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A REPRESENTATION THEOREM FOR BISHOP-PHELPS CONES*

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Abstract: The paper presents a theorem for representation a given cone as a Bishop–Phelps cone in normed spaces and studies interior and separation properties for Bishop–Phelps cones. The representation theorem is formulated in the form of necessary and sufficient condition and provides relationship between the parameters (the linear functional, the norm, and the scalar coefficient of the norm) determining the Bishop-Phelps cone. The necessity is given in reflexive Banach spaces. The representation theorem is used to establish the theorem on interior of the Bishop–Phelps cone representing a given cone, and the theorem on the separation property. It is shown that every Bishop–Phelps cone in finite dimensional space satisfies the separation property for the nonlinear separation theorem. The theorems on the representation, on the interior and on the separation property studied in this paper are comprehensively illustrated on examples in finite and infinite dimensional spaces.

Key words: *Bishop–Phelps cone, representation theorem, augmented dual cone, nonlinear separation theorem*

Mathematics Subject Classification: *65K10, 90C26, 90C29, 90C30, 46N10*

1 Introduction

In this paper we present a theorem for representation a given cone as a Bishop-Phelps cone (BP cone for short) in normed spaces.

This cone was introduced by Bishop and Phelps in 1962 (see [2,16]). Since then BP cones played a crucial role in many investigations on characterization of supporting elements of certain subsets of normed linear spaces (e.g, see [1, 3, 17]).

Most remarkable characteristics for BP cones were given in [6,8,15]. Petschke has shown that every nontrivial convex cone C with a closed and bounded base in a real normed space is representable as a BP cone [15, Theorem 3.2]. Another basic result which follows from this theorem says that [15, Theorem 3.4], every nontrivial convex cone in a finite dimensional space is representable as a BP cone if and only if it is closed and pointed (see also [6, Theorem 4.4] and [8, Proposition 2.18, Proposition 2.19]).

In this paper, we present a necessary and sufficient condition for representation of a given cone as a BP cone, where we do not impose any condition on the existence of a base or on a base. Generally, not every cone in infinite dimensional spaces has a base, or some cones may have a base which is unbounded.

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The representation theorem presented in this paper uses the given norm of the normed space and there is no need to prove the existence of an additional equivalent norm.

This theorem guarantees not only the availability for a representation of some class of cones as a BP cone, but also provides relationship between the parameters (the linear functional, the norm, and the scalar coefficient of the norm) determining this BP cone, and explicitly defines the BP cone which equals the given cone. This is of great importance, because it provides an analytical expression for the given cone and thus provides a convenient mathematical tool in investigations. There are many existence and characterization theorems for optimal solutions in the literature where the objective space is assumed to be partially ordered by a BP cone (see e.g. [4, 5, 9]).

In this paper we also prove theorems on interior of the BP cone and by using the representation theorem we present a detailed discussion on the relationship between the augmented dual cones and the representation of the interior of BP cones.

Finally, by using the representation theorem we show that every BP cone and its conic neighborhood satisfy the nonlinear separation property in finite dimensional spaces. This property was suggested by R. Kasimbeyli in [10] where he proved that two cones satisfying the separation property, can be separated by some BP cone. Such a BP cone is defined by some element from the augmented dual cone. Note that, the augmented dual cones, BP cones and the nonlinear separation theorem are used to develop optimality conditions and solution approaches for a certain class of nonconvex optimization problems in both single objective optimization theory and in vector optimization (see e.g, [4, 9, 11–14]).

The theorems on the representation, on the interior and on the separation property studied in this paper are comprehensively illustrated on examples in finite and infinite dimensional spaces.

The paper is organized as follows. Section 2 gives some preliminaries. The general nonlinear separation property and separation theorems are given in Section 3. In this section a sufficient condition for separation property is also presented. The representation theorem, the theorems on the interior of BP cones and the relationship between the nonlinear separation property and the BP cones are given in Section 4. Section 5 presents illustrative examples and detailed discussions of the representation and characterization theorems in finite and infinite dimensional spaces. Finally, Section 6 draws some conclusions from the paper.

2 Preliminaries

In this section, we recall some concepts of cones, separability and proper efficiency. Throughout the paper, we will assume always, unless stated specifically otherwise, that:

(i) Y is a reflexive Banach space with dual space Y^*, and $C \subsetneq Y$ is a cone which contains nonzero elements;

(ii) $\mathsf{cl}(S)$, $\mathsf{bd}(S)$, $\mathsf{int}(S)$, and $\mathsf{co}(S)$ denote the *closure* (in the norm topology), the *boundary*, the *interior*, and the *convex hull* of a set S, respectively;

(iii) R_+ and R_{++} denote the sets of nonnegative and positive real numbers, respectively;

The unit sphere and unit ball of Y are denoted by

$$U = \{y \in Y : \|y\| = 1\} \tag{2.1}$$

and

$$B = \{y \in Y : \|y\| \leq 1\},$$

respectively.

A nonempty subset C of Y is called a *cone* if

$$y \in C, \lambda \geq 0 \Rightarrow \lambda y \in C.$$

Pointedness of C means that

$$C \cap (-C) = \{0_Y\}.$$

$$\mathsf{cone}(S) = \{\lambda s : \lambda \geq 0 \text{ and } s \in S\}$$

denotes the cone *generated* by a set S.

$C_U = C \cap U = \{y \in C : \|y\| = 1\}$ denotes the *base norm* of the cone C . The term *base norm* is justified by the obvious assertion that $C = \text{cone}(C_U)$, and is firstly used in [18].

Recall that the dual cone C^* of C and its quasi-interior $C^\#$ are defined by

$$C^* = \{y^* \in Y^* : y^*(y) \geq 0 \text{ for all } y \in C\} \tag{2.2}$$

and

$$C^\# = \{y^* \in Y^* : y^*(y) > 0 \text{ for all } y \in C \setminus \{0\}\}, \tag{2.3}$$

respectively.

The following three cones called augmented dual cones of C were introduced in [10].

$$C^{a*} = \{(y^*, \alpha) \in C^\# \times R_+ : y^*(y) - \alpha\|y\| \geq 0 \text{ for all } y \in C\}, \tag{2.4}$$

$$C^{a\circ} = \{(y^*, \alpha) \in C^\# \times R_+ : y^*(y) - \alpha\|y\| > 0 \text{ for all } y \in \text{int}(C)\}, \tag{2.5}$$

and

$$C^{a\#} = \{(y^*, \alpha) \in C^\# \times R_+ : y^*(y) - \alpha\|y\| > 0 \text{ for all } y \in C \setminus \{0\}\}, \tag{2.6}$$

where C is assumed to have a nonempty interior in the definition of $C^{a\circ}$.

3 Separation Property

In this section, we recall the separation property introduced by R. Kasimbeyli in [10]. This property enables to separate two cones (which are not necessarily convex, having only the vertex in common) by a level set of some monotonically increasing (with respect to the ordering cone) sublinear function.

A general sufficient condition for two cones to satisfy the separation property is proved in [12]. In this section, we present this theorem without proof.

Definition 3.1. Let C and K be closed cones of a normed space $(Y, \|\cdot\|)$ with base norms C_U and K_U, respectively. Let $K_U^\partial = K_U \cap bd(K)$, and let $\widetilde{C}$ and $\widetilde{K}^\partial$ be the closures of the sets $co(C_U)$ and $co(K_U^\partial \cup \{0_Y\})$, respectively. The cones C and K are said to have the separation property with respect to the norm $\|\cdot\|$ if

$$\widetilde{C} \cap \widetilde{K}^\partial = \emptyset. \tag{3.1}$$

Definition 3.2. Let C and K be nonempty cones of a normed space $(Y, \|\cdot\|)$ with $\text{int}(K) \neq \emptyset$. A cone K is called a conic neighborhood of C if $(C \setminus \{0_Y\}) \subset \text{int}(K)$. For a positive real number ε, a cone $C_\varepsilon = cone(C_U + \varepsilon B)$ is called an ε-conic neighborhood of C.

