Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 13, No. 1, pp. 43–53, January 2017

NONSMOOTH MULTIOBJECTIVE SEMI-INFINITE PROBLEMS WITH MIXED CONSTRAINTS

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Abstract: A class of multiobjective semi-infinite optimization problems with inequality and equality constraints is considered where the objective and constraints functions are locally Lipschitz. We present necessary optimality conditions for weak efficient solutions of a nonsmooth multiobjective semi-infinite problem. Moreover, sufficient conditions of optimality are provided under assumptions of generalized convexity.

Key words: *semi-infinite programming, multiobjective optimization, constraint qualification, optimality conditions, Clarke subdifferential*

Mathematics Subject Classification: *90C34, 90C40, 49J52*

1 Introduction

In the theory of optimization problems, necessary optimality conditions for semi-infinite programming problems (SIP, briefly) are important theoretically as well as computationally and can be formulated in several different ways. Optimality conditions of SIP have been studied by many authors; see for instance [9, 16] in linear case, [17] in convex case, [10] in smooth case, and [12–14, 19] in locally Lipschitz case.

A multiobjective semi-infinite programming (MOSIP in brief) is an optimization problem where two or more objectives are to be minimized on a set of feasible solutions described by infinitely many constraint functions. Under smooth conditions (say, linearity and convexity as well as differentiability) optimality conditions for MOSIP have been studied by some scholars; see, e.g., [1, 3, 4]. Kanzi and Nobakhtian established in [15] the Karush-Kuhn-Tucker (KKT, in brief) types necessary and sufficient optimality conditions for non convex nondifferentiable and non-convex MOSIP under three different qualification conditions called Abadie, Basic, and Regular constraint qualifications.

Convexity plays an important role in deriving sufficient optimality conditions. Generalized convexity can be viewed as a weaker version of the notion of convexity. Therefore, from the viewpoint of optimization, the descriptions of the optimality conditions in terms of generalized convexity provide sharper results, see for instance [7, 8]. In this work, sufficient optimality conditions are stated under generalized convexity hypothesis.

*The second-named author was partially supported by the Center of Excellence for Mathematics, University of Shahrekord, Iran and a grant from IPM [grant number 93900423].

In all of references cited above, the feasible set described by only inequality constraints. This paper focuses mainly on KKT type necessary optimality conditions for the problem (P), defined as follows:

$$\begin{array}{lll} \text{(P)} & \inf\big(f_1(x), f_2(x), \ldots, f_m(x)\big) & \\ \text{s.t.} & g_j(x) \le 0 & j \in J, \\ & h_t(x) = 0 & t \in T, \end{array}$$

where $f_i, i \in I := \{1, 2, \ldots, m\}$ and $g_j,\ j \in J$ and $h_t,\ t \in T$ are locally Lipschitz functions from $\mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$, and the index sets J and T are arbitrary, not necessarily finite, with $J \cup T \neq \emptyset$.

We organize the paper as follows. In the next section, we provide the preliminary results to be used in the rest of the paper. In Section 3, we introduce two different qualification conditions for (P). Then, we apply these constraint qualifications to derive necessary optimality conditions for (P). Finally in Section 4, sufficient conditions of optimality are established under assumptions of generalized convexity.

2 Notations and Preliminaries

In this section we present definitions and auxiliary results that will be needed in the sequel.

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz. The Clarke directional derivative of φ at $\hat{x} \in \mathbb{R}^n$ in the direction $v \in \mathbb{R}^n$, and the Clarke subdifferential of φ at $\hat{x}$ introduced in [5] are respectively given by

$$\varphi^\circ(\hat{x}; v) := \limsup_{y \to \hat{x},\ t \downarrow 0} \frac{\varphi(y + tv) - \varphi(y)}{t},$$

$$\partial^c \varphi(\hat{x}) := \big\{\xi \in \mathbb{R}^n \mid \langle \xi, v\rangle \le \varphi^\circ(\hat{x}; v) \qquad \text{for all } v \in \mathbb{R}^n\big\}.$$

The Clarke subdifferential is a natural generalization of the derivative since it is known that when function φ is continuous differentiable at $\hat{x}$, $\partial^c \varphi(\hat{x}) = \{\nabla \varphi(\hat{x})\}$. Moreover when a function φ is convex, the Clarke subdifferential coincides with the subdifferential in the sense of convex analysis.

