Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 13, No. 2, pp. 197–217, April 2017

A REGULARIZED INEXACT SMOOTHING NEWTON METHOD FOR CIRCULAR CONE COMPLEMENTARITY PROBLEM*

Xiaoni Chi†, Jiyuan Tao, Zhibin Zhu, and Fujian Duan

Abstract: In this paper, we show that the regularized Fischer-Burmeister smoothing function is coercive under suitable assumptions, which plays an important role in the global convergence of our algorithm. Consequently, we develop a regularized inexact smoothing Newton algorithm for solving the circular cone complementarity problem (CCCP) via the regularized Fischer-Burmeister smoothing function. In addition, in our algorithm, the regularized parameter is viewed as an independent variable so that it is simpler and more easily implemented than many existing algorithms. Also, our algorithm solves only one linear system of equations approximately and performs only one line search at each iteration. Moreover, our algorithm is shown to possess global and local quadratic convergence properties without strict complementarity. Finally, some numerical results illustrate the effectiveness of our algorithm for solving the CCCP.

Key words: *Circular cone complementarity problem, regularized inexact smoothing Newton method, regularized Fischer-Burmeister smoothing function, global convergence, local quadratic convergence*

Mathematics Subject Classification: *90C25, 90C30, 65K05*

1 Introduction

The circular cone complementarity problem (CCCP) is to find a vector $x \in R^n$ such that

$$x \in C_\theta^n, \; F(x) \in (C_\theta^n)^*, \; \langle x, F(x) \rangle = 0, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ represents the Euclidean inner product, $F : R^n \to R^n$ is a continuously differentiable function, and $C_\theta^n = C_{\theta_1}^{n_1} \times C_{\theta_2}^{n_2} \times \cdots \times C_{\theta_m}^{n_m}$ with $n = n_1 + n_2 + \cdots + n_m$ is the Cartesian product of circular cones. The set $C_{\theta_i}^{n_i}, i = 1, \ldots, m$ is the circular cone of dimension n_i defined by

$$C_{\theta_i}^{n_i} := \{x_i = (x_{i0}, x_{i1}) \in R \times R^{n_i - 1} : \; \cos\theta_i \|x_i\| \le x_{i0}\}$$

with the half-aperture angle $\theta_i \in (0, \frac{\pi}{2})$, where $\|\cdot\|$ refers to the Euclidean norm. The dual cone and the interior of $C_{\theta_i}^{n_i}, i = 1, \ldots, m$ are given by [31]

$$(C_{\theta_i}^{n_i})^* := \{v_i \in R^{n_i} | \langle v_i, x_i \rangle \ge 0, \forall x_i \in C_{\theta_i}^{n_i}\} = C_{\frac{\pi}{2} - \theta_i}^{n_i},$$

*"This research was supported in part by the National Natural Science Foundation of China (Nos. 11401126, 11361018), Guangxi Natural Science Foundation (Nos. 2016GXNSFBA380102, 2014GXNSFFA118001), Guangxi Key Laboratory of Cryptography and Information Security (No. GCIS201618), and Guangxi Key Laboratory of Automatic Detecting Technology and Instruments (No. YQ15112), China. This work of the second author was supported by Loyola Summer Research Grant 2016.

†Corresponding author.

© 2017 Yokohama Publishers

and

$$\text{int}C_{\theta_i}^{n_i} := \{x_i = (x_{i0}, x_{i1}) \in R \times R^{n_i-1} :\ \cos\theta_i\|x_i\| < x_{i0}\},$$

respectively. The circular cone [13] is a pointed closed convex cone having hyperspherical sections orthogonal to its axis of revolution around which the cone is invariant to rotation. When the rotation angle $\theta_i = \frac{\pi}{4}$, the circular cone $C_{\theta_i}^{n_i}, i = 1, \dots, m$ reduces to the second-order cone (SOC) K^{n_i} [1] given by

$$K^{n_i} := \{x_i = (x_{i0}, x_{i1}) \in R \times R^{n_i-1} :\ \|x_{i1}\| \le x_{i0}\},$$

and the interior of the SOC K^{n_i} can be defined as

$$\text{int}K^{n_i} := \{x_i = (x_{i0}, x_{i1}) \in R \times R^{n_i-1} :\ \|x_{i1}\| < x_{i0}\}.$$

Thus, the CCCP includes the second-order cone complementarity problem (SOCCP) as a special case. We note that $(C_\theta^n)^*$ and C_θ^n in (1.1) are not the same cone when $\theta \neq \frac{\pi}{4}$, so smoothing Newton algorithms are not applied to (1.1) directly.

