly apply Lemma 2.2 to function $i_{g^{-1}(-S)}$ to reach (2.3). The reason is that λg is not necessary lower semi-continuous for each $\lambda \in S^+$ even if g is S-epi-closed convex. In fact, if g is a S-epi-closed convex function whose domain is not closed, then for $\lambda = 0_{Y^*}$, we have $\lambda g = i_{\text{dom } g}$, which is not lower semi-continuous.

3 Worst Value Versus Robust Value

In this section, we give a necessary and sufficient condition for the equality between the worst value and the robust value of the uncertain problem (P) with attainment of the worst value:

$$\inf(RP) = \max_{u \in U} \inf(P_u). \tag{3.1}$$

For each $u \in U$, let F_u be the feasible set of (P_u) that is

$$F_u = \{x \in \text{dom } g_u \ : \ g_u(x) \in -S\}.$$

We denote by F the feasible set of the problem (RP) :

$$F := \{x \in X \ : \ x \in \text{dom } g_u \ , \ \ g_u(x) \in -S \ , \ \forall u \in U\} = \bigcap_{u \in U} F_u,$$

and define the function $p : X \to \mathbb{R} \cup \{+\infty\}$ by

$$p = \sup_{u \in U}(f + i_{F_u}) = f + \sup_{u \in U} i_{F_u} = f + i_F.$$

We have $\operatorname{dom} p = F \cap \operatorname{dom} f$, and $\inf_X p = \inf(RP)$.

Proposition 3.1. *It holds that*

$$\sup(Q) \leq \inf(RP). \tag{3.2}$$

Proof.

$$\sup(Q) = \sup_{u \in U} \inf(P_u) = \sup_{u \in U} \inf_{x \in X}(f + i_{F_u})(x) \leq \inf_{x \in X} \sup_{u \in U}(f + i_{F_u})(x) = \inf(RP).$$

□

We present an example to show that the inequality (3.2) may be strict even if f is linear, the set of uncertain parameters U is finite and $\inf(RP)$ is reached.

Example 3.2. Consider the following uncertain conical convex programming problem:

$$(P) \qquad \min\ x_1 + x_2 \quad s.t. \quad \frac{1}{2}[(2 - u_1)x_1^2 + (1 + u_2)x_2^2] \leq 1,$$

where $(u_1, u_2) = (1, 2)$ or $(u_1, u_2) = (\frac{1}{2}, 1)$. In this example, one has:

$$X = \mathbb{R}^2, \quad f(x_1, x_2) = x_1 + x_2, \quad Y = \mathbb{R}, \quad S = \mathbb{R}_+, \quad U = \left\{(1, 2), \left(\frac{1}{2}, 1\right)\right\},$$

$$g_u(x_1, x_2) = \frac{1}{2}\left[(2 - u_1)x_1^2 + (1 + u_2)x_2^2\right] - 1.$$

For each $u \in U$, the Slater qualification condition holds for (P_u). By solving the corresponding Karush-Kuhn-Tucker system, one gets

$$\min(P_u) = -\sqrt{2\left(\frac{1}{2 - u_1} + \frac{1}{1 + u_2}\right)} = \begin{cases} -\sqrt{\frac{8}{3}} & \text{if } u = (1, 2) \\ -\sqrt{\frac{7}{3}} & \text{if } u = (\frac{1}{2}, 1). \end{cases}$$

Therefore

$$\max(Q) = -\sqrt{\frac{7}{3}}.$$

On the other hand, the robust counterpart of (P) is given by:

$$(RP) \qquad \min\ x_1 + x_2 \quad s.t. \quad \begin{cases} x_1^2 + 3x_2^2 \leq 2 \\ 3x_1^2 + 4x_2^2 \leq 4. \end{cases}$$

Slater qualification condition holds for (RP). By solving the corresponding Karush-Kuhn-Tucker system, one gets

$$\min(RP) = -\frac{2 + \sqrt{2}}{\sqrt{5}} > -\sqrt{\frac{7}{3}} = \max(Q).$$

Let us consider the opposite of the problem (Q) namely:

$$(-Q) \qquad \inf_u \ \sup\{-f(x) \ : \ g_u(x) \in -S\} \quad s.t. \quad u \in U.$$