In the following theorem we summarize some important properties of the Clarke directional derivative and the Clarke subdifferential from [5] which are widely used in what follows.

Theorem 2.1. *Let $\phi_i : \mathbb{R}^n \to \mathbb{R}, i = 1, 2$ be Lipschitz near $\hat{x}$. Then, the following assertions hold:*

(i) *One has*

$$\begin{aligned} &\phi_1^\circ(\hat{x}; v) = \max\big\{\ \langle \xi, v\rangle \mid \xi \in \partial^c \phi_1(\hat{x})\big\}, \\ &\partial^c\big(\max\{\phi_1, \phi_2\}\big)(\hat{x}) \subseteq conv\big(\partial^c \phi_1(\hat{x}) \cup \partial^c \phi_2(\hat{x})\big), \\ &\partial^c(\lambda \phi_1 + \phi_2)(\hat{x}) \subseteq \lambda \partial^c \phi_1(\hat{x}) + \partial^c \phi_2(\hat{x}), \qquad \forall\, \lambda \in \mathbb{R}. \end{aligned}$$

(ii) *The function $v \to \phi_1^\circ(\hat{x}; v)$ is finite, positively homogeneous, and subadditive on $\mathbb{R}^n$, and*

$$\partial\big(\phi_1^\circ(\hat{x}; .)\big)(0) = \partial^c \phi_1(\hat{x}),$$

where ∂ denotes the subdifferential in sense of convex analysis.

(iii) $\partial^c \phi_1(\hat{x})$ *is nonempty, convex, and compact subset of* $\mathbb{R}^n$.

Definition 2.2. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. We say that φ is generalized convex at $\hat{x} \in \mathbb{R}^n$ if for any $x \in \mathbb{R}^n$

$$\varphi(x) - \varphi(\hat{x}) \geq \varphi^\circ(\hat{x}; x - \hat{x}).$$

Definition 2.3. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. We say that φ is pseudoconvex at $\hat{x} \in \mathbb{R}^n$ if for any $x \in \mathbb{R}^n$

$$\varphi^\circ(\hat{x}; x - \hat{x}) \geq 0 \Longrightarrow \varphi(x) \geq \varphi(\hat{x}).$$

We say that φ is pseudoaffine at $\hat{x}$ if both φ and $-\varphi$ are pseudoconvex at $\hat{x}$.

Definition 2.4. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. We say that φ is quasiconvex at $\hat{x} \in \mathbb{R}^n$ if for any $x \in \mathbb{R}^n$

$$\varphi(x) \leq \varphi(\hat{x}) \Longrightarrow \varphi^\circ(\hat{x}; x - \hat{x}) \leq 0.$$

Definition 2.5. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. We say that φ infine at $\hat{x} \in \mathbb{R}^n$ if for any $x \in \mathbb{R}^n$, and any $\xi \in \partial^c \varphi(\hat{x})$

$$\varphi(x) - \varphi(\hat{x}) = \langle \xi, x - \hat{x} \rangle.$$

Let A be a nonempty subset of $\mathbb{R}^n$, denote by $\bar{A}$, $conv(A)$, and $cone(A)$, the closure of A, the convex hull, and the convex cone (containing the origin) generated by A, respectively. Also, the polar cone and strict polar cone of A are defined respectively by:

$$A^* := \{d \in R^n \mid \langle x, d \rangle \leq 0 \quad \forall x \in A\},$$
$$A^s := \{d \in R^n \mid \langle x, d \rangle < 0 \quad \forall x \in A\},$$

where $\langle ., . \rangle$ exhibits the standard inner product in $\mathbb{R}^n$. Notice that A^* is always a closed convex cone. It is easy to show that if $A^s \neq \emptyset$ then $\overline{A^s} = A^*$; see [11].

Let us recall the following theorems which will be used in the sequel.

Theorem 2.6 ([11])**.** *Let A be a nonempty compact subset of* $\mathbb{R}^n$. *Then*

(I) $conv(A)$ *is a closed set.*

(II) $cone(A)$ *is a closed cone, if* $0 \notin conv(A)$.

3 Necessary Conditions

For single objective semi-infinite problems with (only) inequality constraints, the Slater constraint qualification (SCQ in brief) introduced in [17, Definition 3.6] as follows:
We say that the problem satisfies the Slater constraint qualification (SCQ), if

- for all $j \in J$, g_j is a convex function,
- $J \subseteq \mathbb{R}^p$ is a compact set,
- $g_j(x)$ is a continuous function of (j, x) in $J \times \mathbb{R}^n$
- there is a point $x_0 \in \mathbb{R}^n$ such that $g_j(x_0) < 0$, for all $j \in J$.