It was shown in [31] that for any $x_i = (x_{i0}, x_{i1}) \in R \times R^{n_i-1}$ and $y_i = (y_{i0}, y_{i1}) \in R \times R^{n_i-1}, i = 1, \dots, m$, the algebraic relationship between the circular cone $C_{\theta_i}^{n_i}$ and the SOC K^{n_i} is given by:

$$x_i \in K^{n_i} \Leftrightarrow T_i^{-1}x_i \in C_{\theta_i}^{n_i},\ y_i \in K^{n_i} \Leftrightarrow T_i y_i \in (C_{\theta_i}^{n_i})^*, \tag{1.2}$$

where $T_i = \begin{pmatrix} \tan\theta_i & 0 \\ 0 & I_{n_i-1} \end{pmatrix}$ and T_i^{-1} denotes the inverse matrix of T_i. Thus based on (1.2), the CCCP (1.1) can be reformulated as the SOCCP, which is to find an element $x \in R^n$ such that

$$x \in K,\ T^{-1}F(T^{-1}x) \in K,\ \langle x, T^{-1}F(T^{-1}x)\rangle = 0, \tag{1.3}$$

where $K = K^{n_1} \times K^{n_2} \times \cdots \times K^{n_m}$ with $n = n_1 + n_2 + \cdots + n_m$ is the Cartesian product of the SOCs, and $T = T_1 \oplus T_2 \oplus \cdots \oplus T_m$.

As a special nonsymmetric cone optimization problem, circular cone optimization has been paid more attention. Recently, Zhou et al. [31, 32] investigated some properties for the circular cone. Miao et al. [23] constructed complementarity functions for the CCCP and proposed a few merit functions for solving such a nonsymmetric cone complementarity problem. Alzalg [2] studied primal-dual path-following algorithms for circular programming. Bai et al. [4,5] presented primal-dual interior-point algorithms for circular cone optimization. Che et al. [8] proposed a smoothing inexact Newton method for solving the P_0 nonlinear complementarity problem, which is shown to possess global convergence and local superlinear convergence.

Motivated by their results, in this paper, we propose a regularized inexact smoothing Newton method for the CCCP in this paper. Our algorithm is shown to possess the following good properties:

(i) the algorithm is well-defined if F has the Cartesian P_0-property. This is a weaker assumption than the monotonicity assumption usually used in the CCCP;

(ii) unlike interior point methods, our algorithm does not have restrictions regarding its starting point;

(iii) the algorithm only needs to solve one linear system of equations and to perform one line search at each iteration;

(iv) in absence of strict complementarity, our algorithm is globally and locally quadratically convergent based on the coerciveness of the regularized Fischer-Burmeister smoothing function;

(v) the regularized parameter is viewed as an independent variable, and hence our algorithm is conceptually simpler and more easily implemented than most existing algorithms.

The organization of this paper is as follows. In Section 2, we review some basic concepts. In Section 3, we propose a regularized inexact smoothing Newton method for the CCCP. Moreover, we show that our algorithm is well-defined and the regularized Fischer-Burmeister smoothing function is coercive under suitable assumptions, which plays an important role in the analysis on the global convergence of our algorithm. In Section 4, We establish the global convergence and local quadratic convergence of our algorithm. In Section 5, we give some numerical results.

The following notations are used throughout this paper. R^n (respectively, R) denotes the space of n-dimensional real column vectors (respectively, the set of real numbers). The symbol $\|\cdot\|$ denotes the Euclidean norm defined by $\|x\| := \sqrt{x^T x}$ for a vector x. For convenience, we often use $x = (x_0, x_1)$ for the column vector $x = (x_0, x_1^T)^T \in R \times R^{n-1}$. For two matrices A and B, we define

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

For any $\alpha, \beta > 0$, we write $\alpha = O(\beta)$ (respectively, $\alpha = o(\beta)$) to mean that α/β is uniformly bounded (respectively, tends to zero) as $\beta \to 0$.

2 Preliminaries

In this section, we review Euclidean Jordan algebra associated with the SOC K^n [16], the concept of semsismoothness [27], and the Cartesian P_0-properties [19], which are used in the subsequent analysis.

2.1 Euclidean Jordan algebra associated with the SOC K^n

For any $x = (x_0, x_1)$ and $s = (s_0, s_1) \in R \times R^{n-1}$, the Jordan product is defined as

$$x \circ s = (x^T s, x_0 s_1 + s_0 x_1).$$

The element $e = (1, 0, \cdots, 0) \in R^n$ is the unit element of this algebra. For any $x = (x_0, x_1) \in R \times R^{n-1}$, we define the symmetric matrix

$$L_x = \begin{pmatrix} x_0 & x_1^T \\ x_1 & x_0 I \end{pmatrix}.$$

It is easy to verify that

$$x \circ s = L_x s = L_s x, \quad \forall s \in R^n.$$

Moreover, L_x is positive definite (and hence invertible) if and only if $x \in \text{int}K^n$. More details can be found in [1, 16].