The perturbation of the objective function of $(-Q)$ by adding a linear continuous form leads to define the value function $q : X^* \to \bar{\mathbb{R}}$ by:

$$q(x^*) := \inf_{u\in U} \sup_{x\in F_u} \{\langle x, x^*\rangle - f(x)\} = \inf_{u\in U} (f + i_{F_u})^* (x^*).$$

From Property 2.1 we have

$$q^* = \sup_{u\in U} (f + i_{F_u})^{**} \leq \sup_{u\in U} (f + i_{F_u}) = p.$$

Therefore

$$p^* \ \leq \ q^{**} \leq q. \tag{3.3}$$

Let us introduce the condition

$$(\mathscr{H}) \qquad \begin{cases} f \in \Gamma(X) \\ F \cap \operatorname{dom} f \neq \emptyset \text{ (that is } \inf(RP) < +\infty) \\ g_u \quad \text{is S-level-closed convex, } \forall u \in U. \end{cases}$$

Lemma 3.3. *Assume $(\mathscr{H})$ holds. Then*

$$\operatorname{epi} p^* = \overline{\operatorname{co}} \left(\bigcup_{u\in U} \operatorname{epi} (f + i_{F_u})^* \right). \tag{3.4}$$

Proof. If $(\mathscr{H})$ holds then $f + i_{F_u} \in \Gamma(X)$ for each $u \in U$, and one has:

$$q^* = \sup_{u\in U} (f + i_{F_u})^{**} = \sup_{u\in U} (f + i_{F_u}) = p.$$

Therefore, $q^{**} = p^*$ and $\operatorname{dom} q^* = F \cap \operatorname{dom} f \neq \emptyset$. Applying Lemma 2.2 to the function $p = \sup_{u\in U}(f + i_{F_u})$ one gets (3.4). □

Theorem 3.4. *Assume $(\mathscr{H})$ holds. For each $x^* \in X^*$, the following statements are equivalent:*

i) $p^*(x^*) = \min_{u\in U} \sup_{x\in F_u} \{\langle x, x^*\rangle - f(x)\}$;

ii) $\bigcup_{u\in U} \operatorname{epi} (f + i_{F_u})^*$ *is weak***-closed convex regarding* $\{x^*\} \times \mathbb{R}$.

Proof. Since $\operatorname{dom} p \neq \emptyset$, the conjugate function p^* does not take the value $-\infty$.
Let $x^* \in X^*$. We first consider the case where $p^*(x^*) = +\infty$. Since $p^* \leq q$, we have $q(x^*) = +\infty$ and i) holds. On the other hand, by Lemma 3.3, we have:

$$\overline{\operatorname{co}} \left(\bigcup_{u\in U} \operatorname{epi} (f + i_{F_u})^* \right) \bigcap \Big(\{x^*\} \times \mathbb{R} \Big) = \operatorname{epi} p^* \bigcap \Big(\{x^*\} \times \mathbb{R} \Big) = \emptyset.$$

Consequently, if $p^*(x^*) = +\infty$ then i) and ii) are both satisfied.
Assume now that $p^*(x^*) \in \mathbb{R}$. We first prove that ii) $\Longrightarrow$ i). By Lemma 3.3 it holds that

$$(x^*, p^*(x^*)) \in \operatorname{epi} p^* \bigcap \Big(\{x^*\} \times \mathbb{R}\Big) = \overline{\text{co}} \left(\bigcup_{u \in U} \text{epi } (f + i_{F_u})^* \right) \bigcap \Big(\{x^*\} \times \mathbb{R}\Big).$$

By ii) it follows that

$$(x^*, p^*(x^*)) \in \left(\bigcup_{u \in U} \text{epi } (f + i_{F_u})^* \right) \bigcap \Big(\{x^*\} \times \mathbb{R}\Big).$$

Consequently, there exists $\bar{u} \in U$ such that:

$$\inf_{u \in U} (f + i_{F_u})^*(x^*) = q(x^*) \leq (f + i_{F_{\bar{u}}})^*(x^*) \leq p^*(x^*).$$