The following two theorems proved in [10, Theorems 4.3 and 4.4] concern the existence of a pair $(y^*, \alpha) \in C^{a\#}$ for which the corresponding sublevel set $S(y^*, \alpha)$ of the strongly monotonically increasing sublinear function $g(y) = y^*(y) + \alpha\|y\|$ separates the given cones C and K, where $S(y^*, \alpha)$ is defined as

$$S(y^*, \alpha) = \{y \in Y : y^*(y) + \alpha\|y\| \leq 0.\} \tag{3.2}$$

Theorem 3.3. *Let C and K be closed cones in a reflexive Banach space $(Y, \|\cdot\|)$. Assume that the cones $-C$ and K satisfy the separation property defined in Definition* 3.1*,*

$$-\widetilde{C} \cap \widetilde{K}^{\partial} = \emptyset, \tag{3.3}$$

Then, $C^{a\#} \neq \emptyset$, and there exists a pair $(y^*, \alpha) \in C^{a\#}$ such that the corresponding sublevel set $S(y^*, \alpha)$ of the strongly monotonically increasing sublinear function $g(y) = y^*(y) + \alpha\|y\|$ separates the cones $-C$ and $bd(K)$ in the following sense:

$$y^*(y) + \alpha\|y\| < 0 \leq y^*(z) + \alpha\|z\| \tag{3.4}$$

for all $y \in -C \setminus \{0_Y\}$, and $z \in bd(K)$. In this case the cone $-C$ is pointed.

Conversely, if there exists a pair $(y^*, \alpha) \in C^{a\#}$ such that the corresponding sublevel set $S(y^*, \alpha)$ of the strongly monotonically increasing sublinear function $g(y) = y^*(y) + \alpha\|y\|$ separates the cones $-C$ and $bd(K)$ in the sense of (3.4) *and if either the cone C is closed and convex or $(Y, \|\cdot\|)$ is a finite dimensional space, then the cones $-C$ and K satisfy the separation property* (3.3)*.*

Remark 3.4. It follows from Theorem 3.3 that two cones satisfying the separation property (3.3) can be separated by a BP cone defined for some pair $(y^*, \alpha) \in C^{a\#}$, and conversely, if there exists a pair $(y^*, \alpha) \in C^{a\#}$ such that the corresponding BP cone separates the given cones, then these cones satisfy the separation property (3.3).

Theorem 3.5. *Let C be a closed cone of a reflexive Banach space $(Y, \|\cdot\|_Y)$, and let C_ε be its ε-conic neighborhood for a real number $\varepsilon \in (0, 1)$. Suppose that C and C_ε satisfy the separation property given in Definition* 3.1*. Then, there exists a pair $(y^*, \alpha) \in C^{a\#}$ such that*

$$-C \setminus \{0_Y\} \subset \text{int}(S(y^*, \alpha)) \subset -C_\varepsilon, \tag{3.5}$$

where $\text{int}(S(y^*, \alpha))$ can be defined as

$$\text{int}(S(y^*, \alpha)) = \{y \in Y : y^*(y) + \alpha\|y\| < 0\}. \tag{3.6}$$

Remark 3.6. It follows from definition of the augmented dual cone that every nontrivial cone $C \subset Y$ is a subset of the BP cone

$$C(y^*, \alpha) = \{y \in Y : \alpha\|y\| \leq y^*(y)\}$$

if $(y^*, \alpha) \in C^{a*}$.. Theorem 3.5 strengthens this assertion by saying that for a cone C satisfying conditions of this theorem, there exists a BP cone which contains the given cone being contained in the ε-conic neighborhood of C for a real number $\varepsilon \in (0, 1)$. In other words, under the conditions of Theorem 3.5, there exists a BP cone which is as close to the given cone as possible.

The following theorem is presented in [12, Lemma 3] and gives a general sufficient condition for the separation property in R^n.

Theorem 3.7. *Let C be a closed convex cone in R^n. Assume that there exist a pair $(y^*, \alpha) \in R^n \times R_{++}$ such that,*

$$cl(co(C_U)) = \{y \in B : y^*(y) \geq \alpha\}. \tag{3.7}$$

Then for an arbitrary closed cone $K \subset R^n$ with $C \cap K = \{0\}$, the cones C and K satisfy the separation property given in Definition 3.1.

4 Main Results

In this section, we show that the condition (3.7) of Theorem 3.7 is necessary and sufficient for the representation of a given cone as a BP cone in reflexive Banach spaces. Moreover, this BP cone is defined for the same norm (which is the given norm of the normed space) and the same pair $(y^*, \alpha) \in Y^* \times R_{++}$ used in condition (3.7). Thus, the theorem presented in this paper guarantees not only the availability of a representation of some class of cones as a BP cone, but also gives its exact expression by explaining properties of parameters determining this BP cone.

The following definition for BP cones is used in this paper:

Definition 4.1. Let $(Y, \|\cdot\|)$ be a real normed space. For some positive number $\alpha > 0$ and some continuous linear functional y^* from the dual space Y^* the cone

$$C(y^*, \alpha) = \{y \in Y : \alpha\|y\| \leq y^*(y)\} \tag{4.1}$$

is called Bishop-Phelps cone. In this definition, the triple $(y^*, \alpha, \|\cdot\|)$ will be referred to as parameters determining the given BP cone.

In the original definition of Bishop and Phelps, it is required that $\|y^*\|_* = 1$ and $\alpha \in (0, 1]$.

Some authors (see for example, [4, 6, 8]) do not use the constant α and the assumption $\|y^*\|_* = 1$. This paper follows Definition 4.1. It easily follows from the definition that every BP cone is closed and pointed [8, 15].

We first present a sufficient condition for characterizing interior of every BP cone. Then, the representation theorem will be presented. We begin with the following lemma characterizing the quasi-interior of the augmented dual cone.

Lemma 4.2. *Let $C \in Y$ be a given nonempty cone. If $(y^*, \alpha) \in C^{a\#}$ then $\|y^*\|_* > \alpha$.*

Proof. Let $(y^*, \alpha) \in C^{a\#}$ and let $y \in C \setminus \{0\}$. Then

$$0 < y^*(y) - \alpha\|y\| \leq \|y^*\|_*\|y\| - \alpha\|y\| = \|y\|(\|y^*\|_* - \alpha),$$

which completes the proof. □

The following theorem characterizes interior of BP cones. Note that this theorem is given in [10] in a slightly different setting, therefore we present this theorem without the proof for which we refer reader to [10, Lemma 3.6].

Theorem 4.3. *Let $C(y^*, \alpha) = \{y : y^*(y) \geq \alpha\|y\|\}$ be a given BP cone for some pair $(y^*, \alpha) \in C^{a*}$. If $(y^*, \alpha) \in C^{a\#}$ then*

$$int(C(y^*, \alpha)) = \{y : y^*(y) > \alpha\|y\|\} \neq \emptyset. \tag{4.2}$$

Remark 4.4. A sufficient condition on the characterization of interior of BP cones was also presented in [8, Theorem 2.5(b)], which is equivalent to that of Theorem 4.3. Below we present examples which demonstrate that the condition of Theorem 4.3 is not necessary in general (see, Section 5 and Remark 5.2).