For any $x = (x_0, x_1) \in R \times R^{n-1}$, its spectral factorization is defined as

$$x = \lambda_1(x) u^{(1)}(x) + \lambda_2(x) u^{(2)}(x).$$

Here $\lambda_1(x),\ \lambda_2(x)$ are the spectral values given by

$$\lambda_i(x) = x_0 + (-1)^i \|x_1\|, \ i = 1, 2,$$

and $u^{(1)}(x),\ u^{(2)}(x)$ are the associated spectral vectors given by

$$u^{(i)}(x) = \begin{cases} \frac{1}{2}(1, (-1)^i \frac{x_1}{\|x_1\|}) & \text{if } x_1 \neq 0, \\ \frac{1}{2}(1, (-1)^i \omega) & \text{otherwise,} \end{cases} \quad i = 1, 2,$$

with $\omega \in R^{n-1}$ being any vector satisfying $\|\omega\| = 1$. By using the spectral factorization, we may extend scalar functions to SOC functions. For example, we define

$$x^2 = \lambda_1^2(x)u^{(1)}(x) + \lambda_2^2(x)u^{(2)}(x), \quad \forall x \in R^n.$$

Since both $\lambda_1(x)$ and $\lambda_2(x)$ are nonnegative for any $x \in K^n$, we define

$$\sqrt{x} = \sqrt{\lambda_1(x)}u^{(1)}(x) + \sqrt{\lambda_2(x)}u^{(2)}(x), \quad \forall x \in K^n.$$

2.2 Semismoothness

Definition 2.1. Suppose that $G : R^m \to R^n$ is locally Lipschitz continuous around $x \in R^m$. G is said to be semismooth at x if G is directionally differentiable at x and for any $V \in \partial G(x + \triangle x)$,

$$G(x + \triangle x) - G(x) - V(\triangle x) = o(\|\triangle x\|),$$

where ∂G stands for the generalized Jacobian of G in the sense of Clarke [12]. G is said to be p-order $(0 < p < \infty)$ semismooth at x if G is semismooth at x and

$$G(x + \triangle x) - G(x) - V(\triangle x) = O(\|\triangle x\|^{1+p}).$$

In particular, G is said to be strongly semismooth at x if G is said to be 1-order semismooth at x.

A function $G : R^m \to R^n$ is said to be a semismooth (respectively, p-order semismooth) function if it is semismooth (respectively, p-order semismooth) everywhere in R^m. Semismooth functions include smooth functions, piecewise smooth functions, and convex and concave functions. The composition of (strongly) semismooth functions is still a (strongly) semismooth function [24].

Semismoothness is closely connected with the local convergence analysis of our algorithm. The concept of semismooth was originally introduced by Mifflin [24] for functionals and extended by Qi and Sun [27] to vector-valued functions.

2.3 Cartesian P_0-properties

Definition 2.2. A nonlinear mapping $f = (f_1, ..., f_m)$ with $f_i : R^n \to R^{n_i}$ is said to have

(a) the Cartesian $P-$property, if for any $x, y \in R^n$ with $x \neq y$, there exists an index $\nu \in \{1, 2, ..., m\}$ such that

$$\langle x_\nu - y_\nu, f_\nu(x) - f_\nu(y)\rangle > 0;$$

(b) the Cartesian P_0-property, if for any $x, y \in R^n$ with $x \neq y$, there exists an index $\nu \in \{1, 2, ..., m\}$ such that

$$x_\nu \neq y_\nu \text{ and } \langle x_\nu - y_\nu, f_\nu(x) - f_\nu(y)\rangle \geq 0.$$

It is obvious that a monotone function must have the Cartesian P_0-property. Therefore the assumption that a function with the Cartesian P_0-property is a weaker assumption than the monotonicity assumption usually used in the CCCP.

Definition 2.3. A matrix $M \in R^{n\times n}$ is said to have

(a) the Cartesian $P-$property, if for every nonzero $z = (z_1, ..., z_m) \in R^n$ with $z_i \in R^{n_i}$, there exists an index $\nu \in \{1, 2, ..., m\}$ such that

$$\langle z_\nu, (Mz)_\nu\rangle > 0;$$

(b) the Cartesian P_0-property, if for every nonzero $z = (z_1, ..., z_m) \in R^n$ with $z_i \in R^{n_i}$, there exists an index $\nu \in \{1, 2, ..., m\}$ such that

$$z_\nu \neq 0 \text{ and } \langle z_\nu, (Mz)_\nu\rangle \geq 0.$$

In addition, if a continuously differentiable mapping f has the Cartesian $P-$property (or Cartesian P_0-property), then its transposed Jacobian matrix $\nabla f(x)$ at any $x \in R^n$ has the corresponding Cartesian $P-$property (or Cartesian P_0-property) [25].