Definition 3.1 below generalizes the concept of SCQ for the problem (P).

Definition 3.1. We say that (P) satisfies the *weak Slater constraint qualification* (WSCQ, briefly) if the following assertions hold:

(i) g_j and h_t are respectively pseudoconvex and infine functions for all $(j, t) \in J \times T$.

(ii) $J \subseteq \mathbb{R}^p$ is a compact set.

(iii) $g_j(x)$ is a u.s.c. function of (j, x) in $J \times \mathbb{R}^n$.

(iv) There is a point $x_0 \in \mathbb{R}^n$ such that

- $g_j(x_0) < 0 \qquad \forall j \in J,$
- $h_t(x_0) = 0 \qquad \forall t \in T.$

For a given $\hat{x} \in S := \{x \in \mathbb{R}^n \mid g_j(x) \leq 0,\ h_t(x) = 0 \ \ \forall (j, t) \in J \times T\}$, let $J(\hat{x})$ denotes the index set of all active inequality constraints at $\hat{x}$, i.e.,

$$J(\hat{x}) := \{j \in J \mid g_j(\hat{x}) = 0\}.$$

A point $\hat{x}$ is said to be a weakly efficient solution to (P) if there is no $x \in S$ satisfies $f_i(x) < f_i(\hat{x})$ for all $i \in I$.

For each $x \in S$ take

$$\begin{aligned}
\mathcal{A}(x) &:= \bigcup_{i \in I} \partial^c f_i(x),\\
\mathcal{B}(x) &:= \bigcup_{j \in J(x)} \partial^c g_j(x),\\
\mathcal{C}(x) &:= \{d \in \mathbb{R}^n \mid h_t^\circ(x; d) = 0 \quad \forall t \in T\},\\
\mathcal{D}(x) &:= \Big(\bigcup_{t \in T} \partial^c h_t(x)\Big) \cup \Big(\bigcup_{t \in T} -\partial^c h_t(x)\Big).
\end{aligned}$$

Theorem 3.2. *Suppose that $\hat{x}$ is a weakly efficient solution of problem (P), and that WSCQ is satisfied. Then, there exist $\alpha_i \geq 0$ for $i \in I$ with $\sum_{i=1}^m \alpha_i = 1$, such that*

$$0 \in \sum_{i=1}^m \alpha_i \partial^c f_i(\hat{x}) + \overline{cone}\big(\mathcal{B}(\hat{x}) + \mathcal{D}(\hat{x})\big).$$

Proof. Take

$$\begin{aligned}
G_j(d) &:= g_j^\circ(\hat{x}; d) && \forall j \in J,\\
H_t(d) &:= h_t^\circ(\hat{x}; d) && \forall t \in T,\\
G(x) &:= \max_{j \in J} g_j(x) && \forall x \in S.
\end{aligned} \tag{3.1}$$

Owning to the [5, Theorem 2.8.2, Step 1] we know G is a locally Lipschitz function and the following inequality holds:

$$G^\circ(\hat{x}; d) \leq \widetilde{G}(d) := \max_{j \in J(\hat{x})} G_j(d) \qquad \forall d \in \mathbb{R}^n.$$

Observe that $\widetilde{G}$ is a convex function. Let $\xi \in \partial^c G(\hat{x})$. The last inequality implies that for each $d \in \mathbb{R}^n$ we have

$$\langle \xi, d - 0 \rangle = \langle \xi, d \rangle \leq \widetilde{G}(d) = \widetilde{G}(d) - \widetilde{G}(0) \quad \Longrightarrow \quad \xi \in \partial \widetilde{G}(0).$$

From the well-known Pshenichnyi-Levin-Valadire Theorem [11, pp. 267] we deduce that

$$\partial \widetilde{G}(0) = conv\Big(\bigcup_{j \in \widetilde{J}(0)} \partial G_j(0) \Big),$$

where, $\widetilde{J}(0) := \{ j \in J(\hat{x}) \mid G_j(0) = \widetilde{G}(0) = 0 \}$. Since $\widetilde{J}(0) = J(\hat{x})$ and $\partial G_j(0) = \partial^c g_j(\hat{x})$ and ξ is an arbitrary element of $\partial^c G(\hat{x})$, we conclude that

$$\partial^c G(\hat{x}) \subseteq conv\big(\mathcal{B}(\hat{x})\big). \tag{3.2}$$