Now we present the representation theorem.

Theorem 4.5. *Let C be a nonempty closed convex cone of a real normed space $(Y, \|\cdot\|)$. Assume that*

$$cl(co(C_U)) = \{y \in B : y^*(y) \geq \alpha\} \tag{4.3}$$

for some $(y^*, \alpha) \in Y^* \times R_{++}$. Then C is representable as a Bishop–Phelps cone with the same norm and the same pair (y^*, α) defining the condition (4.3). Conversely, if $C = \{y \in Y : y^*(y) - \alpha\|y\| \geq 0\}$ is a Bishop–Phelps cone of a reflexive Banach space $(Y, \|\cdot\|)$, then C satisfies condition (4.3).

Proof. **Necessity.** Let $y^* \in Y^*$ and let $\alpha > 0$ be a real number, and let $C = \{y \in Y : y^*(y) - \alpha\|y\| \geq 0\}$ be a given Bishop–Phelps cone in $(Y, \|\cdot\|)$. Show that C satisfies condition (4.3) with the same $y^* \in Y^*$, $\alpha > 0$ and the same norm.

Let

$$\widetilde{C} = \text{cl}(\text{co}(C_U)). \tag{4.4}$$

It is clear that the base norm of C can be represented as

$$C_U = \{y \in U : y^*(y) - \alpha\|y\| \geq 0\} = \{y \in U : y^*(y) - \alpha \geq 0\}. \tag{4.5}$$

As $\alpha > 0$, in particular, it follows from the definition that C is convex and pointed.

We define the following set

$$D = \{y \in B : \ y^*(y) \geq \alpha\}. \tag{4.6}$$

First we show that

$$\text{co}(C_U) = D. \tag{4.7}$$

Let $y \in \text{co}(C_U)$. Then, by definition of convex hull, there exists a set of nonnegative numbers β_i, $i \in I$ such that, y can be represented as

$$y = \sum_{i \in I} \beta_i y_i, \quad \text{where } y_i \in C_U \text{ and } \sum_{i \in I} \beta_i = 1.$$

Clearly, $y \in B$. On the other hand

$$y^*(y) = \sum_{i \in I} \beta_i y^*(y_i) \geq \alpha.$$

Then, from (4.6) we have $y \in D$; that is, $\text{co}(C_U) \subset D$.

Now, let $y \in D$. We will show that $y \in \text{co}(C_U)$.

If $\|y\| = 1$ then $y \in U$ and the inclusion $y \in C_U \subset \text{co}(C_U)$ follows from (4.5).

Consider the case $\|y\| < 1$, that is $y \in \text{int}(B)$. Denote $\nu = y^*(y)$. Clearly $\nu \geq \alpha$. Take any non-zero vector $b \in Y$ satisfying $y^*(b) = 0$. Consider

$$y_\lambda = y + \lambda b, \quad \lambda \in (-\infty, \infty).$$

We have

$$y^*(y_\lambda) = y^*(y) + \lambda y^*(b) = \nu \geq \alpha. \tag{4.8}$$

As $b \neq \mathbf{0}$, we have $\|y_\lambda\| \to \infty$ if $|\lambda| \to \infty$ which means that $y \notin B$ for sufficiently large values of λ. On the other hand, since $y \in \mathrm{int}(B)$, the inclusion $y_\lambda \in \mathrm{int}(B)$ holds for sufficiently small in absolute value numbers $\lambda > 0$ and $\lambda < 0$. Then, since $\|y_\lambda\|$ is a weakly upper semicontinuous function of λ, and B is weakly compact, there exist numbers $\lambda_1 > 0$ and $\lambda_2 < 0$ such that the corresponding points $y_1 \doteq y_{\lambda_1}$ and $y_2 \doteq y_{\lambda_2}$ belong to the boundary of B (as maximum values of $\|y_\lambda\|$ w.r.t. $\lambda > 0$ and $\lambda < 0$ respectively). That is,

$$y_i \in U, \quad i = 1, 2.$$

These inclusions together with (4.8) and (4.5) imply that $y_i \in C_U, \quad i = 1, 2.$

Finally, denoting $\lambda' = \lambda_1/(\lambda_1 - \lambda_2)$, it is not difficult to check that,

$$\lambda' \in (0, 1) \text{ and } y = (1 - \lambda')y_1 + \lambda' y_2.$$

Therefore, $y \in \mathrm{co(C_U)}$, which means that $D \subset \mathrm{co(C_U)}$.

Thus, we have shown that the relation (4.7) is true. From this relation, we have

$$\widetilde{C} = \{y \in B : \ y^*(y) \geq \alpha\},$$

and the proof of Necessity is completed.

Sufficiency. Now let C be a nonempty closed convex cone of Y, and suppose that condition (4.3) is satisfied for some $(y^*, \alpha) \in Y^* \times R$ with $\alpha > 0$. Show that C is representable as a Bishop–Phelps cone, that is show that $C = C(y^*, \alpha)$.

Let $y \in C \setminus \{0\}_Y$. Then there exists a positive real number β such that $\beta y \in C_U$, and hence $\beta y \in cl(co(C_U))$. Then by condition (4.3) we have:

$$y^*(\beta y) \geq \alpha.$$

Then, since $\beta y \in C_U$, we have $\alpha = \alpha\|\beta y\|$, and $y^*(\beta y) \geq \alpha\|\beta y\|$. Thus, $y^*(y) \geq \alpha\|y\|$, which means that $C \subset C(y^*, \alpha)$.

Now let $y \in C(y^*, \alpha)$. Then for every $y \in C(y^*, \alpha)$ there exists a scalar $\beta > 0$ such that $\beta y \in U \cap C(y^*, \alpha)$ and therefore

$$y^*(\beta y) \geq \alpha\|\beta y\| = \alpha,$$

which implies by condition (4.3) that $\beta y \in cl(co(C_U))$. Since C is a closed and convex cone, we obtain $y \in C$, which establishes the inclusion $C(y^*, \alpha) \subset C$, and the proof of the theorem is completed. □

The next theorem establishes an additional property for parameters of the BP cone representing the given cone.

Theorem 4.6. *Let $C \subset Y$ be a given cone which is representable as a BP cone. If $C(y^*, \alpha)$ is a BP cone representing the given cone C, then $(y^*, \alpha) \in C^{a*} \setminus C^{a\#}$.*

Proof. Assume that $C = C(y^*, \alpha)$ for some pair $(y^*, \alpha) \in C^{a*}$. Then $C = C(y^*, \alpha) = \{y \in Y : y^*(y) - \alpha\|y\| \geq 0, \}$ and clearly $(y^*, \alpha) \in C^{a*}$ by the definition of C^{a*}. Obviously, the set $\{y \in Y : y^*(y) - \alpha\|y\| = 0\}$ represents the boundary of C, and since C is assumed to contain nonzero elements, there exists some $y \in C \setminus \{0\}$ in the boundary with $y^*(y) - \alpha\|y\| = 0$ which completes the proof. □

Remark 4.7. Theorem 4.6 explains an interesting property of the BP cone representing the given cone. Since Theorem 4.3 gives only sufficient condition, it is not clear whether the interior of the BP cone representing the given cone can be represented by formula (4.2) or not. It seems that if the pair $(y^*, \alpha) \in C^{a*}$ does not belong to $C^{a\#}$ then there is no guarantee that the interior of this cone can be described by the set $\{y : y^*(y) > \alpha\|y\|\}$ which may be empty. In the next section we present examples which illustrate this situation. The following theorem explains an easy way, how the given cone C can be included in the interior of a (closest to C) BP cone whose interior is represented by (4.2).