The concepts of P_0-properties on Cartesian products in R^n were first established by Facchinei and Pang [15]. Recently, Chen, Qi [9] and Kong, Tuncel, Xiu [19] extended the concepts of Cartesian P_0-properties to the setting of positive semidefinite cones and Euclidean Jordan algebra, respectively.

3 Regularized Inexact Smoothing Newton Method

In this section, we show that under suitable assumptions the regularized Fischer-Burmeister smoothing function is coercive, which will play an important role in the global convergence of our algorithm. Also, we propose a regularized inexact smoothing Newton method for the CCCP based on the regularized Fischer-Burmeister smoothing function. Moreover, we prove that our algorithm is well-defined if F has the Cartesian P_0-property. This is a weaker assumption than the monotonicity assumption usually used in the SOCCP.

Since the circular cone C^n_θ is a closed convex cone in R^n, we obtain from Theorem 2.4.4 in [15] the following existence result of the CCCP. If the continuous mapping F is pseudo monotone, then the CCCP (1.1) has a nonempty compact solution set S^* if and only if the CCCP (1.1) is strictly feasible, i.e., there exists an $\bar{x} \in C^n_\theta$ and $F(\bar{x}) \in \text{int}(C^n_\theta)^*$.

In the following analysis, we assume that the CCCP (1.1) is strictly feasible, which guarantees the nonemptiness and boundedness of the solution set S^*. We consider the regularized inexact smoothing method for the SOCCP (1.3). Our algorithm solves a sequence of SOCCPs (F_μ):

$$x \in K,\ F_\mu(x) \in K,\ \langle x, F_\mu(x)\rangle = 0, \tag{3.1}$$

where $\mu > 0$ is a regularization parameter tending to zero, and F_μ is given by

$$F_\mu(x) = T^{-1}F(T^{-1}x) + \mu x. \tag{3.2}$$

By Definition 2.2, we have the following result.

Lemma 3.1. *For any $\mu > 0$, let F_μ be given as in (3.2). If F has the Cartesian P_0-property, F_μ has the Cartesian $P-$property.*

Now, we consider the smoothing SOC complementarity function [7] $\phi : R_+ \times R^n \times R^n \to R^n$ defined by

$$\phi(\mu, x, s) = (1 + \mu)(x + s) - \sqrt{(x + \mu s)^2 + (\mu x + s)^2 + 2\mu^2 e}, \tag{3.3}$$

which is the smoothing function of the vector-valued Fischer-Burmeister function $\phi_{\rm FB}$: $R^n \times R^n \to R^n$ given by [17]

$$\phi_{\rm FB}(x,s) := x + s - \sqrt{x^2+s^2}.$$

In fact, the smoothing function (3.3) is a regularized version of the smoothing Fischer-Burmeister SOC complementarity function [17]

$$\hat{\phi}(\mu,x,s) := x + s - \sqrt{x^2+s^2+2\mu^2 e}. \tag{3.4}$$

Notice that [7]

$$\phi(\mu,x,s) = 0 \Leftrightarrow x \circ s = \mu^2 e,\ x \in {\rm int}K,\ s \in {\rm int}K.$$

Especially, we have

$$\phi(0,x,s) = 0 \Leftrightarrow x \circ s = 0,\ x \in K,\ s \in K. \tag{3.5}$$

Let $z := (\mu,x,s) \in R \times R^n \times R^n$. By using the smoothing function (3.3), we define

$$H(z) := \begin{pmatrix} \mu \\ \Phi_\mu(x,s) \end{pmatrix}, \tag{3.6}$$

$$\Phi_\mu(x,s) := \begin{pmatrix} T^{-1}F(T^{-1}x) + \mu x - s \\ \phi(\mu,x,s) \end{pmatrix}, \tag{3.7}$$

and $\Psi(z) := \|H(z)\|^2$. Then it follows from (1.1), (1.2), (1.3), (3.1), (3.2), (3.5), (3.6) and (3.7) that

$$H(z) = 0 \Leftrightarrow (x,s) \text{ solves the SOCCP (1.3)} \Leftrightarrow (T^{-1}x, Ts) \text{ solves the CCCP (1.1)}.$$

To implement our algorithm, we need to reformulate the SOCCP (1.3) as a nonlinear system of equations $H(z) = 0$, which does not contain any explicit inequality constraints like $x \in K, s \in K$ or $x \in {\rm int}K, s \in {\rm int}K$. So we apply Newton's method to this system approximately at each iteration. By driving $\|H(z)\| \downarrow 0$, we have a solution of the SOCCP (1.3), and hence a solution of the CCCP (1.1).

