Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 13, No. 1, pp. 75–103, January 2017

WEAK CONJUGATE DUALITY FOR NONCONVEX VECTOR OPTIMIZATION

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Abstract: In this work, weak conjugate map, weak biconjugate map and weak subdifferential of a set-valued map are defined by using notions of supremum/infimum of a set and vectorial norm, and relationships among these notions are examined. Furthermore, necessary and sufficient conditions for weakly subdifferentiability of a set-valued map are given. It is proved that under some assumptions Lipschitz set-valued maps are weakly subdifferentiable. By using these notions a dual of unconstrained nonconvex vector optimization problems is constructed, and weak duality theorem is presented. Stability of primal problem is defined and it is proved that the stability of primal problem implies the strong duality. Furthermore, some stability conditions are presented. By using a special perturbation function weak Fenchel dual problem of constrained vector optimization problem is constructed and at the end, an example of a nonconvex constrained vector optimization problem which can not be solved by using Lagrange dual problem [25] but can be solved by using weak Fenchel dual problem is given.

Key words: *weak conjugate map, weak conjugate duality, weak subdifferential, nonconvex vector optimization*

Mathematics Subject Classification: *90C46, 49N15, 28B99, 49J52, 65K10*

1 Introduction

Conjugate duality theory, based on conjugate maps of perturbation functions, provides a frame work to duality in optimization. In scalar optimization, Fenchel [12] and Rockafellar [30] presented early results of this theory. Ekeland and Temam [11] and Zălinescu [36] improved this theory in topological vector spaces. In addition to these, Boţ, Grad and Wanka [7–9] constructed three dual problems, namely the Lagrange, Fenchel and Fenchel-Lagrange dual problems, obtained by considering special perturbation functions for optimization problems in finite dimensional spaces.

The notion of subdifferential which is based on to support the epigraph of a function at a given boundary point by an hyperplane was defined by Rockafellar [30] and Moreau [28] and was developed by Ekeland and Temam [11]. Some optimality conditions were given for convex functions by using this notion [11, 30]. However, it may be impossible to support from below the epigraph of most of the nonconvex functions by hyperplanes. So, the classical subdifferential theory is insufficient to give optimality conditions for nonconvex optimization problems. Thus, some researchers used supporting cones instead of supporting hyperplanes [5, 6, 16–18, 29]. In the light of to support a nonconvex set with supporting conic surfaces instead of hyperplanes Kasimov [15] defined weak conjugate maps and weak subdifferential by using the set of concave functions in place of linear functionals in definitions of conjugate function and subdifferential. Moreover, by using these functions and weak subdifferential, Azimov and Gasimov [5,6] constructed a conjugate dual problem, presented necessary and sufficient optimality conditions and duality theorems involving lower Lipschitz functions. In addition, Küçük et al. [21], by using two special perturbation functions in the construction of dual problems given by Azimov and Kasimov [6], defined two dual problems namely weak Fenchel and weak Fenchel-Lagrange dual problems and gave optimality conditions for nonconvex optimization problems.

Tanino and Sawaragi [34] extended the conjugate duality theory to vector optimization by defining conjugate maps of vector functions using maximal element of a set in a partially ordered finite dimensional space. Furthermore, Tanino [33] extended this theory to convex vector optimization problems in partially ordered topological vector spaces by using the concept of supremum of a set and constructed a conjugate dual problem for convex vector optimization problems. Moreover, the notion generalized weak subdifferential was defined for nonconvex functions with values in an ordered vector space and optimality condition for nonconvex vector optimization problems were studied by Küçük et al. [22–24]. Conjugate duality theory was extended to set-valued vector optimization problems by Song [31, 32] and Kawasaki [19]. Furthermore, Li et. al. [25] constructed two dual problems for constrained set-valued optimization problems. For further development on this area one can see [1, 3, 4, 7–10, 13, 20, 26, 27].

In this article, motivated by works [5, 6, 33] weak conjugate map, weak biconjugate map and weak subdifferential of a set-valued map in a partially ordered topological vector space are defined. Relationships among weak conjugate map, weak biconjugate map and weak subdifferential are examined. In addition, necessary and sufficient conditions for weak subdifferentiability of set-valued maps are presented. Moreover, it is proved that under some assumptions Lipschitz set-valued maps are weakly subdifferentiable. Then a weak conjugate dual problem of an unconstrained vector optimization problem of perturbation function is constructed. Relations between the optimal objective maps of primal and dual problem are investigated. Weak duality theorem is presented and stability of primal problem is defined. Moreover, it is proved that the stability of a primal problem implies the strong duality. Furthermore, conditions for stability of a primal problem are given. Finally, by choosing a special perturbation function in the construction of weak conjugate dual problem weak Fenchel dual problem for constrained vector optimization problem is constructed and an example of a nonconvex constrained vector optimization problem which can not be solved by using Lagrange dual problem [25] but can be solved by using weak Fenchel dual problem is given.

This paper is organized as follows: in Section 2, some notions and preliminary results are given, in Section 3 weak conjugate map, weak biconjugate map of a set-valued map are defined and the relations between these notions are examined, in Section 4, weak subdifferential for set-valued maps is defined, weak subdifferentiability conditions are presented and relations among weak subdifferential, weak conjugate map and weak biconjugate map are examined. In Section 5, by using the weak conjugate map of the perturbation function, a new dual problem for the given vector optimization problem is constructed. The inclusion relationships between the image sets of primal problem and dual problem are given. In addition, weak duality and strong duality assertions are proved. In Section 6, by using a special perturbation function weak Fenchel dual problem is obtained.

2 Mathematical Preliminaries

In this section, by using the concepts of supremum and infimum of a set and vectorial norm, we define weak conjugate map and weak biconjugate map for a set-valued map. Furthermore, we examine relations among a set-valued map, weak conjugate and weak biconjugate maps of this set-valued map. First, let us remind the basic notions and preliminary results.

Let Y be a real topological vector space which is partially ordered by a pointed, closed, convex cone C with nonempty interior $intC$ in Y. We use the following ordering relations:

$$y_1 \underset{C}{\leq} y_2 \iff y_2 - y_1 \in C \text{ and } y_1 \underset{C}{<} y_2 \iff y_2 - y_1 \in intC.$$

We add two imaginary points $+\infty$, $-\infty$ which satisfy the following:

$$-\infty \underset{C}{<} y \underset{C}{<} +\infty, \quad (\pm\infty) + y = y + (\pm\infty) = (\pm\infty) \; for \; all \; y \in Y$$

$$(\pm\infty)+(\pm\infty) = (\pm\infty) \quad \lambda(\pm\infty) = (\pm\infty) \; for \; all \; \lambda > 0 \text{ and } \lambda(\pm\infty) = (\mp\infty) \; for \; all \; \lambda < 0$$

to Y and denote the extended space by $\overline{Y}$.

The sum $+\infty - \infty$ is not considered since we can avoid it.

Definition 2.1 ([33]). Given a set $Z \subset \overline{Y}$, the set $A(Z)$ of all points above Z and the set $B(Z)$ of all points below Z are defined by

$$A(Z) = \{y \in \overline{Y} \mid y \underset{C}{>} y' \text{ for some } y' \in Z\}$$

$$B(Z) = \{y \in \overline{Y} \mid y \underset{C}{<} y' \text{ for some } y' \in Z\},$$

respectively.

Definition 2.2 ([33]). Given a set $Z \subset \overline{Y}$,

i) a point $\bar{y} \in \overline{Y}$ is said to be weakly maximal point of Z if $\bar{y} \in Z$ and $\bar{y} \notin B(Z)$, i.e. if $\bar{y} \in Z$ and there is no $y' \in Z$ such that $\bar{y} \underset{C}{<} y'$. The set of all weakly maximal points of Z is called the weak maximum of Z and is denoted by wmax$Z$.

ii) a point $\bar{y} \in \overline{Y}$ is said to be weakly minimal point of Z if $\bar{y} \in Z$ and $\bar{y} \notin A(Z)$, i.e. if $\bar{y} \in Z$ and there is no $y' \in Z$ such that $y' \underset{C}{<} \bar{y}$. The set of all weakly minimal points of Z is called the weak minimum of Z and is denoted by wmin$Z$.

iii) a point $\bar{y} \in \overline{Y}$ is said to be a supremal point of Z if $\bar{y} \notin B(Z)$ and $B(\bar{y}) \subset B(Z)$, i.e. if there is no $y \in Z$ such that $\bar{y} \underset{C}{<} y$ and if the relation $y' \underset{C}{<} \bar{y}$ implies the existence of some $y \in Z$ such that $y' \underset{C}{<} y$. The set of all supremal points of Z is called the supremum of Z and denoted by Sup$Z$.

iv) a point $\bar{y} \in \overline{Y}$ is said to be an infimal point of Z if $\bar{y} \notin A(Z)$ and $A(\bar{y}) \subset A(Z)$, i.e. if there is no $y \in Z$ such that $y \underset{C}{<} \bar{y}$ and if the relation $\bar{y} \underset{C}{<} y'$ implies the existence of some $y \in Z$ such that $y \underset{C}{<} y'$. The set of all infimal points of Z is called infimum of Z and denoted by Inf$Z$.

Definition 2.3 ([14]). Let X and Y be real linear spaces, and let C be a convex cone in Y. A map $\|\!| \cdot |\!\|: X \to C$ is called a vectorial norm, if the following conditions are satisfied for all $x,\ z \in X$ and all $\lambda \in \mathbb{R}$:

(a) $\|\!|x|\!\| = 0_Y \Leftrightarrow x = 0_X$;

(b) $\|\!|\lambda x|\!\| = |\lambda| \|\!|x|\!\|$;

(c) $\|\!|x+z|\!\| \underset{C}{\leq} \|\!|x|\!\| + \|\!|z|\!\|$ (Triangle inequality)

In particular, if $Y = \mathbb{R}$ and $C = \mathbb{R}_+$ then the map $\|\!| \cdot |\!\|$ is called a norm and denoted $\| \cdot \|$.

Lemma 2.4 shows that characterizations of supremum and infimum in $\mathbb{R}$ can be extended to partially ordered topological vector spaces.

Lemma 2.4. *Let $\emptyset \neq Z \subset Y$ be a given set and $\bar{x} \in Y$ be a given point. Then*

i) $\bar{x} \in \mathrm{Inf}Z$ *if and only if there is no* $z \in Z$ *such that* $z \underset{C}{<} \bar{x}$ *and for each* $0 \underset{C}{<} \varepsilon$ *there exists* $x(\varepsilon) \in Z$ *such that* $x(\varepsilon) \underset{C}{<} \bar{x} + \varepsilon$.

ii) $\bar{x} \in \mathrm{Sup}Z$ *if and only if there is no* $z \in Z$ *such that* $\bar{x} \underset{C}{<} z$ *and for each* $0 \underset{C}{<} \varepsilon$ *there exists* $x(\varepsilon) \in Z$ *such that* $\bar{x} - \varepsilon \underset{C}{<} x(\varepsilon)$.

Proof. i) Let $\bar{x} \in \mathrm{Inf}Z$ and $0 \underset{C}{<} \varepsilon$. From the definition of infimum $\bar{x} \notin A(Z)$, i.e. there is no $z \in Z$ such that $z \underset{C}{<} \bar{x}$. So, the first condition is satisfied.

Since $0 \underset{C}{<} \varepsilon$ we have $\bar{x} \underset{C}{<} \bar{x} + \varepsilon$. Hence, $\bar{x} + \varepsilon \in A(\bar{x})$. Then from the definition of infimum $\bar{x} + \varepsilon \in A(\bar{x}) \subset A(Z)$ which means there is $x(\varepsilon) \in Z$ such that $x(\varepsilon) \underset{C}{<} \bar{x} + \varepsilon$. So the second statement is satisfied.

Conversely, assume that there is no $z \in Z$ such that $z \underset{C}{<} \bar{x}$ and for each $0 \underset{C}{<} \varepsilon$ there exists $x(\varepsilon) \in Z$ such that $x(\varepsilon) \underset{C}{<} \bar{x} + \varepsilon$. The first condition implies $\bar{x} \notin A(Z)$. Now, we will show $A(\bar{x}) \subset A(Z)$. Let $z \in A(\bar{x})$, i.e. $\bar{x} \underset{C}{<} z$. By setting $\varepsilon = z - \bar{x}$, from the hypothesis we obtain the existence of an element $x(\varepsilon) \in Z$ which satisfies $x(\varepsilon) \underset{C}{<} \bar{x} + \varepsilon = z$ that means $z \in A(Z)$. Hence, $\bar{x} \in \mathrm{Inf}Z$.

ii) The characterization for supremum can be proved similarly. □

Proposition 2.5 ([33]). *Let $Z \subset \overline{Y}$ be a given set. Then $B(Z) = B(\mathrm{Sup}Z)$ and $A(Z) = A(\mathrm{Inf}Z)$.*

Proposition 2.6 ([33]). *For $F_1, F_2 : X \rightrightarrows \overline{Y}$ where X is an arbitrary set*

$$\mathrm{Sup} \bigcup_{x \in X} [F_1(x) + F_2(x)] = \mathrm{Sup} \bigcup_{x \in X} [F_1(x) + \mathrm{Sup}F_2(x)]$$

where the sum $+\infty - \infty$ is assumed not to occur.