Theorem 4.8. *Let $C \subset Y$ be a given cone and let $(y^*, \alpha) \in C^{a*}$ with $\alpha > 0$, be the pair for which the representation property (4.3) is satisfied. Let $C(y^*, \alpha)$ be the BP cone representing the given cone C. Then $(y^*, \beta) \in C^{a\#}$ for every $\beta \in (0, \alpha)$, and*

$$(C \setminus \{0\}) \subset int(C(y^*, \beta)) = \{y : y^*(y) > \beta\|y\|\} \neq \emptyset.$$

Proof. Let $C = C(y^*, \alpha)$, where $(y^*, \alpha) \in C^{a*}$ with $\alpha > 0$. Then by Lemma [10, Lemma 3.2 (ii)] $(y^*, \beta) \in C^{a\#}$ for every $\beta \in (0, \alpha)$. Now let $y \in C(y^*, \alpha)$, and $\beta \in (0, \alpha)$ be arbitrary elements. Then

$$y^*(y) - \beta\|y\| > y^*(y) - \alpha\|y\| \geq 0,$$

which means by Theorem 4.3 that $y \in int(C(y^*, \beta))$ and the proof is completed. □

The following theorem establishes that every BP cone in R^n, satisfies the separation property together with its ε conic neighborhood.

Theorem 4.9. *Let C be a BP cone in R^n. Then for every $\varepsilon \in (0, 1)$, cones C and $bd(C_\varepsilon)$ satisfy the separation property given in Definition 3.1.*

Proof. Since $C \cap bd(C_\varepsilon) = \{0\}$, the proof follows from theorems 4.5 and 3.7 and the definition of the ε conic neighborhood of a cone (see Definition 3.2). □

5 Illustrative Examples

In this section we present illustrative examples for the representation and separation theorems, and for the theorem on interior of BP cones in both finite and infinite dimensional spaces.

5.1 Example 1

The example presented in this section illustrates the representation theorem for a given convex and pointed cone, where different norms are analyzed. In the case when the representation is available, we also present the analysis of the theorem on interior of the BP cone. The relation between the separation property and the representation theorem is analyzed by considering some nonconvex cone.

Let $Y = R^2$, $C = \{(s, s) : \ s \geq 0\}$ and let $K = \{(s, t) : s \leq 0 \text{ or } t \leq 0\}$. Since the separation property and the representation theorem both depend on the norm induced, we will investigate this example with respect to all three norms l_1, l_2, and l_∞ in R^2. Let $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ be the l_1, l_2 and l_∞ norms respectively, where $\|y\|_1 = |s| + |t|$, $\|y\|_2 = \sqrt{s^2 + t^2}$, and $\|y\|_\infty = max\{|s|, |t|\}$, for every $y = (s, t) \in R^2$.

For $y^* = (1,1)$, it is easy to calculate $\alpha = max\{y^*(\frac{y}{\|y\|}) : y \in R^2.\}$ It is clear that $\alpha_1 = 1$, $\alpha_2 = \sqrt{2}$, and $\alpha_\infty = 2$, where α_1, α_2 and α_∞ are the maximum values of α calculated with respect to l_1, l_2 and l_∞ norms respectively.

Let B_1, B_2 and B_∞ be the unit balls w.r.t. l_1, l_2 and l_∞ norms respectively. Denote by C_{U_1}, C_{U_2} and C_{U_∞} the base norms of C w.r.t. l_1, l_2 and l_∞ norms respectively. Then, we have

$$\widetilde{C}_1 = \text{cl}(\text{co}(C_{U_1})) = \{(1/2, 1/2)\},$$

$$\widetilde{C}_2 = \text{cl}(\text{co}(C_{U_2})) = \{(\sqrt{2}/2, \sqrt{2}/2)\}$$

and

$$\widetilde{C}_\infty = \text{cl}(\text{co}(C_{U_\infty})) = \{(1,1)\}.$$

5.1.1 Illustration of the Separation Property

Noting that $K^\partial_{U_1} = K^\partial_{U_2} = K^\partial_{U_\infty} = \{(0,1),(1,0)\}$, one has $\widetilde{K}^\partial = \{(s,t) : s \geq 0, t \geq 0 \text{ and } s+t \leq 1\}$. Since $\widetilde{K}^\partial \cap \tilde{C}_1 = \{(1/2,1/2)\} \neq \emptyset$, the separation property is not satisfied for these cones for the case of l_1 norm. It is easy to see that for $y^* = (1,1)$ and $\alpha_1 = 1$,

$$\{y \in B_1 : y^*(y) \geq \alpha_1\} = \{(s,t) : s+t = 1, s \geq 0, t \geq 0\} \neq \tilde{C}_1,$$

which shows that C does not satisfy assumption (3.7) for l_1 norm.

On the other hand, $\tilde{C}_2 = \{y \in B_2 : \ y^*(y) \geq \alpha_2\} = \{(\sqrt{2}/2, \sqrt{2}/2)\}$. Hence C satisfies assumption (3.7) for l_2 norm and since $C \cap K = \{0\}$, by Theorem 3.7 the cones C and K must satisfy the separation property given in Definition 3.1. It is easy to check that $\widetilde{K}^\partial \cap \tilde{C} = \emptyset$ and hence C and K satisfy the separation property given in Definition 3.1 for l_2 norm.

It can be shown in a similar way that $\tilde{C}_\infty = \text{cl}(\text{co}(C_{U_\infty})) = \{y \in B_\infty : y^*(y) \geq \alpha_\infty\} = \{(1,1)\}$. Hence C satisfies assumption (3.7) and therefore the separation property is satisfied for the cones C and K in the case of l_∞ norm.

5.1.2 Illustration of the Representation Theorem

In this subsection we discuss the representation condition (4.3) and show that similar to the separation property for this example, it is not satisfied in the case of l_1 norm, but is satisfied for l_2, and l_∞ norms.

For $y^* = (1,1)$ and $\alpha_1 = 1$, it has been shown in the previous subsection that $\text{cl}(\text{co}(C_{U_1})) \neq \{y \in B_1 : y^*(y) \geq \alpha_1\}$. Therefore, condition (4.3) of the representation theorem is not satisfied for C in the case of l_1 norm. In such a case Theorem 4.5 says that C can not be represented as a Bishop–Phelps cone with the parameters $y^* = (1,1), \alpha_1 = 1$, and the l_1 norm. Actually we can conclude by the construction that, there does not exist a pair (y^*, α) with $C = \{y = (s,t) : y^*(y) \geq \alpha\|y\|_1\}$. It seems that the smallest Bishop–Phelps cone (with $y^* = (1,1), \alpha_1 = 1$ and l_1 norm) containing C is a cone $\{(s,t) : s+t \geq |s| + |t|\}$ which equals R^2_+.