In what follows, we study the Lipschitzian, semismoothness and differential properties of the function $H(z)$ and derive the computable formulas for its Jacobian. These properties and formulas play a fundamental role in developing and analyzing our algorithm.

Theorem 3.2. *Let $H(z)$ be defined as in (3.6) and (3.7). Then the following results hold.*

(i) *H is Lipschitz continuous and semismooth everywhere in $R_+ \times R^n \times R^n$. If $F'(\cdot)$ is Lipschitz continuous in R^n, H is strongly semismooth everywhere in $R_+ \times R^n \times R^n$.*

(ii) *If $\mu > 0$, H is continuously differentiable everywhere in $R_{++} \times R^n \times R^n$ with its Jacobian*

$$H'(z) = \begin{pmatrix} 1 & 0 & 0 \\ x & T^{-1}F'(T^{-1}x)T^{-1} + \mu I & -I \\ \phi'_\mu(z) & \phi'_x(z) & \phi'_s(z) \end{pmatrix}, \tag{3.8}$$

where

$$\phi'_\mu(z) = x + s - L_w^{-1}(L_u s + L_v x + 2\mu e),$$

$$\phi'_x(z) = (1+\mu)I - L_w^{-1}(L_u + \mu L_v),\ \phi'_s(z) = (1+\mu)I - L_w^{-1}(\mu L_u + L_v),$$
$$u := u(\mu, x, s) = x + \mu s,\ v := v(\mu, x, s) = \mu x + s,$$
$$w := w(\mu, x, s) = \sqrt{u^2 + v^2 + 2\mu^2 e}.$$

(iii) *If F has the Cartesian P_0-property, the matrix $H'(z)$ is nonsingular for any $\mu > 0$.*

Proof. (i) By Theorem 3.2 in [29], it is not difficult to see that $H(z)$ is Lipschitz continuous and semismooth everywhere in $R_+ \times R^n \times R^n$. If $F'(\cdot)$ is Lipschitz continuous in R^n, $F(\cdot)$ is strongly semismooth everywhere in R^n. Therefore, H is strongly semismooth everywhere in $R_+ \times R^n \times R^n$.

(ii) Now we prove (3.8). For any $\mu > 0$, it follows from Theorem 3.2 in [29] that $\phi(\mu, x, s)$ is continuously differentiable and its Jacobian is given by

$$\phi'_\mu(z) = x + s - L_w^{-1}(L_u s + L_v x + 2\mu e),$$

$$\phi'_x(z) = (1+\mu)I - L_w^{-1}(L_u + \mu L_v),$$
$$\phi'_s(z) = (1+\mu)I - L_w^{-1}(\mu L_u + L_v),$$

where

$$u := u(\mu, x, s) = x + \mu s,\ v := v(\mu, x, s) = \mu x + s,$$
$$w := w(\mu, x, s) = \sqrt{u^2 + v^2 + 2\mu^2 e}.$$

Thus, we have the desired Jacobian formula (3.8).

(iii) Let $\Delta z := (\Delta\mu, \Delta x, \Delta s) \in R \times R^n \times R^n$ be a vector in the null space of $H'(z)$. It is sufficient to show $\Delta z = 0$. By (3.8), we have

$$\Delta\mu = 0,$$

$$x\Delta\mu + [T^{-1}F'(T^{-1}x)T^{-1} + \mu I]\Delta x - \Delta s = 0, \tag{3.9}$$

$$\phi'_\mu(z)\Delta\mu + \phi'_x(z)\Delta x + \phi'_s(z)\Delta s = 0. \tag{3.10}$$

Now we assume that $\Delta x \neq 0$. Thus, by (3.10) we have

$$[(1+\mu)I - L_w^{-1}(L_u + \mu L_v)]\Delta x + [(1+\mu)I - L_w^{-1}(\mu L_u + L_v)]\Delta s = 0. \tag{3.11}$$

Applying L_w to both sides of (3.11) yields

$$[(1+\mu)L_w - (L_u + \mu L_v)]\Delta x + [(1+\mu)L_w - (\mu L_u + L_v)]\Delta s = 0.$$

Thus, we have for $i = 1, 2, ..., m$ that

$$[(1+\mu)L_{w_i} - (L_{u_i} + \mu L_{v_i})]\Delta x_i + [(1+\mu)L_{w_i} - (\mu L_{u_i} + L_{v_i})]\Delta s_i = 0. \tag{3.12}$$