On the other hand, by definition of WSCQ and feasibility of $\hat{x}$ we have

$$\begin{cases} g_j(x_0) < 0 = g_(\hat{x}) & \forall j \in J(\hat{x}) \\ h_t(x_0) = 0 = h_t(\hat{x}) & \forall t \in T. \end{cases}$$

Then, the pseudoconvexity of g_j for $j \in J$, and the infinness of h_t for $t \in T$, imply that

$$\begin{cases} g_j^\circ(\hat{x}; x_0 - \hat{x}) < 0 & \forall j \in J(\hat{x}) \\ h_t^\circ(\hat{x}; x_0 - \hat{x}) = 0 & \forall t \in T, \end{cases}$$

and hence

$$x_0 - \hat{x} \in \big(\mathcal{B}(\hat{x})\big)^s \cap \mathcal{C}(\hat{x}). \tag{3.3}$$

The last inclusion implies $\big(\mathcal{B}(\hat{x})\big)^s \cap \mathcal{C}(\hat{x}) \neq \emptyset$. Let $d \in \big(\mathcal{B}(\hat{x})\big)^s \cap \mathcal{C}(\hat{x})$. By (3.2) and the fact that

$$d \in \big(\mathcal{B}(\hat{x})\big)^s \cap \mathcal{C}(\hat{x}) = \Big(conv\big(\mathcal{B}(\hat{x})\big)\Big)^s \cap \mathcal{C}(\hat{x}) \subseteq \big(\partial^c G(\hat{x})\big)^s \cap \mathcal{C}(\hat{x}),$$

we conclude that,

$$\begin{cases} \langle \xi, d \rangle < 0 & \forall \xi \in \partial^c G(\hat{x}) \\ h_t^\circ(\hat{x}; d) = 0 & \forall t \in T, \end{cases} \quad \Longrightarrow \quad \begin{cases} G^\circ(\hat{x}; d) < 0 \\ h_t^\circ(\hat{x}; d) = 0 & \forall t \in T. \end{cases}$$

Now by definition of $G^\circ(\hat{x}; d)$ we can find $\delta > 0$ such that

$$\begin{cases} G(\hat{x} + \beta d) - G(\hat{x}) < 0 & \forall \beta \in (0, \delta) \\ h_t^\circ(\hat{x}; d) = 0 & \forall t \in T. \end{cases}$$

Thus for all $\beta \in (0, \delta)$, and for all $(j, t) \in J \times T$ we have

$$\begin{cases} g_j(\hat{x} + \beta d) \leq G(\hat{x} + \beta d) < 0 \\ h_t^\circ\big(\hat{x}; (\hat{x} + \beta d) - \hat{x}\big) = \beta h_t^\circ(\hat{x}; d) = 0, \end{cases} \quad \Longrightarrow \quad \begin{cases} g_j(\hat{x} + \beta d) < 0 \\ h_t(\hat{x} + \beta d) = h_t(\hat{x}) = 0, \end{cases}$$

which insures $\hat{x}+\beta d \in S$ for all $\beta > 0$ small enough. This implies $d \in \Gamma(S,\hat{x})$, where $\Gamma(S,\hat{x})$ denotes the contingent cone of S at $\hat{x}$, i.e.,

$$\Gamma(S,\hat{x}) := \Big\{ v \in \mathbb{R}^n \mid \exists \{(r_k, v_k)\} \to (0^+, v), \text{ such that } \hat{x} + r_k v_k \in S \ \ \forall k \in \mathbb{N} \Big\}.$$

From [15, Theorem 3.4, Step 1] we conclude that $d \notin \big(\mathcal{A}(\hat{x})\big)^s$. Since d is an arbitrary element of $\big(\mathcal{B}(\hat{x})\big)^* \cap \mathcal{C}(\hat{x})$, the last relation yields

$$\widetilde{F}(d) := \max_{1 \le i \le m} f_i^\circ(\hat{x}; d) \ge 0 \qquad \forall d \in \big(\mathcal{B}(\hat{x})\big)^* \cap \mathcal{C}(\hat{x}).$$