Corollary 2.7 ([33]). *Let X be a set, $F : X \rightrightarrows \overline{Y}$ be a set-valued map. Then*

$$\text{Sup} \bigcup_{x\in X} F(x) = \text{Sup} \bigcup_{x\in X} \text{Sup}F(x).$$

Lemma 2.8. *Let $D, E \subset Y$ be given sets. If* $\text{Sup}E \subseteq \text{Sup}D \cup A(\text{Sup}D)$ *and* $\text{Sup}E \subseteq \text{Sup}D \cup B(\text{Sup}D)$*, then* $\text{Sup}D = \text{Sup}E$.

Proof. From inclusions in the hypothesis and as $A(\text{Sup}D) \cap B(\text{Sup}D) = \emptyset$ (Proposition 4.5 in [35]) we have

$$\begin{aligned} \text{Sup}E &\subseteq (\text{Sup}D \cup A(\text{Sup}D)) \cap (\text{Sup}D \cup B(\text{Sup}D)) \\ &= \text{Sup}D \cup \emptyset \\ &= \text{Sup}D. \end{aligned} \tag{2.1}$$

Let us show that $\text{Sup}D \subseteq \text{Sup}E$. Assume the contrary that $\bar{y} \notin \text{Sup}E$ for some $\bar{y} \in \text{Sup}D$. Then either $\bar{y} \in A(\text{Sup}E)$ or $\bar{y} \in B(\text{Sup}E)$. We claim that $\bar{y} \notin B(\text{Sup}E)$. Otherwise, there exists $a \in SupE$ such that $\bar{y} \underset{C}{<} a$. As $a \in \text{Sup}E \subseteq SupD$ we have $\bar{y} \in B(\text{Sup}D)$ which contradicts to $\bar{y} \in \text{Sup}D$. So, $\bar{y} \in A(\text{Sup}E)$. Thus, there exists $a \in \text{Sup}E$ such that $a \underset{C}{<} \bar{y}$. Because $a \in \text{Sup}E$ and $\text{Sup}E \subseteq \text{Sup}D$ we get $a \in \text{Sup}D$. Hence, $\bar{y} \in A(\text{Sup}D)$ which contradicts to $\bar{y} \in \text{Sup}D$. Thus, $\bar{y} \in \text{Sup}E$. So we obtain

$$\text{Sup}D \subseteq \text{Sup}E. \tag{2.2}$$

From (2.1) and (2.2) we have $\text{Sup}D = \text{Sup}E$. □

Lemma 2.9. *Let X, Y, Z be topological vector spaces, Y be partially ordered by closed, convex, pointed cone C with nonempty interior int$C$, let $F : X \rightrightarrows \overline{Y}$ be a set-valued map and $a, b : Z \times X \to \overline{Y}$ be vector valued functions. If $a(z, x) \underset{C}{\leq} b(z, x)$ for all $z \in Z$ and for all $x \in X$, then*

$$\begin{aligned} \text{Sup}\bigcup_{z\in Z} \text{Inf} \bigcup_{x\in X} [b(z, x) + F(x)] \quad \subset \quad &(\text{Sup} \bigcup_{z\in Z} \text{Inf} \bigcup_{x\in X} [a(z, x) + F(x)])\cup \\ &A(\text{Sup} \bigcup_{z\in Z} \text{Inf} \bigcup_{x\in X} [a(z, x) + F(x)]). \end{aligned}$$

Proof. Assume the contrary that there exists $\bar{y} \in \text{Sup} \bigcup_{z\in Z} \text{Inf} \bigcup_{x\in X} [b(z, x) + F(x)]$ such that $\bar{y} \notin \text{Sup} \bigcup_{z\in Z} \text{Inf} \bigcup_{x\in X} [a(z, x) + F(x)] \cup A(\text{Sup} \bigcup_{z\in Z} \text{Inf} \bigcup_{x\in X} [a(z, x) + F(x)])$. Thus, from Proposition 4.5 in [28] we have

$$\bar{y} \in B(\text{Sup} \bigcup_{z\in Z} \text{Inf} \bigcup_{x\in X} [a(z, x) + F(x)] = B(\bigcup_{z\in Z} \text{Inf} \bigcup_{x\in X} [a(z, x) + F(x)]).$$

Hence, there exist $\bar{z} \in Z$ and $y \in \text{Inf} \bigcup_{x\in X} [a(\bar{z}, x) + F(x)]$ such that

$$\bar{y} \underset{C}{<} y. \tag{2.3}$$

Now, let us prove $y \in B(\bigcup_{z\in Z} \text{Inf} \bigcup_{x\in X} [b(z,x)+F(x)])$. Assume the contrary that $y \notin B(\bigcup_{z\in Z} \text{Inf} \bigcup_{x\in X} [b(z,x)+F(x)]))$ So,

$$y \notin B(\text{Inf} \bigcup_{x\in X} [b(z,x)+F(x)])$$

for all $z \in Z$. In particular,

$$y \notin B(\text{Inf} \bigcup_{x\in X} [b(\bar{z},x)+F(x)]).$$

Hence, we have either $y \in A(\text{Inf} \bigcup_{x\in X} [b(\bar{z},x)+F(x)])$ or $y \in \text{Inf} \bigcup_{x\in X} [b(\bar{z},x)+F(x)]$.

If $y \in A(\text{Inf} \bigcup_{x\in X} [b(\bar{z},x)+F(x)]) = A(\bigcup_{x\in X} [b(\bar{z},x)+F(x)])$, then there exist $x \in X$ and $y' \in F(x)$ such that $b(\bar{z},x)+y' \underset{C}{<} y$. As $a(\bar{z},x) \underset{C}{\leq} b(\bar{z},x)$ we have

$$a(\bar{z},x)+y' \underset{C}{\leq} b(\bar{z},x)+y' \underset{C}{<} y$$

that means

$$y \in A(\bigcup_{x\in X} [a(\bar{z},x)+F(x)]) = A(\text{Inf} \bigcup_{x\in X} [a(\bar{z},x)+F(x)]).$$

This contradicts to $y \in \text{Inf} \bigcup_{x\in X} [a(\bar{z},x)+F(x)]$.
Hence, $y \notin A(\text{Inf} \bigcup_{x\in X} [b(\bar{z},x)+F(x)])$.

If $y \in \text{Inf} \bigcup_{x\in X} [b(\bar{z},x)+F(x)]$, then from (2.3) we get $\bar{y} \in B(\bigcup_{z\in Z} \text{Inf} \bigcup_{x\in X} [b(z,x)+F(x)])$ that contradicts to $\bar{y} \in \text{Sup} \bigcup_{z\in Z} \text{Inf} \bigcup_{x\in X} [b(z,x)+F(x)]$.
Thus, $y \notin \text{Inf} \bigcup_{x\in X} [b(\bar{z},x)+F(x)]$.

Hence, we obtain $y \in B(\bigcup_{z\in Z} \text{Inf} \bigcup_{x\in X} [b(z,x)+F(x)])$.

Because $y \in B(\bigcup_{z\in Z} \text{Inf} \bigcup_{x\in X} [b(z,x)+F(x)])$ there exists $\tilde{y} \in \bigcup_{z\in Z} \text{Inf} \bigcup_{x\in X} [b(z,x)+F(x)]$ such that $y \underset{C}{<} \tilde{y}$. From (2.3) we have $\bar{y} \underset{C}{<} y \underset{C}{<} \tilde{y}$ .Thus,

$$\bar{y} \in B(\bigcup_{z\in Z} \text{Inf} \bigcup_{x\in X} [b(z,x)+F(x)]) = B(\text{Sup} \bigcup_{z\in Z} \text{Inf} \bigcup_{x\in X} [b(z,x)+F(x)])$$

that contradicts to $\bar{y} \in \text{Sup} \bigcup_{z\in Z} \text{Inf} \bigcup_{x\in X} [b(z,x)+F(x)]$.

Hence, $\bar{y} \notin B(\mathrm{Sup} \bigcup_{z\in Z} \mathrm{Inf} \bigcup_{x\in X} [a(z,x) + F(x)]$ which means

$$\bar{y} \in \mathrm{Sup} \bigcup_{z\in Z} \mathrm{Inf} \bigcup_{x\in X} [a(z,x) + F(x)] \cup A(\mathrm{Sup} \bigcup_{z\in Z} \mathrm{Inf} \bigcup_{x\in X} [a(z,x) + F(x)]).$$

□

3 Weak Conjugate Maps

In this section, we define weak conjugate map and weak biconjugate map for a set-valued map by using the concepts of supremum/infimum of a set and vectorial norm. Furthermore, we examine relations among a set-valued map, weak conjugate and weak biconjugate maps of this set-valued map.

Throughout this article, we assume that X, Y are topological vector spaces, Y is partially ordered by closed, convex, pointed cone C with nonempty interior $intC$, $F : X \rightrightarrows \overline{Y}$ is a set-valued map and $\|\!| \cdot |\!\| : X \to C$ is a vectorial norm.

Definition 3.1. Under assumptions given above

a) A set-valued map $F^w : X \times L(X,Y) \times \mathbb{R}_+ \rightrightarrows \overline{Y}$ defined by

$$F^w(x_0, U, c) := \mathrm{Sup} \bigcup_{x\in X} [-c|\!|\!| x - x_0 |\!|\!| + c|\!|\!| x_0 |\!|\!| + U(x) - F(x)]$$

for all $(x_0, U, c) \in X \times L(X,Y) \times \mathbb{R}_+$ is called the weak conjugate map of F.

b) A set-valued map $F^{ww} : X \rightrightarrows \overline{Y}$ defined by

$$F^{ww}(x) := \mathrm{Sup} \bigcup_{(x_0,U,c)\in X\times L(X,Y)\times\mathbb{R}_+} [-c|\!|\!| x - x_0 |\!|\!| + c|\!|\!| x_0 |\!|\!| + U(x) - F^w(x_0, U, c)]$$

for all $x \in X$ is called the weak biconjugate map of F.

Weak biconjugate map F^{ww} can be represented more simply in the following way.

Proposition 3.2. *For each* $x \in X$

$$F^{ww}(x) = \mathrm{Sup} \bigcup_{(U,c)\in L(X,Y)\times\mathbb{R}_+} [c|\!|\!| x |\!|\!| + U(x) - F^w(x, U, c)].$$

Proof. As $\bigcup_{\substack{c\in\mathbb{R}_+ \\ U\in L(X,Y)}} [c|\!|\!| x |\!|\!| + U(x) - F^w(x, U, c)] \subseteq \bigcup_{\substack{x_0\in X \\ U\in L(X,Y) \\ c\in\mathbb{R}_+}} [-c|\!|\!| x - x_0 |\!|\!| + c|\!|\!| x_0 |\!|\!| + U(x) - F^w(x_0, U, c)]$

and the relation $Y_1 \subseteq Y_2$ implies the relation $\mathrm{Sup} Y_1 \subseteq \mathrm{Sup} Y_2 \cup B(\mathrm{Sup} Y_2)$ we have

$$\begin{aligned}\mathrm{Sup} \bigcup_{\substack{c\in\mathbb{R}_+ \\ U\in L(X,Y)}} [c|\!|\!| x |\!|\!| + U(x) - F^w(x, U, c)] &\subseteq \mathrm{Sup} \bigcup_{\substack{x_0\in X \\ U\in L(X,Y) \\ c\in\mathbb{R}_+}} [-c|\!|\!| x - x_0 |\!|\!| + c|\!|\!| x_0 |\!|\!| + U(x) - F^w(x_0, U, c)] \\ &\cup B(\mathrm{Sup} \bigcup_{\substack{x_0\in X \\ U\in L(X,Y) \\ c\in\mathbb{R}_+}} [-c|\!|\!| x - x_0 |\!|\!| + c|\!|\!| x_0 |\!|\!| + U(x) - F^w(x_0, U, c)]) \\ &= F^{ww}(x) \cup B(F^{ww}(x)). \qquad (3.1)\end{aligned}$$

From definitions of F^w and F^{ww} we get

$$\begin{aligned} F^{ww}(x) &= \text{Sup} \bigcup_{\substack{x_0 \in X \\ U \in L(X,Y) \\ c \in \mathbb{R}_+}} [-c\|x - x_0\| + c\|x_0\| + U(x) - F^w(x_0, U, c)] \\ &= \text{Sup} \bigcup_{\substack{x_0 \in X \\ U \in L(X,Y) \\ c \in \mathbb{R}_+}} [-c\|x - x_0\| + c\|x_0\| + U(x) \\ & \quad -\text{Sup} \bigcup_{y \in X} [-c\|y - x_0\| + c\|x_0\| + U(y) - F(y)]] \\ &= \text{Sup} \bigcup_{\substack{x_0 \in X \\ U \in L(X,Y) \\ c \in \mathbb{R}_+}} \text{Inf} \bigcup_{y \in X} [-c\|x - x_0\| + c\|y - x_0\| + U(x) - U(y) + F(y)]. \end{aligned}$$