Let

$$\overline{L}_i := (1+\mu)L_{w_i} - (L_{u_i} + \mu L_{v_i}),$$
$$\underline{L}_i := (1+\mu)L_{w_i} - (\mu L_{u_i} + L_{v_i}),$$

for $i = 1, 2, ..., m$. Direct calculations yield

$$[(1+\mu)w_i]^2 - [(u_i + \mu v_i)^2 + (\mu u_i + v_i)^2] = 2\mu(u_i - v_i)^2 + 2\mu^2(1+\mu)^2 e_i \in \text{int}K^{n_i}, \tag{3.13}$$

It follows from (3.13) and Lemma 3.5 in [17] that $\overline{L}_i$, $\underline{L}_i$ and the symmetric part of $\overline{L}_i\underline{L}_i$ are all positive definite and hence they are all nonsingular. Multiplying both sides of (3.12) by $\Delta x_i^T \underline{L}_i^{-1}$ from the left yields

$$\Delta x_i^T \underline{L}_i^{-1}\overline{L}_i \Delta x_i + \langle \Delta x_i, \Delta s_i \rangle = 0, \ i = 1, 2, ..., m,$$

or equivalently by (3.9)

$$\overline{\Delta x}_i^T \overline{L}_i \underline{L}_i \overline{\Delta x}_i + \langle \Delta x_i, [(T^{-1}F'(T^{-1}x)T^{-1} + \mu I)\Delta x]_i \rangle = 0, \ i = 1, 2, ..., m, \tag{3.14}$$

where $\overline{\Delta x}_i := \underline{L}_i^{-1}\Delta x_i$. Since the symmetric part of $\overline{L}_i\underline{L}_i$ is positive definite, we have

$$\overline{\Delta x}_i^T \overline{L}_i \underline{L}_i \overline{\Delta x}_i \geq 0, \ i = 1, 2, ..., m. \tag{3.15}$$

Since $F(x)$ has the Cartesian P_0-property, $F_\mu(x)$ has the Cartesian $P-$property by Lemma 3.1. Therefore, the transposed Jacobian matrix $\nabla F_\mu(x)$ at any $x \in R^n$ has the corresponding Cartesian $P-$property, i.e., there exists an index $\nu \in \{1, 2, ..., m\}$ such that

$$\langle \Delta x_\nu, [(T^{-1}F'(T^{-1}x)T^{-1} + \mu I)\Delta x]_\nu \rangle > 0.$$

The last inequality together with (3.15) implies that

$$\overline{\Delta x}_\nu^T \overline{L}_\nu \underline{L}_\nu \overline{\Delta x}_\nu + \langle \Delta x_\nu, [(T^{-1}F'(T^{-1}x)T^{-1} + \mu I)\Delta x]_\nu \rangle > 0,$$

which contradicts to (3.14). Thus we have $\Delta x = 0$ and hence $\Delta s = 0$ from (3.9). This completes the proof □

Now, we show that under suitable assumptions the regularized Fischer-Burmeister smoothing function ϕ defined by (3.3) is coercive, which will play an important role in the global convergence of our algorithm.

Lemma 3.3. *Let the smoothing Fischer-Burmeister SOC complementarity function $\hat{\phi}$: $R_+ \times R^n \times R^n \to R^n$ be defined by (3.4) and $\{(\mu_k, x^k, s^k)\} \subset R_+ \times R^n \times R^n$, where $\{\mu_k\}$ is a bounded sequence. Then*

(i) *If $\lambda_1(x^k) \to -\infty$ or $\lambda_1(s^k) \to -\infty$ as $k \to \infty$, then $\|\hat{\phi}(\mu_k, x^k, s^k)\| \to +\infty$.*

(ii) *If $\{\lambda_1(x^k)\}$ and $\{\lambda_1(s^k)\}$ are bounded below and $\lambda_2(x^k) \to +\infty$, $\lambda_2(s^k) \to +\infty$, and $\langle \bar{x}^*, \bar{s}^* \rangle > 0$, where $\bar{x}^*$ and $\bar{s}^*$ are any given accumulation points of $\left\{\frac{x^k}{\|x^k\|}\right\}$ and $\left\{\frac{s^k}{\|s^k\|}\right\}$ as $k \to \infty$, then $\|\hat{\phi}(\mu_k, x^k, s^k)\| \to +\infty$.*