Taking into account that $0 \in \big(\mathcal{B}(\hat{x})\big)^* \cap \mathcal{C}(\hat{x})$ and $F(0) = 0$, we can conclude that $\hat{d} := 0$ is an optimal solution of

$$\min \widetilde{F}(d) \qquad \text{subject to} \qquad d \in \big(\mathcal{B}(\hat{x})\big)^* \cap \mathcal{C}(\hat{x}).$$

Combining this and the fact that

$$\big(\mathcal{B}(\hat{x})\big)^* \cap \mathcal{C}(\hat{x}) = \big\{ v \in \mathbb{R}^n \mid G_j(v) \le 0, \ H_t(v) = 0 \quad \forall (j,t) \in J(\hat{x}) \times T \big\},$$

imply that $\hat{d}$ is an optimal solution of the following convex semi-infinite programming:

$$\begin{array}{lll} (\hat{\mathrm{P}}) & \min \widetilde{F}(d) & \\ \text{s.t.} & G_j(d) \le 0, & j \in J(\hat{x}), \\ & H_t(d) = 0, & t \in T, \\ & d \in \mathbb{R}^n. & \end{array}$$

From [11, Page 137] we have $\Gamma\Big(\big(\mathcal{B}(\hat{x})\big)^* \cap \mathcal{C}(\hat{x}), \hat{d}\Big) = \big(\mathcal{B}(\hat{x})\big)^* \cap \mathcal{C}(\hat{x})$, it follows that the problem $(\hat{\mathrm{P}})$ satisfies the Abadie qualification condition at $\hat{d}$ in sense of [14, Definition 4.3]. Thus by [14, Theorem 4.8] we get:

$$0 \in \partial \widetilde{F}(0) + \overline{cone}\big(\mathcal{B}(\hat{x}) \cup \mathcal{D}(\hat{x})\big).$$

Combining this with

$$\begin{aligned} \partial \widetilde{F}(0) \quad &\subseteq \quad conv\Big(\bigcup_{i=1}^m \partial f_i^\circ(\hat{x}; .)(0) \Big) = conv\Big(\bigcup_{i=1}^m \partial^c f_i(\hat{x}) \Big) \\ &= \quad \Big\{ \sum_{i=1}^m \alpha_i \xi_i \mid \alpha_i \ge 0, \ \ \xi_i \in \partial^c f_i(\hat{x}) \quad \forall i \in I, \ \ \text{and} \ \ \sum_{i=1}^m \alpha_i = 1 \Big\}, \end{aligned}$$

yield the result. □

At this point the necessary condition of KKT type for problem (P) can be stated as follows.

Theorem 3.3. *In the hypotheses of Theorem 3.2 if* $cone\big(\mathcal{B}(\hat{x}) \cup \mathcal{D}(\hat{x})\big)$ *is a closed set in* $\mathbb{R}^n$*, then there exist* $\alpha_i \ge 0$ *for* $i \in I$ *with* $\sum_{i=1}^m \alpha_i = 1$*, and* $(\beta_j, \gamma_t) \in \mathbb{R}_+ \times \mathbb{R}$ *for* $(j,t) \in J(\hat{x}) \times T$*, with finitely many of them being nonzero such that*

$$0 \in \sum_{i=1}^m \alpha_i \partial^c f_i(\hat{x}) + \sum_{j \in J(\hat{x})} \beta_j \partial^c g_j(\hat{x}) + \sum_{t \in T} \gamma_t \partial^c h_t(\hat{x}). \tag{3.4}$$

Proof. It follows from Theorem 3.2 and the fact that

$$cone\big(\mathcal{B}(\hat{x}) \cup \mathcal{D}(\hat{x})\big) = cone\big(\mathcal{B}(\hat{x})\big) + span\big(\mathcal{D}(\hat{x})\big) =$$
$$\Big\{ \sum_{j \in J_0} \beta_j \zeta_j \mid \beta_j \geq 0 \;\; \zeta_j \in \partial^c g_j(\hat{x}) \;\; \forall j \in J_0, \quad J_0 \text{ is a finite subset of } J \Big\} +$$
$$\Big\{ \sum_{t \in T_0} \gamma_t \eta_t \mid \eta_t \in \partial^c h_t(\hat{x}) \;\; \forall t \in T_0, \quad T_0 \text{ is a finite subset of } T \Big\}.$$

□

An important particular situation is $T = \emptyset$. In this case, due to the $\mathcal{C}(\hat{x}) = \mathbb{R}^n$ and (3.3), the following relationships are valid:

$$\Big(conv\big(\mathcal{B}(\hat{x})\big)\Big)^s = \big(\mathcal{B}(\hat{x})\big)^s \neq \emptyset.$$

This implies $0 \notin conv\big(\mathcal{B}(\hat{x})\big)$ which ensures by Theorem 2.6 that $cone\big(\mathcal{B}(\hat{x})\big)$ is closed. Thus by Theorem 3.3 we can conclude the next result.