In the case of l_2 norm, C satisfies assumption (4.3). Then by Theorem 4.5 it should be represented as a Bishop–Phelps cone. Indeed, for $y^* = (1,1)$ and $\alpha_2 = \sqrt{2}$ we have

$$C = \{y = (s,t) : y^*(y) \geq \alpha_2\|y\|_2\} = \{(s,t) : s+t \geq \sqrt{2}\sqrt{s^2+t^2}.\} \tag{5.1}$$

The relation

$$\widetilde{C}_\infty = \mathrm{cl}(\mathrm{co}(C_{U_\infty})) = \{y \in B_\infty : y^*(y) \geq \alpha_\infty\} = \{(1,1)\}$$

shows also that C can be represented as a Bishop–Phelps cone in the following form:

$$C = \{y = (s,t) : y^*(y) \geq \alpha_\infty \|y\|_\infty\} = \{(s,t) : s + t \geq 2max\{|s|, |t|\}.\} \tag{5.2}$$

5.1.3 Illustration of the Theorem on Interior

Finally we illustrate Theorem 4.3 on interior of the BP cone. By this theorem, if $(y^*, \alpha) \in C^{a\#}$ then the interior of BP cone $C(y^*, \alpha)$ is representable by (4.2). Since $C = \{(s,s) : s \geq 0\}$, it is clear that $int(C) = \emptyset$. Now use the representation (5.1) of this cone as a BP cone in the case of l_2 norm, and check whether $(y^*, \alpha_2) \in C^{a\#}$ where $y^* = (1,1)$ and $\alpha_2 = \sqrt{2}/2$. For the point $y = (1,1) \in C \setminus \{0\}$ we have

$$y^*(y) - \alpha_2 \|y\|_2 = s + t - \sqrt{2}/2\sqrt{s^2 + t^2} = 1 + 1 - \sqrt{2}\sqrt{1^2 + 1^2} = 0,$$

which shows that $(y^*, \alpha_2) \notin C^{a\#}$. Consequently, since

$$\{y = (s,t) : y^*(y) \geq \alpha_2 \|x\|_2\} = \{(s,t) : s + t \geq \sqrt{2}\sqrt{s^2 + t^2}\} = \{(s,s) : s \geq 0\}$$

we have

$$\{y = (s,t) : y^*(y) > \alpha_2 \|y\|_2\} = \{(s,s) : s + s > \sqrt{2}\sqrt{s^2 + s^2}\} = \emptyset.$$

It can easily be checked that the same interpretation is also valid for the representation (5.2).

5.2 Example 2

Let $C = R^n_+$. Due to Kasimbeyli [10, Theprem 5.9], this cone satisfies the separation property (3.1) with respect to l_1 norm for arbitrary n with $y^* = (1, \ldots, 1)$, and $\alpha = 1$. Then by Theorem 4.9, it satisfies the representation property, and its BP representation is given by

$$C(y^*, \alpha)_{l_1} = \{(y_1, \ldots, y_n) : \sum_{i=1}^n y_i - \sum_{i=1}^n |y_i| \geq 0\}.$$

It is evident that $int(R^n_+) = \{(y_1, \ldots, y_n) : y_i > 0, i = 1, \ldots, n\} \neq \emptyset$. On the other hand, $(y^*, \alpha) \in C^{a*} \setminus C^{a\#}$ (see, Theorem 4.6) and thus the interior of the BP cone $C(y^*, \alpha)_{l_1}$ cannot be represented by

$$\{(y_1, \ldots, y_n) : \sum_{i=1}^n y_i - \sum_{i=1}^n |y_i| > 0\},$$

which is empty set.

The nonnegative orthant R^n_+ has interesting and different interpretations for different values of n and different norms. Therefore we consider each case separately.

In the case $n = 1$, all three norms l_1, l_2, l_∞ have the same formulation and therefore the BP representation of R_+ for all the three norms is given by (see also [6, Example 2.6 (a)])

$$C(1,1) = \{y \in R : y - |y| \geq 0\}.$$

In the case $n = 2$, the l_2 and the l_∞ norms representations of R^2_+ are respectively:

$$C((1,1),1)_{l_2} = \{(y_1, y_2) : y_1 + y_2 - \sqrt{y_1^2 + y_2^2} \geq 0\},$$

and

$$C((1,1),1)_{l_\infty} = \{(y_1, y_2) : y_1 + y_2 - \max(|y_1|, |y_2|) \geq 0\}.$$

It is remarkable that, the condition (4.3) of the representation theorem is not satisfied for R^n_+ with $n \geq 3$ in the cases of l_2 and l_∞ norms. Hence the nonnegative orthant of R^n with $n \geq 3$ can not be represented as a BP cone in the cases of l_2 and l_∞ norms. For example, consider the vector $y = (-1, 2, 2) \in R^3$. Let $y_3^* = (1, 1, 1)$ and let $\alpha = 1$. Then, the relation

$$y_3^*(y) \geq \alpha \|y\|$$

is satisfied for both l_2 and l_∞ norms, but $y \notin R^3_+$.

Remark 5.1. The nonnegative orthant of R^n is also considered in [6, Example 2.6 (c)], where the BP cone representation with l_1 norm and the interpretation on the interior are not correct.

5.3 Example 3

Hamel and Jahn investigated the dual of the cone given in the following example, see [7] and [8, Example 2.15].

Let $Y = R^2$, $C = \{(y_1, y_2) : y_1 \geq 0, y_2 = 0\}$. Jahn represented this cone as a BP cone with $y^* = (1, 0), \alpha = 1$ and the l_2 norm. We analyze this example using condition (4.3) with $y^* = (1, 0), \alpha = 1$ and different norms and explain why the given cone with the given parameters can (or cannot) be represented as a BP cone. We use the similar notation as in subsection 5.1.

5.3.1 The case of l_1 norm

Clearly, $\widetilde{C}_1 \triangleq \mathrm{cl}(\mathrm{co}(C_{U_1})) = \{(1, 0)\}$. On the other hand

$$D_1 \triangleq \{y = (y_1, y_2) \in B_1 : y^*(y) \geq \alpha\} = \{y = (y_1, y_2) : \|y\|_1 \leq 1, y_1 \geq 1\} = \{(1, 0)\}.$$

Hence $\widetilde{C}_1 = D_1$, which means that (4.3) is satisfied and therefore C can be represented as a BP cone:

$$C_1((1,0),1) = \{(y_1, y_2) : y_1 \geq |y_1| + |y_2|\} = \{(y_1, y_2) : y_1 \geq 0, y_2 = 0\}.$$

5.3.2 The case of l_2 norm

By the similar analysis it can easily be shown that $\widetilde{C}_2 = D_2$, which means that (4.3) is satisfied for l_2 norm, and therefore C can be represented as a BP cone in the following form:

$$C_2((1,0),1) = \{(y_1, y_2) : y_1 \geq \sqrt{y_1^2 + y_2^2}\} = \{(y_1, y_2) : y_1 \geq 0, y_2 = 0\}.$$

5.3.3 The case of l_∞ norm

The simple calculation shows that in this case the representation condition (4.3) is not satisfied:

$$D_\infty \triangleq \{y = (y_1, y_2) \in B_\infty : y^*(y) \geq \alpha\} = \{y = (y_1, y_2) : \|y\|_\infty \leq 1, y_1 \geq 1\}$$

$$= \{(y_1, y_2) : -1 \leq y_2 \leq 1, y_1 = 1\} \neq \widetilde{C}_\infty = \{(1, 0)\}.$$

This shows by Theorem 4.5 that C can not be represented as a BP cone with the parameters $y^* = (1, 0), \alpha = 1$ and the l_∞ norm. Indeed, the BP cone $C_\infty((1, 0), 1)$ with these parameters, is different from C, as shown below:

$$C_\infty((1, 0), 1) = \{(y_1, y_2) : y_1 \geq max\{|y_1|, |y_2|\}\}$$

$$= \{(y_1, y_2) : -y_1 \leq y_2 \leq y_1, y_1 \geq 0\} \neq C.$$

The BP cone obtained in this section for l_∞ norm, takes us to another interesting example that is considered in the following section.