Since $p^*(x^*) \leq q(x^*)$, we get i).
Conversely, let us prove i) $\Rightarrow$ ii). Let $(x^*, r) \in \overline{\text{co}} \left(\bigcup_{u \in U} \text{epi } (f + i_{F_u})^* \right)$. By Lemma 3.3 we have $p^*(x^*) \leq r$. By i) there exists $\bar{u} \in U$ such that

$$p^*(x^*) = (f + i_{F_{\bar{u}}})^*(x^*)$$

and finally $(x^*, r) \in \text{epi } (f + i_{F_{\bar{u}}})^*$. □

Corollary 3.5. *Assume* $(\mathscr{H})$ *holds. Then the following statements are equivalent:*

i) $-\infty \leq \max_{u \in U} \inf(P_u) = \inf(RP) < +\infty$ *;*

ii) *the set* $\bigcup_{u \in U} \text{epi } (f + i_{F_u})^*$ *is weak** *− closed convex regarding* $\{0_{X^*}\} \times \mathbb{R}$.

Proof. Since $-p^*(0_{X^*}) = \inf(RP)$, the result follows from Theorem 3.4 applied with $x^* = 0_{X^*}$. □

Corollary 3.6. *Assume* $(\mathscr{H})$ *holds. Then the following statements are equivalent:*

i) $-\infty < p^*(x^*) = \min_{u \in U} (f + i_{F_u})^*(x^*) \leq +\infty, \ \forall x^* \in X^*$*;*

ii) *the set* $\mathbb{A} := \bigcup_{u \in U} \text{epi } (f + i_{F_u})^*$ *is weak***-closed convex .*

Proof. Note that the set $\mathbb{A}$ is weak*-closed convex if and only if $\mathbb{A}$ is weak*-closed convex regarding $\{x^*\} \times \mathbb{R}$ for all $x^* \in X^*$. Then the result follows from Theorem 3.4. □

Let us reinforce condition $(\mathscr{H})$ by assuming that for each $u \in U$, the mapping g_u is S-epi-closed convex, and consider the new condition $(\mathscr{H}')$ given by:

$$(\mathscr{H}') \qquad \begin{cases} f \in \Gamma(X) \\ F \cap \operatorname{dom} f \neq \emptyset \\ g_u \ \text{ is } S\text{-epi-closed convex}, \ \forall u \in U. \end{cases}$$

Corollary 3.7. *Assume ($\mathscr{H}'$) holds. Then the following statements are equivalent:*

i) $\inf(RP) = \max(Q)$;

ii) $\bigcup_{u\in U} \overline{(\operatorname{epi} f^* + K_{g_u})}$ *is weak***-closed convex regarding* $\{0_{X^*}\} \times \mathbb{R}$.

Proof. Since $F \neq \emptyset$, one has $F_u = g_u^{-1}(-S) \neq \emptyset$ for each $u \in U$. Applying Proposition 2.3 to the mapping g_u, one has:

$$\operatorname{epi} i^*_{F_u} = \overline{K_{g_u}}\ , \quad \forall u \in U.$$

By Lemma 2.1 we have, for each $u \in U$,

$$\begin{aligned}\operatorname{epi}(f + i_{F_u})^* &= \overline{(\operatorname{epi} f^* + \operatorname{epi} i^*_{F_u})}\\ &= \overline{(\operatorname{epi} f^* + \overline{K_{g_u}})}\\ &= \overline{(\operatorname{epi} f^* + K_{g_u})}.\end{aligned}$$

Consequently,

$$\bigcup_{u\in U} \operatorname{epi}(f + i_{F_u})^* = \bigcup_{u\in U} \overline{(\operatorname{epi} f^* + K_{g_u})}$$

and the conclusion follows from Corollary 3.5. □

4 Link with Robust Strong Duality Property

For each parameter $u \in U$, we associate to (P_u) the classical Lagrangian dual defined as:

$$(D_u) \qquad \sup_{\lambda} \inf_{x\in X} \big\{ f(x) + \lambda g_u(x) \big\} \quad s.t. \ \ \lambda \in S^+.$$