Let us define $a, b : X \times L(X,Y) \times \mathbb{R}_+ \times X \to Y$

$$\begin{aligned} a(x_0, U, c, y) &= -c\|x - x_0\| + c\|y - x_0\| + U(x) - U(y) \\ b(x_0, U, c, y) &= c\|y - x\| + U(x) - U(y), \end{aligned}$$

respectively. Let $(x_0, U, c) \in X \times L(X,Y) \times \mathbb{R}_+$ be an arbitrary fixed element. As

$$-c\|x - x_0\| + c\|y - x_0\| \underset{C}{\leq} c\|y - x\|$$

for all $y \in X$, we get $a(x_0, U, c, y) \underset{C}{\leq} b(x_0, U, c, y)$ Thus, from Lemma 2.9 we obtain

$$\begin{aligned} & \text{Sup} \bigcup_{\substack{x_0 \in X \\ U \in L(X,Y) \\ c \in \mathbb{R}_+}} \text{Inf} \bigcup_{y \in X} [c\|y - x\| + U(x) - U(y) + F(y)] \subseteq \\ & \text{Sup} \bigcup_{\substack{x_0 \in X \\ U \in L(X,Y) \\ c \in \mathbb{R}_+}} \text{Inf} \bigcup_{y \in X} [-c\|x - x_0\| + c\|y - x_0\| + U(x) - U(y) + F(y)] \\ & \cup A(\text{Sup} \bigcup_{\substack{x_0 \in X \\ U \in L(X,Y) \\ c \in \mathbb{R}_+}} \text{Inf} \bigcup_{y \in X} [-c\|x - x_0\| + c\|y - x_0\| + U(x) - U(y) + F(y)]) \\ & = F^{ww}(x) \cup A(F^{ww}(x)). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} & \text{Sup} \bigcup_{\substack{x_0 \in X \\ U \in L(X,Y) \\ c \in \mathbb{R}_+}} \text{Inf} \bigcup_{y \in X} [c\|y - x\| + U(x) - U(y) + F(y)] \\ & = \text{Sup} \bigcup_{\substack{U \in L(X,Y) \\ c \in \mathbb{R}_+}} \text{Inf} \bigcup_{y \in X} [c\|y - x\| - c\|x\| + c\|x\| + U(x) - U(y) + F(y)] \\ & = \text{Sup} \bigcup_{\substack{U \in L(X,Y) \\ c \in \mathbb{R}_+}} [c\|x\| + U(x) - \text{Sup} \bigcup_{y \in X} [-c\|y - x\| + c\|x\| + U(y) - F(y)] \\ & = \text{Sup} \bigcup_{\substack{U \in L(X,Y) \\ c \in \mathbb{R}_+}} [c\|x\| + U(x) - F^w(x, U, c)] \qquad . \end{aligned}$$

Substituting the last equality into the last inclusion we obtain

$$\operatorname{Sup} \bigcup_{\substack{U\in L(X,Y)\\ c\in\mathbb{R}_+}} [c\|x\| + U(x) - F^w(x,U,c)] \subseteq F^{ww}(x) \cup A(F^{ww}(x)). \tag{3.2}$$

From (3.1), (3.2) and Lemma 2.8 we get

$$\operatorname{Sup} \bigcup_{\substack{U\in L(X,Y)\\ c\in\mathbb{R}_+}} [c\|x\| + U(x) - F^w(x,U,c)] = F^{ww}(x).$$

□

Now, let us find the weak conjugate and weak biconjugate map of a given nonconvex set-valued map.

Example 3.3. Let the set-valued map $F : \mathbb{R} \rightrightarrows \mathbb{R}$ be defined by $F(x) = [-|x|, +\infty)$ for all $x \in \mathbb{R}$. Then weak conjugate of F is

$$F^w(x_0,u,c) = \begin{cases} \{(u-c-1)x_0\} & , \quad |u| \le c-1 \\ \{+\infty\} & , \quad |u| > c-1 \end{cases}$$

and weak biconjugate of F is $F^{ww}(x) = \{-|x|\}$.

Proposition 3.4. *Let $\bar{y} \in Y$ and $(x_0,U,c) \in X \times L(X,Y) \times \mathbb{R}_+$. Then*

i) $(F+\bar{y})^w(x_0,U,c) = F^w(x_0,U,c) - \bar{y}$

ii) $(F+\bar{y})^{ww}(x) = F^{ww}(x) + \bar{y}$.

Next proposition is the generalization of Fenchel inequality given for scalar functions.

Proposition 3.5. *Let $\bar{x} \in X$ and $(x_0,U,c) \in X \times L(X,Y) \times \mathbb{R}_+$. Then*

$$(F(\bar{x}) - U(\bar{x}) + c\|\bar{x} - x_0\| - c\|x_0\|) \cap B(-F^w(x_0,U,c)) = \emptyset.$$

Proof. From the definition of the set $A(F^w(x_0,U,c))$ and as

$$F^w(x_0,U,c) = \operatorname{Sup} \bigcup_{x\in X} [-c\|x-x_0\| + c\|x_0\| + U(x) - F(x)]$$

we have $(-c\|\bar{x}-x_0\| + c\|x_0\| + U(\bar{x}) - F(\bar{x})) \cap A(F^w(x_0,U,c)) = \emptyset$. Because $(F(\bar{x}) + c\|\bar{x}-x_0\| - c\|x_0\| - U(\bar{x})) \cap (-A(F^w(x_0,U,c))) = \emptyset$ and $-A(-Z) = B(Z)$ where Z is an arbitrary subset of Y, we obtain

$$(F(\bar{x}) - U(\bar{x}) + c\|\bar{x} - x_0\| - c\|x_0\|) \cap B(-F^w(x_0,U,c)) = \emptyset.$$

□

By taking $\bar{x} = 0$ in Proposition 3.5 the following corollary is obtained.

Corollary 3.6. *Let $\bar{y} \in F(0)$ and $(x_0,U,c) \in X \times L(X,Y) \times \mathbb{R}_+$. If $y' \in -F^w(x_0,U,c)$, then $\bar{y} \underset{C}{\not<} y'$.*

Corollary 3.7 is the generalization of inequality $F^{ww}(\bar{x}) \leq F(\bar{x})$ where F is scalar function and F^{ww} is weak biconjugate map of F defined by Azimov and Kasimov [6].

Corollary 3.7. *Let $\bar{x} \in X$, $\bar{y} \in F(\bar{x})$ and $y'' \in F^{ww}(\bar{x})$. Then $\bar{y} \underset{C}{\not<} y''$.*

Proof. From Proposition 3.5 we have

$$(F(x) - U(x) + c\|x - x_0\| - c\|x_0\|) \cap B(-F^w(x_0, U, c)) = \emptyset$$

for all $(x_0, U, c) \in X \times L(X, Y) \times \mathbb{R}_+$ and $x \in X$. So, we obtain

$$\begin{aligned} \emptyset &= F(x) \cap (U(x) - c\|x - x_0\| + c\|x_0\| + B(-F^w(x_0, U, c))) \\ &= F(x) \cap B(-c\|x - x_0\| + c\|x_0\| + U(x) - F^w(x_0, U, c)). \end{aligned}$$

Since (x_0, U, c) is an arbitrary element of $X \times L(X, Y) \times \mathbb{R}_+$ and from Proposition 2.5 we get

$$\begin{aligned} \emptyset &= F(x) \cap B(\bigcup_{(x_0,U,c)} [-c\|x - x_0\| + c\|x_0\| + U(x) - F^w(x_0, U, c)]) \\ &= F(x) \cap B(\mathrm{Sup} \bigcup_{(x_0,U,c)} [-c\|x - x_0\| + c\|x_0\| + U(x) - F^w(x_0, U, c)]) \\ &= F(x) \cap B(F^{ww}(x)). \end{aligned}$$

Hence, $y \notin B(F^{ww}(x))$ for all $y \in F(x)$. So, $\bar{y} \underset{C}{\not<} y''$. □

4 Weak Subdifferentials for Set-Valued Maps

In this section, weak subdifferential is defined by using concepts of weak-maximum of a set and vectorial norm. Necessary and sufficient conditions for weakly subdifferentiability of a set-valued map are obtained and under some assumptions it is proved that Lipschitz set-valued maps are weakly subdifferentiable. Furthermore, relationships between weak subdifferential and weak conjugate map are examined and a condition for equality of a set-valued map and weak biconjugate map is obtained.

Definition 4.1. Let $\bar{x} \in X$ and $\bar{y} \in F(\bar{x})$. A pair $(U, c) \in L(X, Y) \times \mathbb{R}_+$ is said to be weak subgradient of F at $(\bar{x}, \bar{y})$ if

$$(U(\bar{x}) - \bar{y}) \in \mathrm{wmax} \bigcup_{x \in X} [U(x) - c\|x - \bar{x}\| - F(x)].$$

The set of all weak subgradients of F at $(\bar{x}, \bar{y})$ is called the weak subdifferential of F at $(\bar{x}, \bar{y})$ and is denoted by $\partial^w F(\bar{x}, \bar{y})$. If $\partial^w F(\bar{x}, \bar{y}) \neq \emptyset$ then F is said to be weakly subdifferentiable at $(\bar{x}, \bar{y})$. If $\partial^w F(\bar{x}, \bar{y}) \neq \emptyset$ for every $\bar{y} \in F(\bar{x})$, then F is said to be weakly subdifferentiable at $\bar{x}$ and the weak subdifferential of F at $\bar{x}$ is denoted by the set $\partial^w F(\bar{x}) = \bigcup_{\bar{y} \in F(\bar{x})} \partial^w F(\bar{x}, \bar{y})$.

In the following example, we find the subdifferential of a nonconvex set-valued map.

Example 4.2. Let the set-valued map $F : \mathbb{R} \rightrightarrows \mathbb{R}$ be defined as $F(x) = [-|x|, +\infty)$, for all $x \in \mathbb{R}$, the ordering cone be $\mathbb{R}_+$. Then the weak subdifferential of F is

$$\partial^w F(\bar{x}, \bar{y}) = \begin{cases} \{(u, c) : c \geq 1,\ |u| \leq c - 1\} &, \quad \bar{y} = -|\bar{x}| \\ \emptyset &, \quad \bar{y} > -|\bar{x}| \end{cases}$$

for all $\bar{x} \in \mathbb{R}$.

Now, we give a characterization for being weakly minimal point of the image set of a set-valued map by using weak subdifferential of this set-valued map.

Proposition 4.3. *Let* $\bar{x} \in X$ *and* $\bar{y} \in F(\bar{x})$. *Then* $\bar{y} \in \operatorname{wmin} \bigcup_{x \in X} F(x)$ *if and only if* $(0,0) \in \partial^w F(\bar{x}, \bar{y})$.

Proof. Proof is straightforward from the definiton of weak subdifferential. □

In Proposition 4.4 we present a characterization for being a weak subgradient by using the weak conjugate map of a set-valued map.

Proposition 4.4. *Let* $\bar{x} \in X$, $\bar{y} \in F(\bar{x})$ *and* $(U, c) \in L(X, Y) \times \mathbb{R}_+$. *Then* $(U, c) \in \partial^w F(\bar{x}, \bar{y})$ *if and only if* $(c\|\bar{x}\| + U(\bar{x}) - \bar{y}) \in F^w(\bar{x}, U, c)$.

Proof. Let $(U, c) \in \partial^w F(\bar{x}, \bar{y})$. Then from the definition of weak subdifferential we have

$$(U(\bar{x}) - \bar{y}) \in \operatorname{wmax} \bigcup_{x \in X} [U(x) - c\|x - \bar{x}\| - F(x)].$$

Thus,

$$\begin{aligned} (c\|\bar{x}\| + U(\bar{x}) - \bar{y}) &\in \operatorname{wmax} \bigcup_{x \in X} [U(x) - c\|x - \bar{x}\| + c\|\bar{x}\| - F(x)] \\ &\subseteq \operatorname{Sup} \bigcup_{x \in X} [U(x) - c\|x - \bar{x}\| + c\|\bar{x}\| - F(x)] \\ &= F^w(\bar{x}, U, c). \end{aligned}$$

Conversely, let $(c\|\bar{x}\| + U(\bar{x}) - \bar{y}) \in F^w(\bar{x}, U, c)$. Then we have

$$(U(\bar{x}) - \bar{y}) \in \operatorname{Sup} \bigcup_{x \in X} [U(x) - c\|x - \bar{x}\| - F(x)].$$

In addition to this, since

$$(U(\bar{x}) - \bar{y}) \in \bigcup_{x \in X} [U(x) - c\|x - \bar{x}\| - F(x)]$$

we get

$$(U(\bar{x}) - \bar{y}) \in \operatorname{wmax} \bigcup_{x \in X} [U(x) - c\|x - \bar{x}\| - F(x)]$$

which means $(U, c) \in \partial^w F(\bar{x}, \bar{y})$. □

Theorem 4.5 gives a condition for equality of a set-valued map and weak biconjugate map of it.

Theorem 4.5. *Let* $\bar{x} \in X$. *If* F *is weakly subdifferentiable at* $\bar{x}$, *then* $F(\bar{x}) \subseteq F^{ww}(\bar{x})$. *In addition, if* $\operatorname{Inf} F(\bar{x}) = F(\bar{x})$, *then* $F(\bar{x}) = F^{ww}(\bar{x})$.