5.4 Example 4

Let

$$C = \{(y_1, y_2) : -y_1 \leq y_2 \leq y_1, y_1 \geq 0\}.$$

We will illustrate the representation theorem for different norms and the theorem on the interior.

5.4.1 The case of l_1 norm

Let $y^* = (1, 0), \alpha_1 = 1/2$. Then it is easy to check that

$$D_1 \triangleq \{y = (y_1, y_2) \in B_1 : y^*(y) \geq \alpha_1\} = \{(y_1, y_2) : |y_1| + |y_2| \leq 1, y_1 \geq 1/2\} = \tilde{C}_1,$$

where $\tilde{C}_1 = \mathrm{cl}(\mathrm{co}(C_{U_1}))$. This means that (4.3) is satisfied and therefore C can be represented as a BP cone:

$$C = C_1((1, 0), 1/2) = \{(y_1, y_2) : y_1 \geq 1/2(|y_1| + |y_2|)\}.$$

Clearly $(y^*, \alpha_1) = ((1, 0), 1/2) \in C^{a*} \setminus C^{a\#}$, nevertheless the interior of $C_1((1, 0), 1/2)$ can be represented by (4.2):

$$int(C) = int(C_1((1, 0), 1/2)) = \{(y_1, y_2) : y_1 > 1/2(|y_1| + |y_2|)\}.$$

Remark 5.2. The example presented in Subsection 5.1 illustrates Theorem 4.3 on interior of the BP cone in the case when interior of the original cone C is empty. This example illustrates the case if $(y^*, \alpha) \notin C^{a\#}$ then interior of BP cone $C(y^*, \alpha)$ is not representable by (4.2).

Example presented in Subsection 5.2 illustrates the representation theorem and the theorem on the interior of BP cones in the case when interior of the original cone C is not empty, but since the BP representation of this cone uses pair (y^*, α) which does not belong to $C^{a\#}$, its interior can not be represented in the form of (4.2).

Example 5.4 illustrates the case when interior of the original cone is not empty and the cone is represented as a BP cone by using some pair (y^*, α) which does not belong to $C^{a\#}$. However the relation (4.2) for the interior is satisfied (see Section 5.4.1). This example demonstrates that the condition (4.2) of Theorem 4.3 is not necessary in general.

5.4.2 The case of l_2 norm

Let $y^* = (1,0), \alpha_2 = \sqrt{2}/2$. Then

$$D_2 \triangleq \{y = (y_1, y_2) \in B_2 : y^*(y) \geq \alpha_2\} = \{(y_1, y_2) : \sqrt{y_1^2 + y_2^2} \leq 1, y_1 \geq \sqrt{2}/2\} = \tilde{C}_2,$$

where $\tilde{C}_2 = \mathrm{cl}(\mathrm{co}(C_{U_2}))$. This means that (4.3) is satisfied and therefore C can be represented as a BP cone in the following form:

$$C = C_2((1,0), \sqrt{2}/2) = \{(y_1, y_2) : y_1 \geq \sqrt{2}/2\sqrt{y_1^2 + y_2^2}\}.$$

Again $(y^*, \alpha_2) = ((1,0), \sqrt{2}/2) \in C^{a*} \setminus C^{a\#}$, and the interior of $C_1((1,0), \sqrt{2}/2)$ is represented by (4.2):

$$int(C) = int(C_1((1,0), \sqrt{2}/2)) = \{(y_1, y_2) : y_1 > (\sqrt{2}/2)\sqrt{y_1^2 + y_2^2}\}.$$

5.4.3 The case of l_∞ norm

Let $y^* = (1,0), \alpha_\infty = 1$. Then it can easily be shown that the conditions of the representation theorem are satisfied and C can be represented as a BP cone by using the l_∞ norm in the form

$$C = C_\infty((1,0), 1) = \{(y_1, y_2) : y_1 \geq max\{|y_1|, |y_2|\}\}.$$

The interior of $C_\infty((1,0), 1)$ is represented in the form:

$$int(C) = int(C_\infty((1,0), 1)) = \{(y_1, y_2) : y_1 > max\{|y_1|, |y_2|\}\}.$$

The BP cones in the following three examples are considered in [6, Example 2.7]. Here we present a detailed analysis and comprehensive illustrations of the theorems given in this paper, on these examples.

5.5 Example 5

Let Y be the Banach space l^1, and let C be the nonnegative orthant of l^1. Then $(l^1)^* = l^\infty$ and taking $y^*(y) = \sum_{i=1}^{\infty} y_i$ and $\alpha = 1$ it can easily be shown that the representation condition (4.3) is satisfied and the BP cone

$$C(y^*, \alpha) = \{y \in l^1 : \sum_{i=1}^{\infty} y_i \geq \sum_{i=1}^{\infty} |y_i|\}$$

is a cone representing the nonnegative orthant of l^1. The augmented dual cone C^{a*} and its quasi interior $C^{a\#}$ can easily be calculated for the nonnegative orthant C of l^1.

By definition of the augmented dual cone, we have

$$\mathbb{C}^{a*} = \left\{((w_1, w_2, \ldots), \alpha) \in \mathbb{C}^\# \times \mathbb{R}_+ : \sum_{i=1}^{\infty} w_i y_i \geq \alpha \sum_{i=1}^{\infty} y_i \quad \text{for all } (y_1, y_2, \ldots) \in \mathbb{C}\right\}$$

or

$$\mathbb{C}^{a*} = \{((w_1, w_2, \ldots), \alpha) : w_i > 0, i = 1, 2, \ldots, 0 \leq \alpha \leq \inf\{w_1, w_2, \ldots\}\}$$

and

$$\mathbb{C}^{a\#} = \{((w_1, w_2, \ldots), \alpha) : 0 \leq \alpha < \inf\{w_1, w_2, \ldots\}\}.$$

It is evident that $(y^*, \alpha) \in \mathbb{C}^{a*} \setminus \mathbb{C}^{a\#}$, where $y^*(y) = \sum_{i=1}^{\infty} y_i$ and $\alpha = 1$.

Note that similar interpretation on the interior presented in subsection 5.2 for R^2_+ with l_1 norm is also valid for the nonnegative orthant of l^1.

In this case we have:

$$\{(y_1, y_2, \ldots) : \sum_{i=1}^{\infty} y_i - \sum_{i=1}^{\infty} |y_i| > 0\} = \emptyset,$$

thus, the relation (4.2) for the interior of a BP cone is not satisfied. Again, by [10, Lemma 3.2 (ii)]), we have that the pair (y^*, β) belongs to $C^{a\#}$ for every $\beta \in (0, 1)$. Then by Theorem 4.3, the interior of BP cone $C(y^*, \beta)$ with $\beta \in (0, 1)$, can be represented in the following form:

$$int(C(y^*, \beta)) = \{(y_1, y_2, \ldots) : \sum_{i=1}^{\infty} y_i - \beta \sum_{i=1}^{\infty} |y_i| > 0\} \neq \emptyset$$

and $C \setminus \{0_{l^1}\} \subset int(C(y^*, \beta))$ for every $\beta \in (0, 1)$.

5.6 Example 6

Let Y be the Banach space $C([0, 1])$ of continuous functions $y(\cdot)$ on $[0, 1]$, and let C be a cone of functions in $C([0, 1])$ defined as follows:

$$C = \{y \in C([0, 1]) : 0 \leq y(1) = \max_{t\in[0,1]} |y(t)|\}.$$

First we will show that C is a closed convex and pointed cone of $C([0, 1])$, then the illustrations of the representation and separation theorems, and the theorem on the interior will be given.