Corollary 3.4. *In the hypotheses of Theorem 3.2 if $T = \emptyset$, then there exist $\alpha_i \geq 0$ for $i \in I$ with $\sum_{i=1}^m \alpha_i = 1$, and $\beta_j \geq 0$ for $j \in J(\hat{x})$, with finitely many of them being nonzero such that*

$$0 \in \sum_{i=1}^m \alpha_i \partial^c f_i(\hat{x}) + \sum_{j \in J(\hat{x})} \beta_j \partial^c g_j(\hat{x}).$$

Let us introduce a new constraint qualification for problem (P) which will be crucial in the sequel.

We present now the generalized Mangasarian-Fromovitz constraint qualification for problem (P), which is considered by Pappalardo in [18] for nondifferentiable programming problems where the objective function is real valued and set constraint is not considered.

We say that (P) satisfies the *Mangasarian-Fromovitz constraint qualification* (MFCQ, briefly) at $\hat{x}$ if the following assertions hold:

(i) $J \subseteq \mathbb{R}^p$ is a compact set.

(ii) $g_j(x)$ is a u.s.c. function of (j, x) in $J \times \mathbb{R}^n$.

(iii) $0 \in \sum_{t \in T} \nu_t \partial^c h_t(\hat{x}) \quad \Longrightarrow \quad \nu = 0,$

(iv) There exists $d^* \in \mathbb{R}^n$ such that

- $g_j^\circ(\hat{x}; d^*) < 0 \qquad \forall j \in J(\hat{x}),$
- $h_t^\circ(\hat{x}; d^*) = 0 \qquad \forall t \in T.$

Remark 3.5. If MFCQ holds at some point $\hat{x} \in \mathbb{R}^n$, then $|T| \leq n$.

Theorem 3.6. *Suppose that $\hat{x}$ is a weakly efficient solution of problem (P), and MFCQ is satisfied at$\hat{x}$. Then there exist $\alpha_i \geq 0$ for $i \in I$ with $\sum_{i=1}^m \alpha_i = 1$, and $(\beta_j, \gamma_t) \in \mathbb{R}_+ \times \mathbb{R}$ for $(j, t) \in J(\hat{x}) \times T$, with finitely many of them being nonzero, such that*

$$0 \in \sum_{i=1}^m \alpha_i \partial^c f_i(\hat{x}) + \sum_{j \in J(\hat{x})} \beta_j \partial^c g_j(\hat{x}) + \sum_{t \in T} \gamma_t \partial^c h_t(\hat{x}).$$

Proof. Regarding Remark 3.5, we conclude that $\hat{x}$ is a solution of the following multiobjective optimization problem with finite number of constraints:

$$\begin{aligned} & \min \ \big(f_1(x), f_2(x), \ldots, f_m(x)\big) \\ \text{s.t.} \quad & G(x) \leq 0, \\ & h_t(x) = 0 \qquad t \in T, \end{aligned}$$

where $G(x)$ is defined in (3.1). Employing Fritz-John Theorem (see e.g., [6, Theorem 3.1]), we find a scalar $\lambda \geq 0$ and vectors $\widehat{\alpha} := (\widehat{\alpha}_i) \in \mathbb{R}^m_+$ and $\widehat{\gamma} := (\widehat{\gamma}_t) \in \mathbb{R}^{|T|}$ such that $(\widehat{\alpha}, \lambda, \widehat{\gamma}) \neq 0$ and

$$0 \in \sum_{i=1}^{m} \widehat{\alpha}_i \partial^c f_i(\hat{x}) + \lambda \partial^c G(\hat{x}) + \sum_{t \in T} \widehat{\gamma}_t \partial^c h_t(\hat{x}).$$