Proof. Let F be weakly subdifferentiable at $\bar{x}$ and $\bar{y}$ be an arbitrary element of $F(\bar{x})$. Then there exists $(U, c) \in \partial^w F(\bar{x}, \bar{y})$. Thus, from Proposition 4.4 we have $(c\|\bar{x}\| + U(\bar{x}) - \bar{y}) \in F^w(\bar{x}, U, c)$. So, we get

$$\bar{y} \in (c\|\bar{x}\| + U(\bar{x}) - F^w(\bar{x}, U, c)) \subseteq \bigcup_{\substack{x_0 \in X \\ (T,d) \in L(X,Y) \times \mathbb{R}_+}} [-d\|\bar{x} - x_0\| + d\|x_0\| + T(\bar{x}) - F^w(x_0, T, d)]. \quad (4.1)$$

From Proposition 3.5 we obtain $\bar{y} \not\leq_C y' + T(\bar{x}) - d\|\bar{x} - x_0\| + d\|x_0\|$ for all $(x_0, T, d) \in X \times L(X,Y) \times \mathbb{R}_+$ and for all $y' \in -F^w(x_0, T, d)$. By using (4.1) we get

$$\begin{aligned} \bar{y} &\in \text{wmax} \bigcup_{(x_0,T,d)\in X\times L(X,Y)\times\mathbb{R}_+} [-d\|\bar{x} - x_0\| + d\|x_0\| + T(\bar{x}) - F^w(x_0, T, d)] \\ &\subseteq \text{Sup} \bigcup_{(x_0,T,d)\in X\times L(X,Y)\times\mathbb{R}_+} [-d\|\bar{x} - x_0\| + d\|x_0\| + T(\bar{x}) - F^w(x_0, T, d)] \\ &= F^{ww}(\bar{x}) \end{aligned}$$

which means $F(\bar{x}) \subseteq F^{ww}(\bar{x})$.

Let $\text{Inf}F(\bar{x}) = F(\bar{x})$, F be weakly subdifferentiable at $\bar{x}$ and $\bar{y}$ be an arbitrary element of $F^{ww}(\bar{x})$. From Proposition 2.5 in [33] we have

$$\begin{aligned} \overline{Y} &= \text{Inf}F(\bar{x}) \cup A(\text{Inf}F(\bar{x})) \cup B(\text{Inf}F(\bar{x})) \\ &= F(\bar{x}) \cup A(F(\bar{x})) \cup B(F(\bar{x})) \end{aligned}$$

and the above three sets in the right hand side are disjoint. From Corollary 3.7 we get $y \not\leq_C \bar{y}$ for all $y \in F(\bar{x})$. Thus, $\bar{y} \notin A(F(\bar{x}))$. Hence, we have either $\bar{y} \in F(\bar{x})$ or $\bar{y} \in B(F(\bar{x}))$.

We claim that $\bar{y} \notin B(F(\bar{x}))$. Otherwise, there exists $y' \in F(\bar{x})$ such that $\bar{y} <_C y'$. As $y' \in F(\bar{x})$ and F is weakly subdifferentiable at $\bar{x}$, F is weakly subdifferentiable at $(\bar{x}, y')$, i.e. $\partial^w F(\bar{x}, y') \neq \emptyset$. So, there exists $(T, c) \in \partial^w F(\bar{x}, y')$. From Proposition 4.4 we have $(-c\|\bar{x}\| - T(\bar{x}) + y') \in -F^w(\bar{x}, T, c)$. Since $y' >_C \bar{y}$ we get $\bar{y} \in B(c\|\bar{x}\| + T(\bar{x}) - F^w(\bar{x}, T, c))$. This contradicts to

$$\bar{y} \in \text{Sup} \bigcup_{(x_0,U,d)\in X\times L(X,Y)\times\mathbb{R}_+} [-d\|\bar{x} - x_0\| + d\|x_0\| + U(\bar{x}) - F^w(x_0, U, d)] = F^{ww}(\bar{x}).$$

Hence, $\bar{y} \notin B(F(\bar{x}))$ which means $\bar{y} \in F(\bar{x})$. So, $F^{ww}(\bar{x}) \subseteq F(\bar{x})$. Then we obtain $F(\bar{x}) = F^{ww}(\bar{x})$. □

Proposition 4.6 gives conditions for weak subdifferentiability of a set-valued map.

Proposition 4.6. *Let $\bar{x} \in X$. If there exists $L > 0$ such that $F(\bar{x}) \not\subset F(x) + L\|x - \bar{x}\| + intC$ for all $x \in X$ and if $\text{wmin}F(\bar{x}) \neq \emptyset$, then F is weakly subdifferentiable at $(\bar{x}, \bar{y})$ for all $\bar{y} \in F(\bar{x}) \setminus \bigcup_{x\in X} [F(x) + L\|x - \bar{x}\| + intC]$.*

Proof. Let $\bar{y} \in F(\bar{x}) \setminus \bigcup_{x\in X} [F(x) + L\|x - \bar{x}\| + intC]$. This implies

$$\bar{y} \in F(\bar{x}) - L\|\bar{x} - \bar{x}\| \setminus \bigcup_{x\in X} [F(x) + L\|x - \bar{x}\| + intC].$$

Hence, $\bar{y} \in \text{wmin} \bigcup_{x\in X} [F(x) + L\|x - \bar{x}\|]$ which means $(0, L) \in \partial^w F(\bar{x}, \bar{y})$. □

Proposition 4.7 gives another necessary condition for weak subdifferentiability of a set-valued map in finite dimensional Euclidean space.

Proposition 4.7. *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a set-valued map, $\mathbb{R}^p$ be partially ordered by $\mathbb{R}^p_+$, the vectorial norm $\|\!|\cdot|\!\| : \mathbb{R}^n \to \mathbb{R}^p_+$ be defined as $\|\!|x|\!\| = (\underbrace{\|x\|, ..., \|x\|}_{p})$ for all $x \in \mathbb{R}^n$ where $\|\cdot\|$ is a norm on $\mathbb{R}^n$, $\bar{x} \in \mathbb{R}^n$ be a given point, $\mathrm{wmin}F(\bar{x}) \neq \emptyset$ and let $F(\bar{x})$ be $\mathbb{R}^p_+$-bounded, i.e. there exists $a \in \mathbb{R}^p_+$ such that $a \underset{\mathbb{R}^p_+}{\leq} y$ for all $y \in F(\bar{x})$. If there exist $L > 0$ and a neighborhood V of $\bar{x}$ such that*

$$F(\bar{x}) \not\subset F(x) + L\|\!|x - \bar{x}|\!\| + int\mathbb{R}^p_+ \quad \text{for all } x \in V,$$

and $p \geq 0$ and $q \in \mathbb{R}^p$ such that

$$-p\|\!|x|\!\| + q \notin F(x) + int\mathbb{R}^p_+ \text{ for all } x \in \mathbb{R}^n,$$

then there exists $M > 0$ satisfies

$$F(\bar{x}) \not\subset F(x) + M\|\!|x - \bar{x}|\!\| + int\mathbb{R}^p_+ \quad \text{for all } x \in \mathbb{R}^n.$$

Proof. Assume the contrary that for any $k \in \mathbb{N}$ there exists $x_k \in \mathbb{R}^n$ such that

$$F(\bar{x}) \subset F(x_k) + k\|\!|x_k - \bar{x}|\!\| + int\mathbb{R}^p_+.$$

So, there exists $\bar{y}_k \in F(x_k)$ such that $y - \bar{y}_k - k\|\!|x_k - \bar{x}|\!\| \in int\mathbb{R}^p_+$ for all $k \in \mathbb{N}$ and for all $y \in F(\bar{x})$. Let choose an arbitrary element $y \in F(\bar{x})$. Thus, we have

$$y^j - \bar{y}^j_k - k\|x_k - \bar{x}\| > 0 \tag{4.2}$$

for all $j \in \{1, 2, \ldots, p\}$. Because $-p\|\!|x|\!\| + q \notin F(x) + int\mathbb{R}^p_+$ for all $x \in \mathbb{R}^n$, particularly $x_k \in \mathbb{R}^n$ satisfies $-p\|\!|x_k|\!\| + q \notin F(x_k) + int\mathbb{R}^p_+$. So, we get $-p\|\!|x_k|\!\| + q - y_k \notin int\mathbb{R}^p_+$ for all $y_k \in F(x_k)$. By using triangle inequality we obtain $-p\|\!|x_k - \bar{x}|\!\| - p\|\!|\bar{x}|\!\| + q - y_k \notin int\mathbb{R}^p_+$ for all $y_k \in F(x_k)$. Hence, there exists $i_k \in \{1, 2, \ldots, p\}$ such that

$$-p\|x_k - \bar{x}\| - p\|\bar{x}\| + q^{i_k} - y^{i_k}_k \leq 0. \tag{4.3}$$

In particular $\bar{y}_k$ in (4.2) satisfies (4.3), i.e.

$$-p\|x_k - \bar{x}\| - p\|\bar{x}\| + q^{i_k} - \bar{y}^{i_k}_k \leq 0. \tag{4.4}$$

Inequality (4.2) is valid for i_k, i.e.

$$y^{i_k} - \bar{y}^{i_k}_k - k\|x_k - \bar{x}\| > 0 \tag{4.5}$$

By adding both sides of inequalities (4.4) and (4.5) we obtain

$$\begin{aligned} 0 &> -p\|x_k - \bar{x}\| - p\|\bar{x}\| + q^{i_k} - \bar{y}^{i_k}_k - y^{i_k} + \bar{y}^{i_k}_k + k\|x_k - \bar{x}\| \\ &= (k-p)\|x_k - \bar{x}\| - p\|\bar{x}\| + q^{i_k} - y^{i_k}. \end{aligned}$$

Therefore, we get

$$(k-p)\|x_k - \bar{x}\| < p\|\bar{x}\| - q^{i_k} + y^{i_k}. \tag{4.6}$$

$k - p > 0$ for k large enough. Since $i_k \in \{1, 2, \ldots, p\}$ for all $k \in \mathbb{N}$ and y^{i_k} is a component of a chosen element y, the sequence $(y^{i_k})_{k\in\mathbb{N}}$ has p elements. So, $(y^{i_k})_{k\in\mathbb{N}}$ is bounded. Then from (4.6) and the boundedness of $(y^{i_k})_{k\in\mathbb{N}}$ we have

$$\|x_k - \bar{x}\| < \frac{p\|\bar{x}\| - q^{i_k} + y^{i_k}}{k-p} \to 0 \text{ as } k \to \infty$$

which means $x_k \to \bar{x}$.

As V is a neighborhood of $\bar{x}$ and $x_k \to \bar{x}$ there exists $k_0 \in \mathbb{N}$ such that $x_k \in V$ for all $k \geq k_0$. From hypothesis $F(\bar{x}) \not\subset F(x_k) + L\|x_k - \bar{x}\| + int\mathbb{R}^p_+$ for all $k \geq k_0$. So, for each $y_k \in F(x_k)$ there exists $z_k \in F(\bar{x})$ such that $z_k - y_k - L\|x_k - \bar{x}\| \notin int\mathbb{R}^p_+$. Hence, there exists $j_k \in \{1, 2, \ldots, p\}$ such that $z_k^{j_k} - y_k^{j_k} - L\|x_k - \bar{x}\| \leq 0$. In particular, for $\bar{y}_k^{j_k}$ there exists $\bar{z}_k^{j_k} \in F(\bar{x})$ such that

$$\bar{z}_k^{j_k} - \bar{y}_k^{j_k} - L\|x_k - \bar{x}\| \leq 0 \tag{4.7}$$

By using inequality (4.2), we obtain

$$y^{j_k} - \bar{y}_k^{j_k} - k\|x_k - \bar{x}\| > 0. \tag{4.8}$$

From inequalities (4.7) and (4.8) we have $\bar{z}_k^{j_k} - y^{j_k} + k\|x_k - \bar{x}\| - L\|x_k - \bar{x}\| < 0$. As $F(\bar{x})$ is $\mathbb{R}^p_+$-bounded,
$(\bar{z}_k^{j_k} - y^{j_k})_{n\in\mathbb{N}}$ is bounded from below. Let b be a lower bound for this sequence. Therefore, for enough large k we have $b + (k - L)\|x_k - \bar{x}\| < 0$ which is not possible. Hence, there exists $M > 0$ such that $F(\bar{x}) \not\subset F(x) + M\|x - \bar{x}\| + int\mathbb{R}^p_+$. □

Corollary 4.8. *Under assumptions of Proposition 4.7, F is weakly subdifferentiable at $(\bar{x}, \bar{y})$ for all $\bar{y} \in F(\bar{x})\setminus \bigcup_{x\in\mathbb{R}^n} [F(x) + M\|x - \bar{x}\| + int\mathbb{R}^p_+]$ where M is positive number obtained in Proposition 4.7.*

Proposition 4.9. *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a set-valued map, $\mathbb{R}^p$ be partially ordered by $\mathbb{R}^p_+$ and let the vectorial norm $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}^p_+$ be defined as $\|x\| = (\underbrace{\|x\|, \ldots, \|x\|}_{p})$ for all $x \in \mathbb{R}^n$ where $\|\cdot\|$ is a norm on $\mathbb{R}^n$. Let $\bar{x} \in \mathbb{R}^n$, $\bar{y} \in F(\bar{x})$ and* wmin*$F(\bar{x}) \neq \emptyset$. If $F$is weakly subdifferentiable at $(\bar{x}, \bar{y})$, then there exists $L > 0$ such that $F(\bar{x}) \not\subset F(x) + L\|x - \bar{x}\| + int\mathbb{R}^p_+$ for all $x \in \mathbb{R}^n$.*

Proof. Because F is weakly subdifferentiable at $(\bar{x}, \bar{y})$ there exists$(U, c) \in L(\mathbb{R}^n, \mathbb{R}^p) \times \mathbb{R}_+$ such that

$$(\bar{y} - U(\bar{x})) \in \text{wmin} \bigcup_{x\in\mathbb{R}^n} [F(x) - U(x) + c\|x - \bar{x}\|].$$