5.6.1 C is a closed convex and pointed cone

The closedness is obvious. To show the convexity, let $x(\cdot)$ and $y(\cdot)$ be elements from C. Then

$$(x + y)(1) = \max_{t\in[0,1]} |x(t)| + \max_{t\in[0,1]} |y(t)| \geq \max_{t\in[0,1]} |x(t) + y(t)|.$$

On the other hand,

$$0 \leq (x + y)(1) \leq \max_{t\in[0,1]} |x(t) + y(t)|,$$

which implies that $x(\cdot) + y(\cdot) \in C$ and thus the convexity is proved. Now let $y(\cdot) \in C$. Then $y(1) = \max_{t\in[0,1]} |y(t)|$. Assuming that $-y(\cdot) \in C$, we obtain $-y(1) = \max_{t\in[0,1]} |-y(t)|$, and hence $y(1) \geq 0$ and $-y(1) \geq 0$ which implies $y(1) = 0$. Finally, the equality $\max_{t\in[0,1]} |y(t)| = 0$ leads $y(\cdot) = 0$, which proves the pointedness of C.

5.6.2 Illustration of the Representation Theorem

Let $y^*(y(\cdot)) = y(1)$ and $\alpha = 1$. Then by the definition,

$$C_U = \{y \in C([0, 1]) : \sup_{t\in[0,1]} |y(t)| = y(1) = 1\},$$

and hence

$$co(C_U) = \{y \in C([0,1]) : y(\cdot) = \lambda y_1(\cdot) + (1-\lambda)y_2(\cdot), y_i(\cdot) \in C_U, \lambda \in [0,1], i = 1,2,\}.$$

Let $y(\cdot) = \lambda y_1(\cdot) + (1-\lambda)y_2(\cdot) \in co(C_U)$ where $\max_{t\in[0,1]} |y_i(t)| = y_i(1) = 1, i = 1,2$ and $\lambda \in [0,1]$. Then

$$\max_{t\in[0,1]} |y(t)| = \max_{t\in[0,1]} |\lambda y_1(t) + (1-\lambda)y_2(t)|$$
$$\leq \lambda \max_{t\in[0,1]} |y_1(t)| + (1-\lambda) \max_{t\in[0,1]} |y_2(t)| = \lambda y_1(1) + (1-\lambda)y_2(1) = 1 = y(1).$$

On the other hand

$$y(1) = \lambda y_1(1) + (1-\lambda)y_2(1) = 1 \leq \max_{t\in[0,1]} |\lambda y_1(t) + (1-\lambda)y_2(t)| = \max_{t\in[0,1]} |y(t)|,$$

which implies $\max_{t\in[0,1]} |y(t)| = y(1) = 1$ and therefore we obtain that $\mathrm{co}(C_U) = C_U$. Now due to closedness of the unit sphere in $C([0,1])$ we obtain:

$$\widetilde{C} = \mathrm{cl}(\mathrm{co}(C_U)) = C_U.$$

Now let

$$D = \{y \in C([0,1]) : \sup_{t\in[0,1]} |y(t)| \leq 1, y^*(y(\cdot)) \geq \alpha\}.$$

Then since $y^*(y(\cdot)) = y(1)$ and $\alpha = 1$ we have:

$$D = \{y \in C([0,1]) : \sup_{t\in[0,1]} |y(t)| \leq 1, y(1) \geq 1\}$$
$$= \{y \in C([0,1]) : \sup_{t\in[0,1]} |y(t)| = y(1) = 1\}.$$

The expressions obtained for $\widetilde{C}$ and D shows that these sets are equal and hence the representation condition (4.3) is satisfied. Thus C can obviously be represented as the following BP cone:

$$C(y^*,\alpha) = \{y \in C([0,1]) : y(1) \geq \sup_{t\in[0,1]} |y(t)|\}. \tag{5.3}$$

5.6.3 Illustration of the Theorem on Interior

Theorem 4.3 on interior of BP cones says that the interior of the BP cone $C(y^*,\alpha)$ can be represented by (4.2) if $(y^*,\alpha) \in C^{a\#}$. However, Theorem 4.6 says that $(y^*,\alpha) \in C^{a*} \setminus C^{a\#}$ for every BP cone $C(y^*,\alpha)$ representing the given cone C. Unfortunately, as it is illustrated in the examples presented in this section, in such a situation the set defined in (4.2) and representing the interior of the BP cone may be empty. Regarding the cone C that we consider in this subsection, it is easy to conclude that the interior of this cone cannot be represented by (4.2) because we have:

$$\{y \in C([0,1]) : y(1) > \sup_{t\in[0,1]} |y(t)|\} = \emptyset.$$

In this situation one can use Theorem 4.8 to construct a BP cone containing the given one (without the zero element) in its interior. That is by Theorem 4.8, we have that $(y^*,\beta) \in C^{a\#}$ for every $\beta \in (0,1)$, where $y^*(y(\cdot)) = y(1)$ and

$$C \setminus \{0\} = C(y^*,1) \setminus \{0\} \subset int(C(y^*,\beta)) = \{y \in C[0,1] : y(1) > \beta \sup_{t\in[0,1]} |y(t)|\}. \tag{5.4}$$

We now give an independent proof of this assertion.

Let $y \in C(y^*, 1) \setminus \{0\}$ and let $\beta \in (0,1)$. Then we have:

$$y(1) - \beta \sup_{t\in[0,1]} |y(t)| > y(1) - \sup_{t\in[0,1]} |y(t)| \geq 0, \tag{5.5}$$

and therefore there exists a positive number ε such that

$$y(1) - \beta \sup_{t\in[0,1]} |y(t)| > \varepsilon. \tag{5.6}$$

Now let ε_1 be a positive number such that $\varepsilon_1 < \varepsilon/(1+\beta)$, and let $x \in B_C = \{x(\cdot) \in C[0,1] : \sup_{t\in[0,1]} |x(t)| \leq 1.\}$ We show that $\tilde{y} = y + \varepsilon_1 x \in C(y^*, \beta)$.

$$\tilde{y}(1) - \beta \sup_{t\in[0,1]} |\tilde{y}(t)| = y(1) + \varepsilon_1 x(1) - \beta \sup_{t\in[0,1]} |y(t) + \varepsilon_1 x(t)|$$

$$\geq y(1) + \varepsilon_1 x(1) - \beta \sup_{t\in[0,1]} |y(t)| - \beta \sup_{t\in[0,1]} |\varepsilon_1 x(t)| > \varepsilon_1 x(1) + \varepsilon - \varepsilon_1 \beta \sup_{t\in[0,1]} |x(t)|$$

$$\geq \varepsilon + \varepsilon_1 (x(1) - \beta) \geq \varepsilon + \varepsilon_1(-1-\beta) > 0,$$

and the assertion (5.4) is proved.

5.6.4 Illustration of the Separation Property

It follows from (5.4) that

$$C(y^*, \beta) = \{y \in C([0,1]) : y(1) \geq \beta \sup_{t\in[0,1]} |y(t)|\}$$

with $0 < \beta < 1$, is a BP cone containing $C \setminus \{0\}$ in its interior and this cone becomes as close to C as β is close to 1. Then it is clear that $C(y^*, \beta)$ is a conic neighborhood of C and its boundary $bd(C(y^*, \beta))$ is a nonconvex cone satisfying $bd(C(y^*, \beta)) \cap C = \{0_{C([0,1])}\}$. Denote $K = bd(C(y^*, \beta))$:

$$K = \{y \in C([0,1]) : y(1) = \beta \sup_{t\in[0,1]} |y(t)|\}.$$

In this subsection we check whether the separation property $\widetilde{C} \cap \widetilde{K}^{\partial} = \emptyset$ given by (3.1) is satisfied for K and C.