The optimistic dual of the uncertain problem (P) is given by ([2],[8],[14],[16])

$$(ODP) \qquad \sup_{(u,\lambda)} \inf_{x\in X} \big\{ f(x) + \lambda g_u(x) \big\} \quad s.t. \ \ (u,\lambda) \in U \times S^+.$$

Proposition 4.1. *It holds that:*

$$\sup(ODP) \leq \sup(Q). \tag{4.1}$$

Proof. By Lagrangian weak duality between (P_u) and (D_u), one has, for each $u \in U$:

$$\sup_{\lambda\in S^+} \inf_{x\in X} \big\{ f(x) + \lambda g_u(x) \big\} \leq \inf(P_u).$$

Taking the supremum over U, one gets

$$\sup(ODP) = \sup_{\substack{u\in U\\ \lambda\in S^+}} \inf_{x\in X} \big\{ f(x) + \lambda g_u(x) \big\} \leq \sup_{u\in U} \inf(P_u) = \sup(Q).$$

□

Remark 4.2. It is worth noticing that, even in the certainty case, say $U = \{\bar{u}\}$, it may happen that (see Lemma 4.5 below):

$$\sup_{\lambda \in S^+} \inf_{x \in X} \{f(x) + \lambda g_{\bar{u}}(x)\} = \sup(ODP) < \sup(Q) = \inf(P_{\bar{u}}).$$

One says that robust strong duality holds for the uncertain conical convex problem (P) whenever the values of the robust counterpart and the optimistic dual coincide with dual attainment, i.e.

$$\inf(RP) = \max(ODP). \tag{4.2}$$

The terminology robust strong duality was introduced in [16]. This property can be found in [2] and [14] under the name "primal worst equals dual best".

Proposition 4.3. *If robust strong duality holds then*

$$\inf(RP) = \max(Q).$$

Proof. Assume that robust strong duality holds. By Juxtaposing Proposition 3.1 and Proposition 4.1, we have

$$\max(ODP) = \sup(Q) = \inf(RP).$$

Consequently, there exists $(\bar{u}, \bar{\lambda}) \in U \times S^+$ such that

$$\inf_{x \in X} \left\{f(x) + \bar{\lambda} g_{\bar{u}}(x)\right\} = \inf(RP) = \sup(Q) \geq \inf(P_{\bar{u}}) \geq \inf_{x \in X} \left\{f(x) + \bar{\lambda} g_{\bar{u}}(x)\right\},$$

where the last inequality results from the weak duality between $(P_{\bar{u}})$ and $(D_{\bar{u}})$. Thus $\inf(RP) = \inf(P_{\bar{u}})$ and we are done. □

In order to obtain robust strong duality from the previous results, we recall a convex composite duality principle.

Lemma 4.4 ([7] Theorem 8.3)**.** *Assume that $f \in \Gamma(X)$, $g : \text{dom}\, g \subset X \to Y$ is S-epi-closed convex and $g^{-1}(-S) \bigcap \text{dom}\, f \neq \emptyset$. Then the following statements are equivalent:*

i) $\inf_{g(x) \in -S}\{f(x) - \langle x, x^* \rangle\} = \max_{\lambda \in S^+} \inf_{g(x) \in -S}\{f(x) - \langle x, x^* \rangle + \lambda g(x)\}$, *for any* $x^* \in X^*$*;*

ii) $\bigcup_{\lambda \in S^+} \text{epi}\,(f + \lambda g)^*$ *is weak** − *closed.*

Lemma 4.5 ([11] Corollary 5)**.** *Assume that $f \in \Gamma(X)$, $g : \text{dom}\, g \subset X \to Y$ is S-epi-closed convex and $g^{-1}(-S) \bigcap \text{dom}\, f \neq \emptyset$. Then the following statements are equivalent:*

i) $\inf_{g(x) \in -S} f(x) = \max_{\lambda \in S^+} \inf_{x \in X}\{f(x) + \lambda g(x)\}$*;*

ii) $\bigcup_{\lambda \in S^+} \text{epi}\,(f + \lambda g)^*$ *is weak***-closed regarding* $\{0_{X^*}\} \times \mathbb{R}$.