Then we get

$$(F(x) - U(x) + c\|x - \bar{x}\| - \bar{y} + U(\bar{x})) \cap (-int\mathbb{R}^p_+) = \emptyset \tag{4.9}$$

for all $x \in \mathbb{R}^n$. We have $\|U\| = \sup_{x\in\mathbb{R}^n\setminus\{0\}} \dfrac{\|U(x)\|}{\|x\|} \geq \dfrac{|U_i(x)|}{\|x\|} \geq \dfrac{U_i(x)}{\|x\|}$ for all $x \in \mathbb{R}^n\setminus\{0\}$ and for all $i \in \{1, 2, \ldots, p\}$ where $U = (U_1, U_2, \ldots, U_p)$ and $U_i : \mathbb{R}^n \to \mathbb{R}$ is linear map for $i = 1, 2, \ldots, p$. Hence, we obtain $\|U\|\|x\| \geq U_i(x)$ for all $x \in \mathbb{R}^n$ and for all $i \in \{1, 2, \ldots, p\}$. In particular, we have $\|U\|\|x - \bar{x}\| \geq U_i(\bar{x} - x)$. Therefore, we get $\|U\|\|x - \bar{x}\| \underset{\mathbb{R}^p_+}{\geq} U(\bar{x} - x)$. From (4.9) we have

$$\begin{aligned} \emptyset &= (F(x) + \|U\|\|x - \bar{x}\| + c\|x - \bar{x}\| - \bar{y}) \cap (-int\mathbb{R}^p_+) \\ &= (F(x) + (\|U\| + c)\|x - \bar{x}\| - \bar{y}) \cap (-int\mathbb{R}^p_+) \end{aligned}$$

for all $x \in \mathbb{R}^n$. Then $y + (\|U\| + c)\|x - \bar{x}\| - \bar{y} \notin -int\mathbb{R}^p_+$ for all $y \in F(x)$. Thus, we get $\bar{y} \notin y + (\|U\| + c)\|x - \bar{x}\| + int\mathbb{R}^p_+$. As y is an arbitrary element of $F(x)$ we have

$$\bar{y} \notin F(x) + (\|U\| + c)\|x - \bar{x}\| + int\mathbb{R}^p_+$$

which means $F(\bar{x}) \not\subset F(x) + (\|U\| + c)\|x - \bar{x}\| + int\mathbb{R}^p_+$. By setting $\|U\| + c = L$ in the last relation we complete the proof. □

Corollary 4.10. *Under assumptions of Proposition 4.9, F is weakly subdifferentiable at $(\bar{x}, \bar{y})$ if and only if there exists $L > 0$ such that $F(\bar{x}) \not\subset F(x) + L\|x - \bar{x}\| + int\mathbb{R}^p_+$ for all $x \in \mathbb{R}^n$ and $\bar{y} \in F(\bar{x}) \setminus \bigcup_{x\in\mathbb{R}^n} [F(x) + L\|x - \bar{x}\| + int\mathbb{R}^p_+]$.*

Proposition 4.11. *Let X, Y be real topological spaces, Y be partially ordered by closed, convex and pointed cone C with nonempty interior $intC$, $\|\cdot\| : X \to C$ be a vectorial norm and $F : X \rightrightarrows \overline{Y}$ be a set-valued map. Let $\bar{x} \in X$ be given point and $\mathrm{wmin}F(\bar{x}) \neq \emptyset$. If there exists $L > 0$ such that $F(\bar{x}) \not\subset F(x) + L\|x - \bar{x}\| + intC$ for all $x \in X$, then there exist $p \geq 0$ and $q \in Y$ such that $-p\|x\| + q \notin F(x) + intC$ for all $x \in X$.*

Proof. Since $F(\bar{x}) - L\|x - \bar{x}\| \not\subset F(x) + intC$ for all $x \in X$ we have $F(\bar{x}) - L\|x\| - L\|\bar{x}\| \not\subset F(x) + intC$. Because, if $F(\bar{x}) - L\|x\| - L\|\bar{x}\| \subset F(x) + intC$, then for every $\bar{y} \in F(\bar{x})$ there exists $y \in F(x)$ such that $\bar{y} - L\|x\| - L\|\bar{x}\| \in y + intC$. Thus, we get

$$y \underset{C}{<} \bar{y} - L\|x\| - L\|\bar{x}\| \underset{C}{\leq} \bar{y} - L\|x - \bar{x}\|.$$

Therefore,

$$F(\bar{x}) - L\|x - \bar{x}\| \subset F(x) + intC$$

which contradicts to assumption. So,

$$F(\bar{x}) - L\|x\| - L\|\bar{x}\| \not\subset F(x) + intC.$$

Hence, there exists $\bar{y} \in F(\bar{x})$ such that $\bar{y} - L\|x\| - L\|\bar{x}\| \notin F(x) + intC$. By setting $p = L$ and $q = \bar{y} - L\|\bar{x}\|$ we obtain

$$-p\|x\| + q \notin F(x) + intC$$

for all $x \in X$. □

Corollary 4.12 gives two characterizations for weak subdifferentiability of a set-valued map by using Corollary 4.8, Corollary 4.10 and Proposition 4.11.

Corollary 4.12. *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a set valued map, $\mathbb{R}^p$ be partially ordered by $\mathbb{R}^p_+$ vectorial norm $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}^p_+$ be defined as $\|x\| = (\underbrace{\|x\|, ..., \|x\|}_{p})$ for all $x \in \mathbb{R}^n$ where $\|\cdot\|$ is a norm on $\mathbb{R}^n$ and $\bar{x} \in \mathbb{R}^n$ be given. Let $\mathrm{wmin}F(\bar{x}) \neq \emptyset$ and $F(\bar{x})$ be $\mathbb{R}^p_+$-bounded. Then the followings are equivalent to each other:*

i) *There exist $L > 0$ and a neighborhood $V \in \mathcal{N}(\bar{x})$ of $\bar{x}$ such that*

$$F(\bar{x}) \not\subset F(x) + L\|x - \bar{x}\| + int\mathbb{R}^p_+$$

for all $x \in V$, in addition there exist $p \geq 0$ and $q \in \mathbb{R}^p$ such that

$$-p\|x\| + q \notin F(x) + int\mathbb{R}^p_+$$

for all $x \in \mathbb{R}^n$ and $\bar{y} \in F(\bar{x}) \setminus \bigcup_{x\in\mathbb{R}^n} [F(x) + M\|x - \bar{x}\| + int\mathbb{R}^p_+]$ where M is the positive number obtained in Proposition 4.7.

ii) *There exists* $L > 0$ *such that* $F(\bar{x}) \not\subset F(x) + L\|x - \bar{x}\| + int\mathbb{R}^p_+$ *for all* $x \in \mathbb{R}^n$ *and* $\bar{y} \in F(\bar{x}) \setminus \bigcup_{x\in\mathbb{R}^n} [F(x) + L\|x - \bar{x}\| + int\mathbb{R}^p_+]$

iii) F *is weakly subdifferentiable at* $(\bar{x}, \bar{y})$.

Proposition 4.13. *Let* $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ *be a set valued map,* $\mathbb{R}^p$ *be partially ordered by* $\mathbb{R}^p_+$*, vectorial norm* $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}^p_+$ *be defined as* $\|x\| = (\underbrace{\|x\|, ..., \|x\|}_{p})$ *for all* $x \in \mathbb{R}^n$ *where* $\|\cdot\|$ *is a norm on* $\mathbb{R}^n$ *and* $\bar{x} \in \mathbb{R}^n$ *be a given point. If* F *is Lipschitz on* V *with Lipschitz constant* L *(in the sense Aubin [2]), where* V *is a neighborhood of* $\bar{x}$*, then*

$$F(\bar{x}) \not\subset F(x) + L\|x - \bar{x}\| + int\mathbb{R}^p_+$$

for all $x \in V \setminus \{\bar{x}\}$.

Proof. Since F is Lipschitz on V, we have $F(x) \subset F(\bar{x}) + L\|x - \bar{x}\|\overline{B}$ for all $x \in V \setminus \{\bar{x}\}$ where $\overline{B}$ denotes the unit ball in $\mathbb{R}^p$. Hence, for all $y \in F(x)$ there exist $\bar{y} \in F(\bar{x})$ and $e \in \overline{B}$ such that $y = \bar{y} + L\|x - \bar{x}\|e$ where $e = (e_1, e_2, \ldots, e_p)$. Since $e_i + 1 \geq 0$ for all $i \in \{1, 2, \ldots, p\}$, we obtain

$$\begin{aligned}
\bar{y} - y - L\|x - \bar{x}\| &= -L\|x - \bar{x}\|e - L\|x - \bar{x}\| \\
&= (-L\|x - \bar{x}\|e_1, \ldots, -L\|x - \bar{x}\|e_p) - (L\|x - \bar{x}\|, \ldots, L\|x - \bar{x}\|) \\
&= (-L\|x - \bar{x}\|(e_1 + 1), \ldots, -L\|x - \bar{x}\|(e_p + 1)) \\
&\in -\mathbb{R}^p_+.
\end{aligned}$$

So we have $\bar{y} - y - L\|x - \bar{x}\| \notin int\mathbb{R}^p_+$, that means

$$\bar{y} \notin F(x) + L\|x - \bar{x}\| + int\mathbb{R}^p_+.$$

As a consequence, we have $F(\bar{x}) \not\subset F(x) + L\|x - \bar{x}\| + int\mathbb{R}^p_+$ for all $x \in V \setminus \{\bar{x}\}$. □

Corollary 4.14 shows that under some assumptions, Lipschitz set-valued maps on a set are weakly subdifferentiable.

Corollary 4.14. *Let* $\mathbb{R}^p$ *be ordered by* $\mathbb{R}^p_+$*,* $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ *be a set-valued map, the vectorial norm* $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}^p_+$ *be defined as* $\|x\| = (\underbrace{\|x\|, ..., \|x\|}_{p})$ *for all* $x \in \mathbb{R}^n$ *where* $\|\cdot\|$ *denotes a norm on* $\mathbb{R}^n$*. Let* $\bar{x} \in \mathbb{R}^n$*,* wmin$F(\bar{x}) \neq \emptyset$ *and* $F(\bar{x})$ *be* $\mathbb{R}^p_+$*-bounded. If* F *is Lipschitz on* $V(\subset domF)$ *with Lipschitz constant* L *and if there exist* $p \geq 0$ *and* $q \in \mathbb{R}^p$ *such that* $-p\|x\| + q \notin F(x) + int\mathbb{R}^p_+$ *for all* $x \in \mathbb{R}^n$*, then there exists* $M > 0$ *such that* $F(\bar{x}) \not\subset F(x) + M\|x - \bar{x}\| + int\mathbb{R}^p_+$ *for all* $x \in \mathbb{R}^n$ *and* F *is weakly subdifferentiable at* $(\bar{x}, \bar{y})$ *for all* $\bar{y} \in F(\bar{x}) \setminus \bigcup_{x\in\mathbb{R}^n} [F(x) + M\|x - \bar{x}\| + int\mathbb{R}^p_+]$.

5 Weak Conjugate Duality in Vector Optimization

In this section, by using weak conjugate of perturbation function a weak conjugate dual problem is defined for unconstrained vector optimization problems and weak duality theorem is proved. Furthermore, stability of primal problem with respect to the perturbation function is defined and it is proved that stability of the primal problem implies the strong duality. Moreover, some necessary conditions for stability of primal problem are presented.

First, let us define the unconstrained vector optimization problem. We assume that X, Y are real normed spaces, Y is partially ordered by closed, convex and pointed cone C with nonempty interior $intC$ and $f : X \to \overline{Y}$ is a vector valued function.

Consider the vector optimization problem

$$(VOP)\begin{cases} \text{minimize } f(x) \\ s.t. \ x \in X \end{cases}.$$

Solving this problem means to find the set

$$\text{Inf}(VOP) = \text{Inf}\{f(x) \mid x \in X\}.$$

We introduce perturbation parameter $z \in Z$ and embed the primal problem (VOP) into a family of vector optimization problems, where Z is another locally convex real topological vector space. Let φ be a function from $X \times Z$ to $Y \cup \{+\infty\}$ such that

$$\varphi(x, 0) := f(x) \ \text{ for all } x \in X.$$

This function is called perturbation function.

Perturbed problem is defined as

$$(P_z)\begin{cases} \text{minimize } \varphi(x, z) \\ s.t. \ x \in X \end{cases}.$$

To construct weak conjugate dual problem of (VOP) let us find the weak conjugate map of φ.

Let $\||\cdot\||_X : X \to C$ and $\||\cdot\||_Z : Z \to C$ be vectorial norms. Weak conjugate of φ, $\varphi^w : X \times L(X, Y) \times \mathbb{R}_+ \times Z \times L(Z, Y) \times \mathbb{R}_+ \rightrightarrows Y$ is defined as

$$\begin{aligned}\varphi^w(x_0, U, c, z_0, V, d) = \text{Sup} \bigcup_{(x,z)\in X\times Z} & [-c\||x - x_0\||_X + c\||x_0\||_X + U(x) \\ & -d\||z - z_0\||_Z + d\||z_0\||_Z + V(z) - \varphi(x, z)]\end{aligned}$$

for all $(x_0, U, c, z_0, V, d) \in X \times L(X, Y) \times \mathbb{R}_+ \times Z \times L(Z, Y) \times \mathbb{R}_+$.