In Section 5.6.2 it is shown that $\widetilde{C} = \mathrm{cl}(\mathrm{co}(C_U)) = C_U$, and hence

$$\widetilde{C} = \{y \in C([0,1]) : \sup_{t\in[0,1]} |y(t)| = y(1) = 1\}.$$

Let $\beta \in (0,1)$ be chosen. Then we have:

$$K_U^{\partial} = K_U \cap bd(K) = K_U = \{x \in C([0,1]) : \sup_{t\in[0,1]} |x(t)| = 1, x(1) = \beta\},$$

and thus

$$\widetilde{K}^{\partial} = co(K_U^{\partial} \cup \{0_{C([0,1])}\}) = \{x \in C([0,1]) : \sup_{t\in[0,1]} |x(t)| = \lambda, x(1) = \lambda\beta, \lambda \in [0,1].\}$$

Since $x(1) = \lambda\beta < 1$ for every $x(\cdot) \in \widetilde{K}^{\partial}$, and $y(1) = 1$ for every $y \in \widetilde{C}$, we obtain $\widetilde{C} \cap \widetilde{K}^{\partial} = \emptyset$ and hence it is shown that the separation property for K and C is satisfied.

Remark 5.3. The BP cone of this subsection is also considered in [6, Example 2.7 (a)], where the interpretation is not correct.

5.7 Example 7

Let Y be the Banach space $L([0,1])$ of Lebesgue integrable functions $y(\cdot)$ on $[0,1]$, and let C be a cone of nonnegative functions in $L([0,1])$ defined as follows:

$$C = \{y \in L([0,1]) : y(t) \geq 0, \text{ for a.e. } t \in [0,1]\}.$$

5.7.1 Illustration of the Representation Theorem

Let $y^*(y(\cdot)) = \int_0^1 y(t)dt$ and let $\alpha = 1$. Then

$$\begin{aligned} C_U &= \{y \in L([0,1]) : \int_0^1 |y(t)|dt = 1, y(t) \geq 0, t \in [0,1]\} \\ &= \{y \in L([0,1]) : \int_0^1 y(t)dt = 1, y(t) \geq 0, t \in [0,1]\}. \end{aligned}$$

Obviously C_U is convex and closed and hence

$$\widetilde{C} = \mathrm{cl}(\mathrm{co}(C_U)) = C_U.$$

Now let

$$D = \{y \in L([0,1]) : \|y(\cdot)\| \leq 1, y^*(y(\cdot)) \geq \alpha\}.$$

Then

$$D = \{y \in L([0,1]) : \int_0^1 |y(t)|dt \leq 1, \int_0^1 y(t)dt \geq 1\},$$

which implies that $\widetilde{C} = D$ and hence the representation condition (4.3) is satisfied. Thus C can be represented as the following BP cone:

$$C(y^*, \alpha) = \left\{ y \in L([0,1]) : \int_0^1 y(t)dt \geq \int_0^1 |y(t)|dt \right\},$$

or

$$C(y^*, \alpha) = \left\{ y \in L([0,1]) : \int_0^1 y(t)dt = \int_0^1 |y(t)|dt \right\}.$$

5.8 Example 8

Let Y be the reflexive Banach space $L^2([0,1])$, and let C be a cone of nonnegative functions in $L^2([0,1])$ defined as follows:

$$C = \{y \in L^2([0,1]) : y(t) \geq 0, \text{ for a.e. } t \in [0,1]\}.$$

Let $y^*(y(\cdot)) = \int_0^1 y(t)dt$ and let $\alpha = 1$. Then the corresponding BP cone $C(y^*, \alpha)$ is defined as follows:

$$C(y^*, \alpha) = \left\{ y \in L^2([0,1]) : \int_0^1 y(t)dt \geq \left(\int_0^1 |y(t)|^2 dt \right)^{1/2} \right\},$$

or, equivalently

$$C(y^*, \alpha) = \left\{ y \in L^2([0,1]) : \int_0^1 y(t)dt = \left(\int_0^1 |y(t)|^2 dt \right)^{1/2} \right\}.$$

It is easy to see that function $y(t) = t$ belongs to the cone C but does not belong to $C(y^*, \alpha)$, which demonstrates that the BP cone $C(y^*, \alpha)$ does not represent the cone C of nonnegative functions in $L^2([0,1])$. In this case, we can conclude by Theorem 4.5 that, the representation condition (4.3) is not satisfied and the sets $cl(cone(C_U))$ and D are not equal, where

$$C_U = \left\{ y \in L^2([0,1]) : \left(\int_0^1 |y(t)|^2 dt \right)^{1/2} = 1, y(t) \geq 0, \text{ for a.e. } t \in [0,1] \right\}$$

and by the definition

$$D = \left\{ y \in L^2([0,1]) : \left(\int_0^1 |y(t)|^2 dt \right)^{1/2} \leq 1, \int_0^1 y(t)dt \geq 1 \right\},$$

or

$$D = \left\{ y \in L^2([0,1]) : \int_0^1 y(t)dt = \left(\int_0^1 |y(t)|^2 dt \right)^{1/2} = 1 \right\}.$$

Note that the cone C of this example has a base $\{y \in C : \int_0^1 y(t)dt = 1\}$ which is unbounded.

6 Conclusions

In this paper, we present a representation theorem which establishes that every cone of a real normed space satisfying condition (4.3) is representable as a Bishop–Phelps cone and conversely, every BP cone of a reflexive Banach space, representing given cone C satisfies this condition. This theorem is formulated without any conditions neither on the existence of a base, nor on a base itself. The condition (4.3) uses the given norm of the normed space (the presented theorem does not need to construct another norm for representation) and gives an explicit formulation of how a given cone can be expressed in the form of a BP cone. Note that such a representation theorem appears in the literature firstly. Earlier a representation theorem was given by Petschke, who showed that every nontrivial convex cone C with a closed and bounded base in a real normed space is representable as a BP cone.

The paper studies two important properties of BP cones in relation with the representation theorem. One of them is the interior of BP cones, the other one is the separation property used in the nonlinear separation theorem for not necessarily convex cones. The paper presents characterization theorems on interior of BP cones. It has been shown that every BP cone satisfies the separation property together with its ε conic neighborhood in R^n. This property is very important in both theoretical investigations and practical applications in nonconvex analysis (see e.g. [4, 5, 9–14]).

The paper presents eight illustrative examples in finite and infinite dimensional spaces where the representation theorem, and the theorems on the interior and theorem on the separation property are all comprehensively investigated. In these examples both the standard positive orthant and special cones are considered. For example, it is shown that the positive orthant of R^n for $n = 1, 2$ satisfies the representation theorem and the BP cone representation of the positive orthant is given explicitly for all three norms l_1, l_2, l_∞. It is shown that the positive orthant of $(R^n, \|\cdot\|_{l_1})$ can be represented as a BP cone for every n, while R^n_+ with l_2 and l_∞ norms, cannot be represented as a BP cone for $n \geq 3$. We have also shown that the positive orthant of l^1 and the one of $L([0,1])$ can be represented as a BP cone, and these representations are given. It has been shown that the positive orthant of $L^2([0,1])$ cannot be represented as a BP cone. The paper also considers some special cones in R^2 and $C([0,1])$ along with the illustrations of theorems given in the paper.

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