Proof. (i) By the definition of the function $\hat{\phi}$, we have that for any $(\mu, x, s) \in R_+ \times R^n \times R^n$,

$$x = \hat{\phi}(\mu, x, s) + (\sqrt{x^2 + s^2 + 2\mu^2 e} - s), \tag{3.16}$$

$$s = \hat{\phi}(\mu, x, s) + (\sqrt{x^2 + s^2 + 2\mu^2 e} - x). \tag{3.17}$$

Since by Proposition 3.4 in [17]

$$\sqrt{x^2 + s^2 + 2\mu^2 e} - s \in K, \quad \sqrt{x^2 + s^2 + 2\mu^2 e} - x \in K$$

for any $(\mu, x, s) \in R_+ \times R^n \times R^n$, it follows from Lemma 14 in [28] that

$$\begin{aligned}\lambda_1(x) &\geq \lambda_1(\hat{\phi}(\mu, x, s)) + \lambda_1(\sqrt{x^2 + s^2 + 2\mu^2 e} - s)\\ &\geq \lambda_1(\hat{\phi}(\mu, x, s)),\end{aligned}$$

$$\begin{aligned}\lambda_1(s) &\geq \lambda_1(\hat{\phi}(\mu, x, s)) + \lambda_1(\sqrt{x^2 + s^2 + 2\mu^2 e} - x)\\ &\geq \lambda_1(\hat{\phi}(\mu, x, s)).\end{aligned}$$

The above inequalities together with (3.16) and (3.17) imply that if $\lambda_1(x^k) \to -\infty$ or $\lambda_1(s^k) \to -\infty$, then $\|\hat{\phi}(\mu_k, x^k, s^k)\| \to +\infty$.

(ii) On the contrary, we assume that $\{\hat{\phi}(\mu_k, x^k, s^k)\}$ is bounded. Let

$$\hat{w}^k := \sqrt{(x^k)^2 + (s^k)^2 + 2(\mu_k)^2 e}$$

for any $k \geq 0$. From the definition of $\hat{\phi}$, we have for any k

$$x^k + s^k = \hat{\phi}(\mu_k, x^k, s^k) + \hat{w}^k.$$

Squaring both sides of the last equality yields

$$2x^k \circ s^k = 2(\mu_k)^2 e + \hat{\phi}(\mu_k, x^k, s^k)^2 + 2\hat{w}^k \circ \hat{\phi}(\mu_k, x^k, s^k). \tag{3.18}$$

Since $\|x^k\| \to +\infty$ and $\|s^k\| \to +\infty$ by the given conditions, we obtain

$$\lim_{k\to\infty} \frac{\hat{w}^k}{\|x^k\|\|s^k\|} = \lim_{k\to\infty} \left[\frac{(x^k)^2 + (s^k)^2 + 2(\mu_k)^2 e}{\|x^k\|^2\|s^k\|^2}\right]^{1/2} = 0, \tag{3.19}$$

which together with the boundedness of $\{\hat{\phi}(\mu_k, x^k, s^k)\}$, implies

$$\lim_{k\to\infty} \frac{\hat{\phi}(\mu_k, x^k, s^k)^2 + 2\hat{w}^k \circ \hat{\phi}(\mu_k, x^k, s^k)}{\|x^k\|\|s^k\|} = 0. \tag{3.20}$$

Since the sequences $\left\{\frac{x^k}{\|x^k\|}\right\}$ and $\left\{\frac{s^k}{\|s^k\|}\right\}$ are bounded, without loss of generality, we assume that $\left\{\frac{x^k}{\|x^k\|}\right\}$ and $\left\{\frac{s^k}{\|s^k\|}\right\}$ are convergent. Let

$$\bar{x}^* := \lim_{k\to\infty} \frac{x^k}{\|x^k\|}, \quad \bar{s}^* := \lim_{k\to\infty} \frac{s^k}{\|s^k\|}.$$

Since both $\{\lambda_1(x^k)\}$ and $\{\lambda_1(s^k)\}$ are bounded below, $\lambda_2(x^k) \to +\infty$ and $\lambda_2(s^k) \to +\infty$, it is easy to verify that

$$\bar{x}^* \in K, \quad \bar{s}^* \in K. \tag{3.21}$$

Thus, by (3.18), (3.19) and (3.20), we obtain

$$\begin{aligned}\bar{x}^* \circ \bar{s}^* &= \lim_{k\to\infty} \tfrac{x^k}{\|x^k\|} \circ \tfrac{s^k}{\|s^k\|}\\ &= \lim_{k\to\infty} \left[\tfrac{(\mu_k)^2 e}{\|x^k\|\|s^k\|} + \tfrac{\hat{\phi}(\mu_k, x^k, s^k)^2 + 2\hat{w}^k \circ \hat{\phi}(\mu_k, x^k, s^k)}{2\|x^k\|\|s^k\|}\right] = 0.\end{aligned} \tag{3.22}$$

From (3.21) and (3.22), we have $\langle \bar{x}^*, \bar{s}^* \rangle = 0$, which contradicts the condition that $\langle \bar{x}^*, \bar{s}^* \rangle > 0$. Therefore, $\|\hat{\phi}(\mu_k, x^k, s^k)\| \to +\infty$ as $k \to \infty$ in this case. This completes the proof.