If we set $x_0 = 0,\ z_0 = 0,\ U = 0,\ c = 0$ in this map we have

$$\varphi^w(\mathbf{0}, V, d) = \text{Sup} \bigcup_{(x,z)\in X\times Z} [-d\||z\||_Z + V(z) - \varphi(x, z)]$$

where $\varphi^w(0, 0, 0, 0, V, d) = \varphi^w(\mathbf{0}, V, d)$. The weak conjugate dual problem of (VOP) is defined as

$$(D^w)\begin{cases} \text{Sup} \bigcup_{(V,d)\in L(Z,Y)\times\mathbb{R}_+} [-\varphi^w(\mathbf{0}, V, d)] \end{cases}.$$

To solve this dual problem means to find the set

$$\text{Sup}(D^w) = \text{Sup} \bigcup_{(V,d)\in L(Z,Y)\times\mathbb{R}_+} [-\varphi^w(\mathbf{0}, V, d)].$$

Weak duality theorem presents that any feasible value of the primal problem is not less than any feasible value of the dual problem.

Theorem 5.1 (Weak Duality Theorem)**.** *For any* $x \in X$ *and* $(V,d) \in L(Z,Y)\times\mathbb{R}_+$ *we have* $\varphi(x,0) \notin B(-\varphi^w(\mathbf{0},V,d))$. *Hence,*

$$\mathrm{Inf}(VOP) \cap B(\mathrm{Sup}(D^w)) = \emptyset.$$

Proof. From Proposition 3.5 we have

$$\varphi(x,0) - 0(x) - V(0) + 0\|x-0\| - 0\|0\| + d\|0-0\| - d\|0\| = \varphi(x,0) \notin B(-\varphi^w(\mathbf{0},V,d))$$

for all $x \in X$ and $(V,d) \in L(Z,Y)\times\mathbb{R}_+$. So, $f(x) \notin B(-\varphi^w(\mathbf{0},V,d))$ for all $x \in X$ and $(V,d) \in L(Z,Y)\times\mathbb{R}_+$. Thus, we have

$$\bigcup_{x\in X} f(x) \cap B(\bigcup_{(V,d)\in L(Z,Y)\times\mathbb{R}_+} [-\varphi^w(\mathbf{0},V,d)]) = \emptyset.$$

Therefore,

$$A(\bigcup_{x\in X} f(x)) \cap B(\bigcup_{(V,d)\in L(Z,Y)\times\mathbb{R}_+} [-\varphi^w(\mathbf{0},V,d)]) = \emptyset.$$

Then we obtain

$$\begin{aligned}\emptyset &= cl(A(\bigcup_{x\in X} f(x))) \cap B(\bigcup_{(V,d)\in L(Z,Y)\times\mathbb{R}_+} [-\varphi^w(\mathbf{0},V,d)]) \\ &= cl(A(\bigcup_{x\in X} f(x))) \cap B(\mathrm{Sup}(D^w))\end{aligned}$$

As wmin $cl(A(\bigcup_{x\in X} f(x))) \subset cl(A(\bigcup_{x\in X} f(x)))$ we get

$$\begin{aligned}\emptyset &= \text{wmin } cl(A(\bigcup_{x\in X} f(x))) \cap B(\mathrm{Sup}(D^w)) \\ &= \mathrm{Inf}(\bigcup_{x\in X} f(x)) \cap B(\mathrm{Sup}(D^w)) \\ &= \mathrm{Inf}(VOP) \cap B(\mathrm{Sup}(D^w)).\end{aligned}$$

□

Corollary 5.2. *Let* $a \in \mathrm{Inf}(VOP)$ *and* $b \in \mathrm{Sup}(D^w)$. *Then* $a \underset{C}{\not<} b$.

Before giving the definition of stability of (VOP) we need the following definition.

Definition 5.3. The set-valued map $\phi : Z \rightrightarrows \overline{Y}$ defined by $\phi(z) := \mathrm{Inf}\{\varphi(x,z) \mid x \in X\}$ for all $z \in Z$ is called the value map for problem (VOP). It is obvious that $\mathrm{Inf}(VOP) = \phi(0)$.

Lemma 5.4. *For any* $(V,d) \in L(Z,Y)\times\mathbb{R}_+$ *we have* $\phi^w(0,V,d) = \varphi^w(\mathbf{0},V,d)$.

Theorem 5.5 implies that the solution set of dual problem can be characterized by the weak biconjugate map of the value map at 0.

Theorem 5.5. $\phi^{ww}(0) = \mathrm{Sup}(D^w)$.

Before giving Strong Duality Theorem we need the following definition.

Definition 5.6. The primal problem is said to be stable if the value map ϕ is weakly subdifferentiable at $0 \in Z$.

Theorem 5.7 (Strong Duality Theorem)**.** *If* (VOP) *is stable, then*

$$\mathrm{Inf}(VOP) = \mathrm{Sup}(D^w).$$

Proof. $\phi(0) = \mathrm{Inf}\{\varphi(x,0) \mid x \in X\} = \mathrm{Inf}\{\mathrm{Inf}\{\varphi(x,0) \mid x \in X\}\} = \mathrm{Inf}\phi(0)$. As (VOP) is stable ϕ is weakly subdifferentiable at 0. Thus, from Theorem 4.5 and Theorem 5.5 we obtain

$$\mathrm{Inf}(VOP) = \phi(0) = \phi^{ww}(0) = \mathrm{Sup}(D^w).$$

□

Proposition 5.8 states a necessary condition for the stability of (VOP).

Proposition 5.8. *If there exist* $L > 0$ *and for any* $\varepsilon \in intC$ *an element* $x(\varepsilon) \in X$ *such that* $-L\|z\| \underset{C}{\leq} \varphi(x,z) - \varphi(x(\varepsilon),0) + \varepsilon$ *for all* $(x,z) \in X \times Z$ *and if* $\mathrm{Inf}(VOP) \neq \emptyset$*, then* ϕ *is weakly subdifferentiable at* 0 *which means* (VOP) *is stable.*

Proof. Let $y \in \phi(0) = \mathrm{Inf} \bigcup_{x\in X} \varphi(x,0)$ be an arbitrary element and $z \in Z$ be a fixed arbitrary element. Then we have $y \notin A(\bigcup_{x\in X} \varphi(x,0))$. Hence,

$$\varphi(x,0) \underset{C}{\not<} y \tag{5.1}$$

for all $x \in X$. In particular, (5.1) is valid for $x(\varepsilon) \in X$, i.e. $\varphi(x(\varepsilon),0) \underset{C}{\not<} y$. As $-L\|z\| \underset{C}{\leq} \varphi(x,z) - \varphi(x(\varepsilon),0) + \varepsilon$ we have

$$\varphi(x,z) + L\|z\| + \varepsilon \underset{C}{\not<} y. \tag{5.2}$$

Therefore,

$$y - \varepsilon \notin \varphi(x,z) + L\|z\| + intC$$

for all $x \in X$. Then we get

$$y - \varepsilon \notin \bigcup_{x\in X} [\varphi(x,z)] + L\|z\| + intC.$$

Now, we will show that

$$y - \varepsilon \notin \mathrm{Inf} \bigcup_{x\in X} [\varphi(x,z)] + L\|z\| + intC.$$

Assume the contrary that

$$y - \varepsilon \in \mathrm{Inf} \bigcup_{x\in X} [\varphi(x,z)] + L\|z\| + intC.$$

Then there exist $a(\varepsilon) \in \mathrm{Inf} \bigcup_{x\in X} \varphi(x,z)$ and $c(\varepsilon) \in intC$ such that $y - \varepsilon = a(\varepsilon) + L\|z\| + c(\varepsilon)$. As $a(\varepsilon) \in \mathrm{Inf} \bigcup_{x\in X} \varphi(x,z)$ we have $A(a(\varepsilon)) \subseteq A(\bigcup_{x\in X} \varphi(x,z))$. Thus, for any a with $a(\varepsilon) \underset{C}{<} a$ there exists $x \in X$ such that $\varphi(x,z) \underset{C}{<} a$. Since, $y - \varepsilon - L\|z\| - c(\varepsilon) = a(\varepsilon)$ we get

$a(\varepsilon) \underset{C}{<} y - \varepsilon - L\|z\|$. Therefore, there exists $\bar{x} \in X$ such that $\varphi(\bar{x}, z) \underset{C}{<} y - \varepsilon - L\|z\|$ which contradicts to (5.2). Hence,

$$y - \varepsilon \notin \operatorname{Inf} \bigcup_{x \in X} [\varphi(x, z)] + L\|z\| + intC. \tag{5.3}$$

Let us show that

$$y \notin \operatorname{Inf} \bigcup_{x \in X} [\varphi(x, z)] + L\|z\| + intC = \phi(z) + L\|z\| + intC.$$

Assume the contrary that

$$y \in \operatorname{Inf} \bigcup_{x \in X} [\varphi(x, z)] + L\|z\| + intC.$$

Then there exist $a \in \operatorname{Inf} \bigcup_{x \in X} [\varphi(x, z)] + L\|z\|$ and $c \in intC$ such that $y = a + c$. Since C is a cone and $c \in intC$ we have $\frac{c}{2} \in intC$. So, we have $y = a + \frac{c}{2} + \frac{c}{2}$. Hence, $y - \frac{c}{2} = a + \frac{c}{2} \in \operatorname{Inf} \bigcup_{x \in X} [\varphi(x, z)] + L\|z\| + intC$ which contradicts to (5.3). Thus,

$$y \notin \operatorname{Inf} \bigcup_{x \in X} [\varphi(x, z)] + L\|z\| + intC = \phi(z) + L\|z\| + intC.$$

Because $y \in \phi(0)$ is an arbitrary element

$$\phi(0) \cap (\phi(z) + L\|z\| + intC) = \emptyset$$

for all $z \in Z$. Moreover, since $\text{wmin}\phi(0) = \operatorname{Inf}(VOP) \neq \emptyset$ from Proposition 4.6 ϕ is weakly subdifferentiable at $(0, y)$. As $y \in \phi(0)$ is an arbitrary element, ϕ is weakly subdifferentiable at 0. □

Proposition 5.9. *Let $\mathbb{R}^p$ be partially ordered by $\mathbb{R}^p_+$, the vectorial norm $\|\cdot\| : \mathbb{R}^m \to \mathbb{R}^p_+$ be defined as $\|x\| = (\underbrace{\|x\|, \ldots, \|x\|}_{p})$ for all $x \in \mathbb{R}^m$ where $\|\cdot\|$ is a norm on $\mathbb{R}^m$ and $f : \mathbb{R}^n \to \mathbb{R}^p$ be a given function. Let $\varphi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p_+$ be a perturbation function, $\phi(0)$ be $\mathbb{R}^p_+$-bounded and $\text{wmin}\phi(0) = \operatorname{Inf}(VOP) \neq \emptyset$ where ϕ is the value map obtained from φ. If (VOP) is stable, then there exist $L > 0$, and for any $\varepsilon \in int\mathbb{R}^p_+$, $x(\varepsilon) \in \mathbb{R}^n$ such that*

$$\varphi(x, z) - \varphi(x(\varepsilon), 0) + \varepsilon \underset{\mathbb{R}^p_+}{\not<} -L\|z\| \text{ for all } z \in \mathbb{R}^m.$$

Proposition 5.10 gives another necessary condition for stability of the primal problem (VOP).

Proposition 5.10. *In addition to assumptions of Proposition 5.9 let there exist $L > 0$, a neighborhood $N(0) \subset \mathbb{R}^m$ of 0 and, for any $\varepsilon \in int\mathbb{R}^p_+$, $x(\varepsilon) \in \mathbb{R}^n$ such that*

$$-L\|z\| \underset{\mathbb{R}^p_+}{\leq} \varphi(x, z) - \varphi(x(\varepsilon), 0) + \varepsilon$$

for all $z \in N(0)$ and $x \in \mathbb{R}^n$. If there exist $p \geq 0$ and $q \in \mathbb{R}^p$ such that

$$-p\|z\| + q \notin \varphi(x, z) + int\mathbb{R}^p_+$$

for all $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$, then ϕ is weakly subdifferentiable at 0.

Proof. Let $y \in \phi(0) = \operatorname{Inf} \bigcup_{x\in\mathbb{R}^n} \varphi(x,0)$ be an arbitrary element and $z \in N(0)$ be an arbitrary fixed element. Then $y \notin A(\bigcup_{x\in\mathbb{R}^n} \varphi(x,0))$. Thus, we get $\varphi(x,0) \underset{\mathbb{R}^p_+}{\not<} y$ for all $x \in \mathbb{R}^n$. In particular, $\varphi(x(\varepsilon),0) \underset{\mathbb{R}^p_+}{\not<} y$. From assumption we have

$$-L\|z\| \underset{\mathbb{R}^p_+}{\leq} \varphi(x,z) - \varphi(x(\varepsilon),0) + \varepsilon$$

for all $x \in \mathbb{R}^n$. Therefore, we obtain $\varphi(x,z) + \varepsilon + L\|z\| \underset{\mathbb{R}^p_+}{\not<} y$ which means $y - \varepsilon \notin \varphi(x,z) + L\|z\| + int\mathbb{R}^p_+$ for all $x \in \mathbb{R}^n$. Hence, we obtain

$$y - \varepsilon \notin \operatorname{Inf} \bigcup_{x\in\mathbb{R}^n} [\varphi(x,z)] + L\|z\| + int\mathbb{R}^p_+.$$

If we take limit as $\varepsilon \downarrow 0$ in the last relation, then we get

$$\begin{aligned} y &\notin \operatorname{Inf} \bigcup_{x\in\mathbb{R}^n} [\varphi(x,z)] + L\|z\| + int\mathbb{R}^p_+ \\ &= \phi(z) + L\|z\| + int\mathbb{R}^p_+ \end{aligned}$$

which means

$$\phi(0) \cap (\phi(z) + L\|z\| + int\mathbb{R}^p_+) = \emptyset.$$

Hence, $\phi(0) \not\subset \phi(z) + L\|z\| + int\mathbb{R}^p_+$ for all $z \in N(0)$.