Since the inclusion (3.2) is an implication of compactness of $J \subseteq \mathbb{R}^p$ and continuousness of the mapping $(j, x) \to g_j(x)$ on $J \times \mathbb{R}^n$, then (3.2) still valid under the MFCQ. Thus due to the last inclusion, the virtue of (3.2), and the structure of a convex hull, we can select a finite index set $\widehat{J} \subseteq J(\hat{x})$ with $\widehat{\beta}_j > 0$ for $j \in \widehat{J}$ such that $\sum_{j \in \widehat{J}} \beta_j = 1$ and

$$0 \in \sum_{i=1}^{m} \widehat{\alpha}_i \partial^c f_i(\hat{x}) + \sum_{j \in \widehat{J}} \lambda \widehat{\beta}_j \partial^c g_j(\hat{x}) + \sum_{t \in T} \widehat{\gamma}_t \partial^c h_t(\hat{x}).$$

We claim that $\widehat{\alpha} = 0$. On the contrary suppose that $\widehat{\alpha} = 0$. Then,

$$\sum_{j \in \widehat{J}} \lambda \widehat{\beta}_j \varsigma_j + \sum_{t \in T} \widehat{\gamma}_t \xi_t = 0, \tag{3.5}$$

for some $\varsigma_j \in \partial^c g_j(\hat{x}),\ j \in \widehat{J}, \xi_t \in \partial^c h_t(\hat{x}), t \in T$. If $\lambda = 0$, then $\widehat{\gamma} = 0$ by (iii). It follows that that $(\widehat{\alpha}, \lambda, \widehat{\gamma}) = 0$, a contradiction. Thus $\lambda \neq 0$, and hence $\lambda \widehat{\beta}_j > 0$ for $j \in \widehat{J}$ by $\widehat{\beta}_j > 0$. Therefore from (3.5), for d^* which satisfies in the definition of MFCQ, we have

$$\sum_{j \in \widehat{J}} \lambda \widehat{\beta}_j \big\langle \varsigma_j, d^* \big\rangle = -\sum_{t \in T} \widehat{\gamma}_t \big\langle \xi_t, d^* \big\rangle.$$

But this is impossible, since the right-hand side of the equation is zero while the left-hand side of equation is nonzero. This contradiction shows that $\widehat{\alpha} \neq 0$. Hence, the result is verified with taking

$$\begin{aligned} \alpha_i &:= \frac{\widehat{\alpha}_i}{\sum\limits_{i=1}^{m} \widehat{\alpha}_i}, \\ \beta_j &:= \begin{cases} \frac{\lambda \widehat{\beta}_j}{\sum_{i=1}^{m} \widehat{\alpha}_i} & \text{if} \quad j \in \widehat{J} \\ 0 & \text{if} \quad j \in J(\hat{x}) \backslash \widehat{J}, \end{cases} \\ \gamma_t &:= \frac{\widehat{\gamma}_t}{\sum\limits_{i=1}^{m} \widehat{\alpha}_i}. \end{aligned}$$

□

Remark 3.7. We point out that if in the definition of Mangasarian-Fromovitz constraint qualification, $h_t, t \in T$ are continuously differentiable, then the conclution in the Theorem 3.6 is also true. In this case condition (iii) becomes as the following

$(iii)'$ Each h_t for $t \in T$ is continuously differentiable at $\hat{x}$ and $\{\nabla h_t(\hat{x}) \mid t \in T\}$ is a linearly independent set.

$\boxed{4}$ Sufficient Optimality Conditions

In this Section we discuss several families of sufficient optimality results under various generalized convexity hypotheses imposed on the involved functions.

Theorem 4.1. *Let $x^* \in S$ and $J(x^*) \cup T \neq \emptyset$. Assume that there exist $\alpha_i \geq 0$ for $i \in I$ with $\sum_{i=1}^{m} \alpha_i = 1$, and $(\beta_j, \gamma_t) \in \mathbb{R}_+ \times \mathbb{R}$ for $(j,t) \in J(\hat{x}) \times T$, with finitely many of them being nonzero, such that (3.4) holds. If all the functions f_i, $i \in \{1, 2, \ldots, m \mid \alpha_i \neq 0\}$, g_j, $j \in \bar{J}(x^*) = \{j \in J(x^*) \mid \beta_j \neq 0\}$ are generalized convex and h_t, $t \in \bar{T} = \{t \in T \mid \gamma_t \neq 0\}$ are infine, then x^* is weakly efficient.*