Concerning the problems (ODP) and (Q), we have:

Proposition 4.6. *Assume that $(\mathscr{H}')$ holds and for each $u \in U$, the set* $\bigcup_{\lambda\in S^+} \operatorname{epi}(f+\lambda g_u)^*$ *is weak***-closed regarding* $\{0_{X^*}\} \times \mathbb{R}$ *. Then,*

$$\sup(ODP) = \sup(Q). \tag{4.3}$$

Proof. Applying Lemma 4.5, we obtain

$$\inf_{x\in F_u} f(x) = \max_{\lambda\in S^+} \inf_{x\in X} \{f(x) + \lambda g_u(x)\}, \quad \forall u \in U.$$

Taking the supremum over U, one obtains

$$\sup(Q) = \sup(ODP).$$

□

Remark 4.7. Consider the Example 3.2. Due to Slater qualification condition, it holds that the set $\bigcup_{\lambda\geq 0} \operatorname{epi}(f + \lambda g_u)^*$ is closed for all $u \in U$ (see for instence [9], Remark 4.3), and we have by Proposition 4.6:

$$\sup(Q) = \sup(ODP) < \inf(RP). \tag{4.4}$$

We now consider robust strong duality property. Let us denote by $Sol(Q)$ the set of optimal solutions of (Q):

$$Sol(Q) = \{u \in U \ : \ \inf(P_u) = \sup(Q)\}.$$

Theorem 4.8. *Assume $(\mathscr{H})$ holds and*

$$\bigcup_{u\in U} \operatorname{epi}(f + i_{F_u})^* \textit{is weak}^* - \textit{closed convex regarding } \{0_{X^*}\} \times \mathbb{R}, \tag{4.5}$$

$$\exists \bar{u} \in Sol(Q) : \begin{cases} g_{\bar{u}} \textit{is S-epi-closed convex and} \\ \bigcup_{\lambda\in S^+} \operatorname{epi}(f + \lambda g_{\bar{u}})^* \textit{is weak}^* - \textit{closed regarding } \{0_{X^*}\} \times \mathbb{R}. \end{cases} \tag{4.6}$$

Then, robust strong duality property holds.

Proof. We have:

$$\begin{aligned} \inf(RP) &= \max(Q) \quad \text{(follows from Corollary 3.5 and (4.5))} \\ &= \inf(P_{\bar{u}}) \quad \text{(for some optimal solution } \bar{u} \text{of } (Q) \text{)} \\ &= \max(D_{\bar{u}}) \quad \text{(by (4.6) and Lemma 4.5)} \\ &\leq \sup(ODP) \quad \text{(by the definition of the optimistic dual problem).} \end{aligned}$$

Applying Proposition 4.1 we get:

$$\sup(ODP) \leq \sup(Q) = \inf(RP) = \max(D_{\bar{u}}) \leq \sup(ODP).$$

Consequently, there exists $(\bar{u}, \bar{\lambda}) \in U \times S^+$ such that

$$\sup(ODP) = \inf_{x\in X} \{f(x) + \bar{\lambda} g_{\bar{u}(x)}\} = \inf(RP),$$

and we are done. □

One says that robust strong duality holds at a given $x^* \in X^*$ if

$$\inf_{x\in F}\{f(x) - \langle x, x^*\rangle\} = \max_{\substack{\lambda\in S^+ \\ u\in U}} \inf_{x\in X}\{f(x) - \langle x, x^*\rangle + \lambda g_u(x)\}. \tag{4.7}$$

If $x^* = 0_{X^*}$ we go back to robust strong duality property. If robust strong duality holds at each $x^* \in X^*$, one says that robust stable strong duality holds for (P).

Corollary 4.9. *Assume ($\mathscr{H}'$) holds and*

$$\bigcup_{u\in U} \overline{\operatorname{epi} f^* + K_{g_u}} \text{ is weak}^*\text{-closed convex}, \tag{4.8}$$

$$\forall u \in U \ , \quad \bigcup_{\lambda\in S^+} \operatorname{epi}(f + \lambda g_u)^* \text{ is weak}^*\text{-closed.} \tag{4.9}$$

Then, robust stable strong duality holds.