As

$$-p\|z\| + q \notin \varphi(x,z) + int\mathbb{R}^p_+$$

for all $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$, we obtain

$$-p\|z\| + q \notin \bigcup_{x\in\mathbb{R}^n} [\varphi(x,z) + int\mathbb{R}^p_+].$$

So, we get

$$\begin{aligned} -p\|z\| + q &\notin \operatorname{Inf} \bigcup_{x\in\mathbb{R}^n} [\varphi(x,z) + int\mathbb{R}^p_+] \\ &= \phi(z) + int\mathbb{R}^p_+ \quad \text{for all } z \in \mathbb{R}^m. \end{aligned}$$

From Theorem 4.12, ϕ is weakly subdifferentiable at 0. □

Proposition 5.11. *In addition to assumptions of Proposition 5.9 let (VOP) be stable. Then there exists $L > 0$ and for any $\varepsilon \in int\mathbb{R}^p_+$ there exists $x(\varepsilon) \in \mathbb{R}^n$ such that*

$$\varphi(x,z) - \varphi(x(\varepsilon),0) + \varepsilon \underset{\mathbb{R}^p_+}{\not<} -L\|z\| \tag{5.4}$$

for all $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$, and there exist $p \geq 0$ and $q \in \mathbb{R}^p$ such that

$$-p\|z\| + q \notin \phi(z) + int\mathbb{R}^p_+ \tag{5.5}$$

for all $z \in \mathbb{R}^m$.

6 Weak Fenchel Dual Problem

In this section, by using a special perturbation function in the construction of weak conjugate dual problem weak Fenchel dual problem is obtained. Finally, an example of a nonconvex constrained vector optimization problem which can not be solved by Lagrange dual problem constructed in [25] but can be solved by weak Fenchel dual problem is presented.

Let us consider the constrained vector optimization problem

$$(VO) \quad \operatorname{Inf}\{f(x) \mid x \in G\}$$

where $f : \mathbb{R}^n \to \mathbb{R}^p$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are vector valued functions, $\mathbb{R}^p$ and $\mathbb{R}^m$ are partially ordered by $\mathbb{R}^p_+$ and $\mathbb{R}^m_+$, respectively, S is a nonempty subset of $\mathbb{R}^n$, the vectorial norm $\|\!| \cdot |\!\| : \mathbb{R}^n \to \mathbb{R}^p_+$ is defined as $|\!\|x\|\!| = (\|x\|, \ldots, \|x\|)$ for all $x \in \mathbb{R}^n$ where $\| \cdot \|$ is a norm on $\mathbb{R}^n$ and $G = \{x \in S \mid g(x) \underset{\mathbb{R}^m_+}{\leq} 0\}$.

The Fenchel perturbation function $\varphi_F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p \cup \{+\infty\}$ is defined as

$$\varphi_F(x, z) = \begin{cases} f(x+z) & , \quad x \in G \\ +\infty & , \quad otherwise \end{cases}$$

for all $x, z \in \mathbb{R}^n$.

The weak conjugate map $\varphi^w_F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \overline{\mathbb{R}}^p$ of φ_F is obtained as

$$\begin{aligned} \varphi^w_F(x_0, U, c, z_0, V, d) \quad = \quad \operatorname{Sup} \bigcup_{(x,z) \in G \times \mathbb{R}^n} & [-c|\!\|x - x_0\|\!| + c|\!\|x_0\|\!| + U(x) \\ & -d|\!\|z - z_0\|\!| + d|\!\|z_0\|\!| + V(z) - f(x+z)]. \end{aligned}$$

for all $(x_0, U, c, z_0, V, d) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$. By taking $x_0 = z_0 = 0$, $U = 0$, $c = 0$, $x + z = r$ and from Proposition 2.6 in [33] we get

$$\begin{aligned} \varphi^w_F(\mathbf{0}, V, d) &= \operatorname{Sup} \bigcup_{(x,z) \in G \times \mathbb{R}^n} [-d|\!\|z\|\!| + V(z) - f(x+z)] \\ &= \operatorname{Sup} \bigcup_{x \in G} \bigcup_{r \in \mathbb{R}^n} [-d|\!\|r - x\|\!| + V(r) - V(x) - f(r)] \\ &= \operatorname{Sup} \bigcup_{x \in G} [-V(x) - d|\!\|x\|\!| + \bigcup_{r \in \mathbb{R}^n} [V(r) + d|\!\|x\|\!| - d|\!\|r - x\|\!| - f(r)]]. \\ &= \operatorname{Sup} \bigcup_{x \in G} [-V(x) - d|\!\|x\|\!| + \operatorname{Sup} \bigcup_{r \in \mathbb{R}^n} [V(r) + d|\!\|x\|\!| - d|\!\|r - x\|\!| - f(r)]]. \\ &= \operatorname{Sup} \bigcup_{x \in G} [-V(x) - d|\!\|x\|\!| + f^w(x, V, d)]. \end{aligned}$$

Hence, by substituting φ^w_F in the weak conjugate dual problem

$$(D^w_F) \qquad \operatorname{Sup} \bigcup_{(V,d) \in \mathbb{R}^n \times \mathbb{R}_+} [-\varphi^w_F(\mathbf{0}, V, d)]$$

we obtain

$$(D^w_F) \qquad \operatorname{Sup} \bigcup_{(V,d) \in \mathbb{R}^n \times \mathbb{R}_+} \operatorname{Inf} \bigcup_{x \in G} [V(x) + d|\!\|x\|\!| - f^w(x, V, d)].$$

Example 6.1 gives a nonconvex constrained vector optimization problem which can not be solved by Lagrange dual problem constructed by Li et al. in [25] but can be solved by weak Fenchel dual problem.

Example 6.1. Let $f : \mathbb{R} \to \mathbb{R}^2$ and $g : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = (x, -|x|)$ and $g(x) = -x - 1 \leq 0$ for all $x \in \mathbb{R}$, respectively, let $S = \mathbb{R}$, $\mathbb{R}^2$ be partially ordered by $\mathbb{R}^2_+$ and the vectorial norm $\|\cdot\| : \mathbb{R} \to \mathbb{R}^2_+$ be defined as $\|x\| = (|x|, |x|)$ for all $x \in \mathbb{R}$. Let us consider the constrained vector optimization problem

$$(VOP)\begin{cases} \text{Inf } f(x) \\ s.t.\ g(x) = -x - 1 \leq 0 \\ x \in S \end{cases}, \quad i.e. \quad (VOP)\begin{cases} \text{Inf } (x, -|x|) \\ s.t.\ x \geq -1,\ x \in \mathbb{R} \end{cases}$$

It is obvious that $\text{Inf}(VOP) = \{-1\}\times[-1,+\infty)\cup[-1,1]\times\{-1\}\cup\{(x,y) \mid x \geq 1, y = -x\}$.

Firstly, let us show that (VOP) can not be solved by Lagrange dual problem constructed in [25].

Lagrange dual problem for (VOP) is defined as

$$\text{Sup}(D_L) = \text{Sup} \bigcup_{\Lambda \in L^+(\mathbb{R},\mathbb{R}^2)} \text{Inf} \bigcup_{x\in\mathbb{R}} [f(x) + \Lambda(g(x))]$$

where

$$\begin{aligned} L^+(\mathbb{R},\mathbb{R}^2) &= \{\Lambda \in L(\mathbb{R},\mathbb{R}^2) \mid \Lambda(z) \underset{\mathbb{R}^2_+}{\geq} 0 \text{ for all } z \geq 0\} \\ &= \{(a,b) \in \mathbb{R}^2 \mid az \geq 0,\ bz \geq 0 \text{ for all } z \geq 0\} \\ &= \mathbb{R}^2_+. \end{aligned}$$

Substituting $L^+(\mathbb{R},\mathbb{R}^2)$ in the dual problem, we obtain

$$\begin{aligned} \text{Sup}(D_L) &= \text{Sup} \bigcup_{\Lambda \in L^+(\mathbb{R},\mathbb{R}^2)} \text{Inf} \bigcup_{x\in\mathbb{R}} [f(x) + \Lambda(g(x))] \\ &= \text{Sup} \bigcup_{(a,b)\in\mathbb{R}^2_+} \text{Inf} \bigcup_{x\in\mathbb{R}} [(x, -|x|) + (a,b)(-x-1))] \\ &= \text{Sup} \bigcup_{(a,b)\in\mathbb{R}^2_+} \text{Inf} \bigcup_{x\in\mathbb{R}} [(x - ax - a, -|x| - bx - b)]. \end{aligned}$$

After some calculations, the set $\text{Inf} \bigcup_{x\in\mathbb{R}} [(x-ax-a, -|x|-bx-b)]$ is obtained as in Figure 1 for all $0 \leq a \leq 1$ and $b \geq 1$, and otherwise it equals $\{-\infty\}$.

As seen in Figure 1 the set of all points below $\bigcup_{(a,b)\in\mathbb{R}^2_+} \text{Inf} \bigcup_{x\in\mathbb{R}} [(x - ax - a, -|x| - bx - b)]$ equals the set of all points below $\{-1\} \times [-1,+\infty) \cup [-1,0] \times \{-1\} \cup \{(x,y) \mid x \geq 0,\ y = -2x - 1\}$. Hence,

$$\begin{aligned} \text{Sup}(D_L) &= \text{Sup} \bigcup_{(a,b)\in\mathbb{R}^2_+} \text{Inf} \bigcup_{x\in\mathbb{R}} [(x - ax - a, -|x| - bx - b)] \\ &= \{-1\} \times [-1,+\infty) \cup [-1,0] \times \{-1\} \cup \{(x,y) \mid x \geq 0,\ y = -2x - 1\}. \end{aligned}$$

So, strong duality is not satisfied.

Now, let us show that strong duality is satisfied for weak Fenchel dual problem. To do this it is enough to prove the weak subdifferentiability of the value map at 0. For this purpose we will follow below steps:

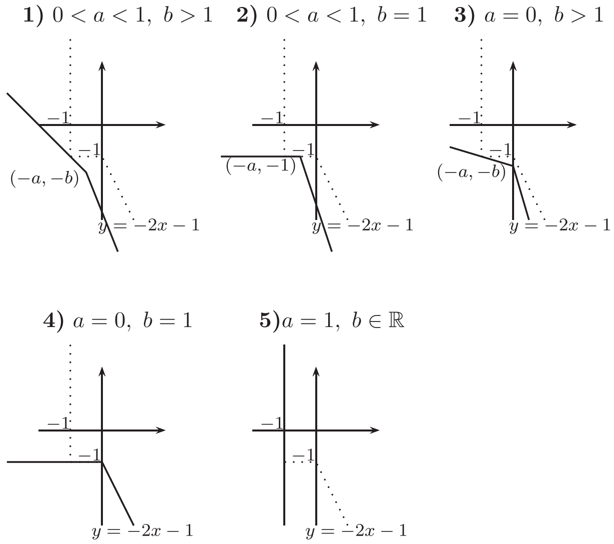

Figure 1: $\text{Inf} \bigcup_{x\in\mathbb{R}} (x-ax-a, -|x|-bx-b)$ for $0\le a\le 1$ and $b\ge 1$

- Determine the perturbation function,
- Determine the value function,
- Show the existence of $L>0$ that satisfies $\phi(0) \not\subset \phi(u) + L\|u\| + int\mathbb{R}^2_+$ for all $u\in\mathbb{R}$.

At the end, we will construct weak Fenchel dual problem and show that strong duality is held for it.

Now, let us determine above steps.

The perturbation function $\varphi_F : \mathbb{R}^2 \to \mathbb{R}^2 \cup \{+\infty\}$ is defined as

$$\varphi_F(x,u) = \begin{cases} f(x+u) & , \quad x\ge -1 \\ +\infty & , \quad otherwise. \end{cases} = \begin{cases} (x+u, -|x+u|) & , \quad x\ge -1 \\ +\infty & , \quad otherwise. \end{cases}$$

for all $(x,u)\in\mathbb{R}^2$.

The value map $\phi : \mathbb{R} \to \overline{\mathbb{R}}^2$ with respect to the perturbation map is defined as

$$\begin{aligned} \phi(u) &= \text{Inf}\{\varphi_F(x,u) \mid x\in\mathbb{R}\} = \text{Inf}\{\varphi_F(x,u) \mid x\ge -1\} \\ &= \begin{cases} \begin{array}{l} \{(x,y) \mid x=-1+u,\ y\ge -1+u\}\cup \\ \{(x,y) \mid y=-1+u,\ -1+u\le x\le 1-u\}\cup \\ \{(x,y) \mid x\ge 1-u,\ y=-x\} \end{array} & , \quad u<1 \\ \\ \begin{array}{l} \{(x,y) \mid x=-1+u,\ y\ge 1-u\}\cup \\ \{(x,y) \mid x\ge -1+u,\ y=-x\} \end{array} & , \quad u\ge 1 \end{cases} \end{aligned}$$

for all $u\in\mathbb{R}$. Image set of $\phi(\cdot)$ is shown in Figure 2 (a) and (b).