Proof. Since (3.4) is satisfied, it follows that there exist $\xi_i \in \partial_c f_i(x^*)$ for $i \in I$, $\zeta_j \in \partial_c g_j(x^*)$ for $j \in \bar{J}(x^*)$, and $\eta_t \in \partial_c h_t(x^*)$ for $t \in \bar{T}$ such that

$$\sum_{i=1}^{m} \alpha_i \xi_i + \sum_{j \in \bar{J}(x^*)} \beta_j \zeta_j + \sum_{t \in \bar{T}} \gamma_t \eta_t = 0. \tag{4.1}$$

Suppose on the contrary that x^* is not a weakly efficient solution for (MOSIP). Then there exists a feasible point $\hat{x}$ for (MOSIP) such that

$$f_i(\hat{x}) < f_i(x^*) \qquad \text{for all } i = 1, \ldots, m.$$

Thus, we have

$$\sum_{i=1}^{m} \alpha_i f_i(\hat{x}) < \sum_{i=1}^{m} \alpha_i f_i(x^*). \tag{4.2}$$

Since $\hat{x}$ is a feasible point for (MOSIP) and $\beta_j g_j(x^*) = 0$ for all $j \in \bar{J}(x^*)$, and $\gamma_t h_t(x^*) = 0$ for all $t \in \bar{T}$ then

$$\sum_{j \in \bar{J}(x^*)} \beta_j g_j(\hat{x}) + \sum_{t \in \bar{T}} \gamma_t h_t(\hat{x}) \leq \sum_{j \in \bar{J}(x^*)} \beta_j g_j(x^*) + \sum_{t \in \bar{T}} \gamma_t h_t(x^*). \tag{4.3}$$

Therefore,

$$\begin{aligned} &\sum_{i=1}^{m} \alpha_i f_i(x^*) + \sum_{j \in \bar{J}(x^*)} \beta_j g_j(x^*) + \sum_{t \in \bar{T}} \gamma_t h_t(x^*) \\ &> \sum_{i=1}^{m} \alpha_i f_i(\hat{x}) + \sum_{j \in \bar{J}(x^*)} \beta_j g_j(\hat{x}) + \sum_{t \in \bar{T}} \gamma_t h_t(\hat{x}) \end{aligned}$$

From the assumptions we obtain

$$\begin{aligned}
&\sum_{i=1}^{m}\alpha_i f_i(x^*)+\sum_{j\in\bar{J}(x^*)}\beta_j g_j(x^*)+\sum_{t\in\bar{T}}\gamma_t h_t(x^*)\\
&>\sum_{i=1}^{m}\alpha_i f_i(\hat{x})+\sum_{j\in\bar{J}(\hat{x})}\beta_j g_j(\hat{x})+\sum_{t\in\bar{T}}\gamma_t h_t(\hat{x})\\
&\geq\sum_{i=1}^{m}\alpha_i f_i(x^*)+\sum_{j\in\bar{J}(x^*)}\beta_j g_j(x^*)+\sum_{t\in\bar{T}}\gamma_t h_t(x^*)\\
&\quad+\sum_{i=1}^{m}\alpha_i\langle\xi_i,x-x^*\rangle+\sum_{j\in\bar{J}(x^*)}\beta_j\langle\zeta_j,x-x^*\rangle+\sum_{t\in\bar{J}(x^*)}\gamma_t\langle\eta_t,x-x^*\rangle\\
&=\sum_{i=1}^{m}\alpha_i f_i(x^*)+\sum_{j\in\bar{J}(x^*)}\beta_j g_j(x^*)+\sum_{t\in\bar{T}}\gamma_t h_t(x^*)
\end{aligned}$$

and the contradiction complete the proof.

□

We can also prove sufficient optimality conditions for (MOSIP) by means of further relaxations on generalized convexity requirements as follows.

Theorem 4.2. *Let $x^* \in M$ and $J(x^*)\cup T \neq \emptyset$. Assume that there exist $\alpha \in \mathbb{R}^p_+$ and scalars $\beta_t \geq 0, t \in T(x^*)$ with $\beta_t \neq 0$ for finitely many indices t, such that (3.4) hold. If all the functions $f_i,\ i \in \{1,2,\ldots,p \mid \alpha_i \neq 0\}$, $g_t,\ j \in \bar{J}(x^*) = \{j \in J(x^*) \mid \beta_j \neq 0\}$ are generalized quasiconvex and $h_t,\ t \in \bar{T} = \{t \in T \mid \gamma_t \neq 0\}$ are infine , then x^* is weakly efficient.*

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Manuscript received 6 December 2014
revised 2 March 2015
accepted for publication 6 March 2015

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