Proof. By Lemma 2.1 and Proposition 2.3, one has:

$$\mathbb{A} := \bigcup_{u\in U} \operatorname{epi}(f + i_{F_u})^* = \bigcup_{u\in U} \overline{\operatorname{epi} f^* + K_{g_u}}.$$

By (4.8) $\mathbb{A}$ is weak*-closed convex . Therefore, for each $x^* \in X^*$, one has:

$$\begin{aligned}
\inf_{x\in F}\{f(x) - \langle x, x^*\rangle\} &= \max_{u\in U} \inf_{x\in F_u}\{f(x) - \langle x, x^*\rangle\} && \text{(by Corollary 3.6)}\\
&= \inf_{x\in F_{\bar u}}\{f(x) - \langle x, x^*\rangle\} && \text{(for some } \bar u \in U)\\
&= \max_{\lambda\in S^+} \inf_{x\in X}\{f(x) - \langle x, x^*\rangle + \lambda g_{\bar u}(x)\} && \text{(by Lemma 4.4 and (4.9))}\\
&= \inf_{x\in X}\{f(x) - \langle x, x^*\rangle + \bar\lambda g_{\bar u}(x)\} && \text{(for some } \bar\lambda \in S^+)\\
&\le \sup_{\substack{\lambda\in S^+ \\ u\in U}} \inf_{x\in X}\{f(x) - \langle x, x^*\rangle + \lambda g_u(x)\}.
\end{aligned}$$

Now, by weak duality (apply Proposition 3.1 and Proposition 4.1 to $f - x^*$), it holds

$$\sup_{\substack{\lambda\in S^+ \\ u\in U}} \inf_{x\in X}\{f(x) - \langle x, x^*\rangle + \lambda g_u(x)\} \le \inf_{x\in F}\{f(x) - \langle x, x^*\rangle\},$$

and we are done. □

Remark 4.10. Let us assume that for each $u \in U$, the maping $g_u : X \to Y$ is continuous and S-epi-convex. For each $\lambda \in S^+$ the function $\lambda g_u = \lambda \circ g_u$ is convex and continuous, and by Moreau-Rockafellar Theorem one has

$$\operatorname{epi}(f + \lambda g_u)^* = \operatorname{epi} f^* + \operatorname{epi}(\lambda g_u)^*.$$

In such a case condition (4.9) turns into

$$\forall u \in U, \quad \operatorname{epi} f^* + K_{g_u} \text{ is } weak^*\text{-closed} . \tag{4.10}$$

Robust strong duality property was established in [16] (Corollary 3.1) for continuous S-epi-convex mappings $g_u : X \to Y$ under the condition

$$\operatorname{epi} f^* + \bigcup_{u\in U} K_{g_u} \text{is weak}^*\text{-closed convex} \tag{4.11}$$

without assuming (4.9) (that is (4.10)) as we did in Corollary 4.9. However, let us observe that:

Proposition 4.11. *The condition (4.8) is weaker than the condition (4.11).*

Proof. We have

$$\operatorname{epi} f^* + \bigcup_{u\in U} K_{g_u} \subset \bigcup_{u\in U} \overline{(\operatorname{epi} f^* + K_{g_u})} \subset \overline{\left(\operatorname{epi} f^* + \bigcup_{u\in U} K_{g_u}\right)} \subset \overline{\operatorname{co}} \left(\operatorname{epi} f^* + \bigcup_{u\in U} K_{g_u}\right).$$

Thus, if (4.11) holds, then all the above inclusions are equalities, and we have in particular

$$\bigcup_{u\in U} \overline{(\operatorname{epi} f^* + K_{g_u})} = \overline{\operatorname{co}} \left(\operatorname{epi} f^* + \bigcup_{u\in U} K_{g_u}\right),$$

which is weak*- closed convex. □

Acknowledgments

The authors are grateful to Professor Michel VOLLE for his constructive comments and helpful suggestions, which have contributed to the preparation of the paper.

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Manuscript received 11 December 2015
revised 2 February 2016, 19 February 2016
accepted for publication 26 March 2016

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