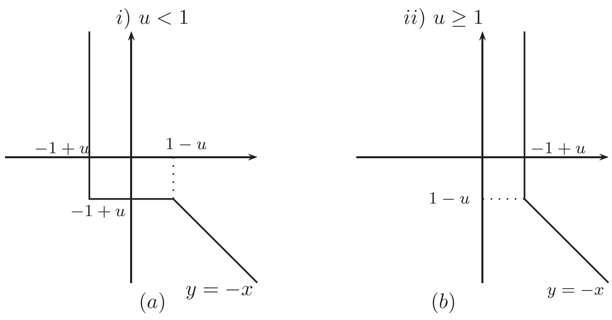

Figure 2: (a) Image set of $\phi(\cdot)$ for $u < 1$ (b) Image set of $\phi(\cdot)$ for $u \geq 1$

Let us show the existence of $L > 0$ satisfies $\phi(0) \not\subset \phi(u) + L\|u\| + int\mathbb{R}^2_+$ for all $u \in \mathbb{R}$. It is obvious that $\phi(0) = \mathrm{Inf}(VOP)$. By choosing $L = 1$ we get

$$(x, y) + L\|u\| = (x + |u|, y + |u|) = (x + |u|, -x + |u|) \in \{(x, y) \mid y > -x\}$$

for all $(x, y) \in \{(x, y) \mid x \geq 1 - u,\ y = -x\} \subset \phi(u)$ which means

$$\phi(0) \not\subset \phi(u) + L\|u\| + int\mathbb{R}^2_+$$

where $u < 1$ and $u \neq 0$. It is clear that

$$\phi(0) \not\subset \phi(u) + L\|u\| + int\mathbb{R}^2_+$$

for $u = 0$ and for all $u \geq 1$. Hence, we have

$$\phi(0) \not\subset \phi(u) + L\|u\| + int\mathbb{R}^2_+$$

for all $u \in \mathbb{R}$ that means $\phi(\cdot)$ is weakly subdifferentiable at $(0, y)$ for all $y \in \phi(0) \backslash \bigcup_{u \in \mathbb{R}} \phi(u) + L\|u\| + int\mathbb{R}^2_+ = \phi(0)$, i.e. $\phi(\cdot)$ is weakly subdifferentiable at 0. From Strong Duality Theorem we obtain $\mathrm{Inf}(VOP) = \mathrm{Sup}(D^w_F)$.

Now, let us construct weak Fenchel dual problem of (VOP). It is defined as

$$(D^w_F) \qquad \mathrm{Sup} \bigcup_{(a,b,d) \in \mathbb{R}^2 \times \mathbb{R}_+} \mathrm{Inf} \bigcup_{x_0 \geq -1} [(ax_0, bx_0) + (d|x_0|, d|x_0|) - f^w(x_0, a, b, d)].$$

After some calculations, sets $\mathrm{Inf} \bigcup_{x_0 \geq -1} [(ax_0, bx_0) + (d|x_0|, d|x_0|) - f^w(x_0, a, b, d)]$ with respect to elements $(a, b, d) \in \mathbb{R}^2 \times \mathbb{R}_+$ are found as in Figure 3 and Figure 4.

As seen in Figure 3 and Figure 4 the set of all points below

$$\bigcup_{(a,b,d) \in \mathbb{R}^2 \times \mathbb{R}_+} \mathrm{Inf} \bigcup_{x_0 \geq -1} [(ax_0, bx_0) + (d|x_0|, d|x_0|) - f^w(x_0, a, b, d)]$$

equals to the set of all points below

$$\{-1\} \times [-1, +\infty) \cup [-1, 1] \times \{-1\} \cup \{(x, y) | x \geq 1, y = -x\}.$$

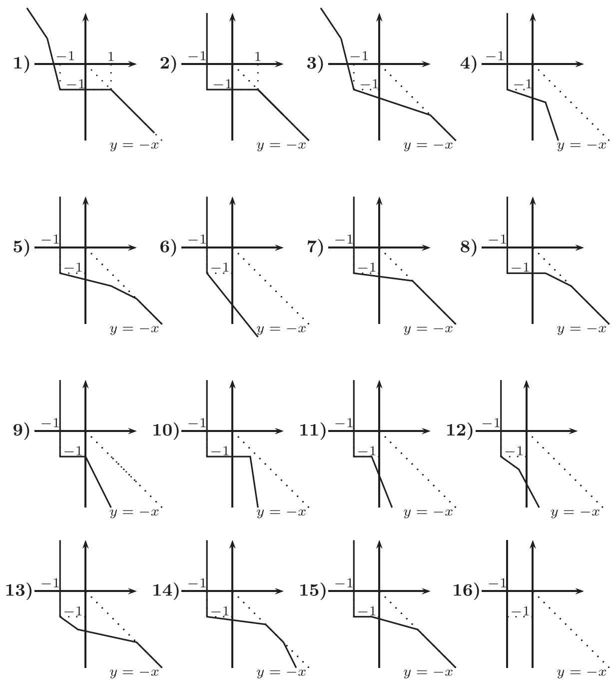

Figure 3: $\text{Inf} \bigcup_{x_0 \geq -1} [(ax_0, bx_0) + (d|x_0|, d|x_0|) - f^w(x_0, a, b, d)]$

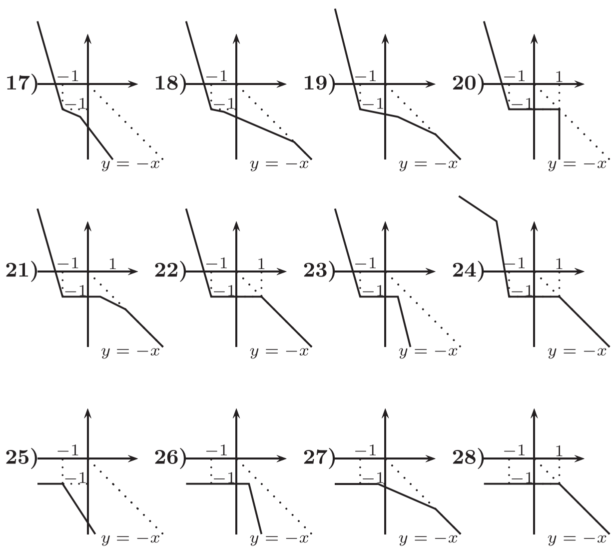

Figure 4: $\text{Inf} \bigcup_{x_0 \geq -1} [(ax_0, bx_0) + (d|x_0|, d|x_0|) - f^w(x_0, a, b, d)]$

Hence, we get

$$\begin{aligned}\mathrm{Sup}(D_F^w) &= \mathrm{Sup} \bigcup_{(a,b,d)\in\mathbb{R}^2\times\mathbb{R}_+} \mathrm{Inf} \bigcup_{x_0\geq -1} [(ax_0, bx_0) + (d|x_0|, d|x_0|) - f^w(x_0, a, b, d)]) \\ &= \{-1\} \times [-1, +\infty) \cup [-1, 1] \times \{-1\} \cup \{(x, y) \mid x \geq 1,\ y = -x\} \\ &= \mathrm{Inf}(VOP).\end{aligned}$$

So, strong duality is held for weak Fenchel dual problem.

7 Concluding Remarks

In this study, by using concepts of supremum, infimum, weak maximum of a set and vectorial norm, we defined weak conjugate maps and weak subdifferentials for set-valued maps. Furthermore, we presented necessary and sufficient conditions for weak subdifferentiability of set-valued maps. These notions enable us to construct a new conjugate dual problem for nonconvex problems. Moreover, we constructed a new conjugate dual problem for unconstrained vector optimization problems by using weak conjugate map of perturbation function and we presented weak and strong duality theorems. Furthermore, by using a special perturbation function for constrained vector optimization problem weak Fenchel dual problem was constructed. By using this dual problem we are able to solve some nonconvex constrained vector optimization problems.

References

[1] L. Altangerel, R.I. Boţ and G. Wanka, Conjugate duality in vector optimization and some applications to the vector variational inequality, *J. Math. Anal. Appl.* 329 (2007) 1010–1035.

[2] J.P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhauser, Boston, 1990.

[3] A.Y. Azimov, Duality for set-valued multiobjective optimization problems I: Mathematical Programming, *J. Optim. Theory Appl.* 137 (2008) 67–74.

[4] A.Y. Azimov, Duality for set-valued multiobjective optimization problems II: Optimal Control *J. Optim. Theory Appl.* 137 (2008) 75–88.

[5] A.Y. Azimov and R.N. Gasimov, Stability and duality of nonconvex problems via augmented Lagrangian, *Cybernet. Systems Anal.* 38 (2002) 412–421.

[6] A.Y. Azimov and R.N. Kasimov, On weak conjugacy, weak subdifferentials and duality with zero gap in nonconvex optimization, *Int. J. Appl. Math.* 1 (1999) 171–192.

[7] R.I. Boţ, S.-M. Grad and G. Wanka, A new constrained qualification and conjugate duality for composed convex optimization problems, *J. Optim. Theory Appl.* 135 (2007) 241–255.

[8] R.I. Boţ, S.-M. Grad and G. Wanka, New regularity conditions for Lagrange and Fenchel-Lagrange duality in infinite dimensional spaces, *Math. Inequal. Appl.* 12 (2009) 171–189.

[9] R.I. Boţ, S.-M. Grad and G. Wanka, *Duality In Vector Optimization*, Springer-Verlag, Berlin-Heidelberg, 2009.

[10] C.R. Chen and S.J. Li, Different conjugate dual problems in vector optimization and their relations, *J. Optim. Theory Appl.* 140 (2009) 443–461.

[11] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, North-Holland Publishing Company, Amsterdam, 1976.

[12] W. Fenchel, On conjugate convex functions, *Canad. J. Math.* 1 (1949) 73–77.

[13] A.H. Hamel, A Fenchel-Rockafellar duality theorem for set-valued optimization, *Optimization* 60 (2011) 1023-1043

[14] J. Jahn, *Vector Optimization*, Springer, Heidelberg, 2004.

[15] R.N. Kasimov, *Duality in Nonconvex Optimization.* Ph.D. Dissertation, Baku State University, Baku, 1992.

[16] R.N. Kasimbeyli and M. Mammadov, On weak subdifferentials, directional derivatives and radial epiderivatives for nonconvex functions, *SIAM J. Optim.* 20 (2009) 841–855.

[17] R.N. Kasimbeyli and M. Mammadov, Optimality conditions in nonconvex optimization via weak subdifferentials, *Nonlinear Anal.* 74 (2011) 2534–2547.

[18] R.N. Kasimbeyli, Radial epiderivatives and set-valued optimization, *Optimization* 58 (2009) 521–534.

[19] H. Kawasaki, Conjugate relations and weak subdifferentials of relations, *Math. Oper. Res.* 6 (1981) 593–607.

[20] H. Kawasaki, A duality theorem in multiobjective nonlinear programming, *Math. Oper. Res.* 7 (1982) 95–110.

[21] Y. Küçük, İ. Atasever and M. Küçük, Weak Fenchel and weak Fencel-Lagrange conjugate duality for nonconvex scalar optimization problems, *J. Global Optim.* 54 (2012) 813–830.

[22] Y. Küçük, İ. Atasever and M. Küçük, Generalized weak subdifferentials, *Optimization* 60 (2011) 537–552.

[23] Y. Küçük, İ. Atasever and M. Küçük, On generalized weak subdifferentials and some properties, *Optimization* 61 (2012) 1369–1381.

[24] Y. Küçük, İ. Atasever and M. Küçük, Some relationships among gw-subdifferential, directional derivative and radial epiderivative for nonconvex vector functions, *Optimization* 64 (2015) 627–640.

[25] S.J. Li, C.R. Chen and S.Y. Wu, Conjugate dual problems in constrained set-valued optimization and applications, *European J. Oper. Res.* 196 (2009) 21–32.

[26] A. Löhne, Optimization with set relations: conjugate duality, *Optimization* 54 (2005) 265–282.

[27] A. Löhne and C. Tammer, A new approach to duality in vector optimization, *Optimization* 56 (2007) 221–239.

[28] J.-J. Moreau, Fonctionelles sous-differentiables, *C.R. Acad. Sci. Paris* 257 (1963) 4117–4119.

[29] R.R. Phelps, Support cones in Banach spaces and their applications, *Adv. Math.* 13 (1974) 1–19.

[30] R.T. Rockafellar, *Convex Analysis* Princeton University Press, New Jersey, 1970.

[31] W. Song, Conjugate duality in set-valued vector optimization, *J. Math. Anal. Appl* 216 (1997) 265–283.

[32] W. Song, A generalization of Fenchel duality in set-valued vector optimization, *Math. Methods Oper. Res.* 48 (1998) 259–272.

[33] T. Tanino, Conjugate duality in vector optimization, *J. Math. Anal. Appl.* 167 (1992) 84–97.

[34] T. Tanino and Y. Sawaragi, Conjugate maps and duality in multiobjective optimization, *J. Optim. Theory Appl* 31 (1980) 473–499.

[35] T. Tanino, On supremum of a set in a multi-dimensional space, *J. Math. Anal. Appl.* 130 (1988) 386–397.

[36] C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific, River Edge, 2002.

Manuscript received 29 July 2013
revised 13 October 2013
accepted for publication 7 January 2014

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