Remark Lemma 3.3 indicates that the smoothing Fischer-Burmeister SOC complementarity function (3.4) is coercive. In Lemma 5.2 [26], Pan and Chen show that the Fischer-Burmeister SOC complementarity function

$$\phi_{\mathrm{FB}}(x, s) = x + s - \sqrt{x^2 + s^2}$$

is coercive. However, we prove (i) in another way, and thus give the detailed proof for completeness. □

Theorem 3.4. *Let the smoothing function ϕ be defined by (3.3), and $\eta, \tau \in R_{++}$ with $\eta < \tau$. Suppose that $\{(\mu_k, x^k, s^k)\} \subset R_{++} \times R^n \times R^n$ is a sequence, where $\{\mu_k\}$ is a non-negative bounded sequence.*

(C1) *$\mu_k \in [\eta, \tau]$ and $\{(x^k, s^k)\}$ is unbounded; and*

(C2) *there is a bounded sequence $\{(a^k, b^k)\}$ such that $\{\langle x^k - a^k, s^k - b^k \rangle\}$ is bounded below.*

Then $\{\phi(\mu_k, x^k, s^k)\}$ is unbounded.

Proof. By the definition of the function ϕ, we have

$$\phi(\mu, x, s) = (x + \mu s) + (\mu x + s) - \sqrt{(x + \mu s)^2 + (\mu x + s)^2 + 2\mu^2 e}.$$

From Lemma 3.3, we see that if there is a subsequence $\{k_n\} \subseteq \{k\}$ such that one of the following conditions holds:

(i) $\lambda_1(x^{k_n} + \mu_{k_n} s^{k_n}) \to -\infty$ as $k \to \infty$;

(ii) $\lambda_1(\mu_{k_n} x^{k_n} + s^{k_n}) \to -\infty$ as $k \to \infty$;

(iii) $\{\lambda_1(x^{k_n} + \mu_{k_n} s^{k_n})\}$ and $\{\lambda_1(\mu_{k_n} x^{k_n} + s^{k_n})\}$ are bounded below, and $\lambda_2(x^{k_n} + \mu_{k_n} s^{k_n}) \to +\infty$, $\lambda_2(\mu_{k_n} x^{k_n} + s^{k_n}) \to +\infty$ as $k \to \infty$, and $\langle \bar{u}^*, \bar{v}^* \rangle > 0$, where $\bar{u}^*$ and $\bar{v}^*$ are any given accumulation points of $\left\{ \frac{x^{k_n} + \mu_{k_n} s^{k_n}}{\|x^{k_n} + \mu_{k_n} s^{k_n}\|} \right\}$ and $\left\{ \frac{\mu_{k_n} x^{k_n} + s^{k_n}}{\|\mu_{k_n} x^{k_n} + s^{k_n}\|} \right\}$, respectively.

then $\{\phi(\mu_k, x^k, s^k)\}$ is unbounded.

Without loss of generality, we assume that cases (i) and (ii) do not appear. Then by following the proof of Theorem 4.1 in [18], we can show that case (iii) must hold. This together with Lemma 3.3 implies that $\|\phi(\mu_{k_n}, x^{k_n}, s^{k_n})\| \to +\infty$ as $k \to \infty$. This completes the proof. □

Theorem 3.5. *Suppose that F is a continuously differentiable monotone function and Φ_μ is defined by (3.7). Then $\Phi_\mu(x, s)$ is coercive in (x, s) for each $\mu > 0$, i.e., $\lim_{\|(x,s)\| \to \infty} \|\Phi_\mu(x, s)\| = +\infty$.*

Proof. On the contrary, we assume that $\Phi_\mu(x, s)$ is not coercive in (x, s) for some $\mu > 0$. Then there exists an unbounded sequence $\{(x^k, s^k)\}$ such that $\{\Phi_\mu(x, s)\}$ is bounded. Since we obtain from (3.7)

$$\|\Phi_\mu(x^k, s^k)\|^2 = \|T^{-1}F(T^{-1}x^k) + \mu x^k - s^k\|^2 + \|\phi(\mu, x^k, s^k)\|^2,$$

the sequences $\{T^{-1}F(T^{-1}x^k) + \mu x^k - s^k\}$ and $\{\phi(\mu, x^k, s^k)\}$ are bounded. Let

$$g(x^k, s^k) := T^{-1}F(T^{-1}x^k) + \mu x^k - s^k.$$

Then $g(x^k, s^k)$ is bounded and

$$s^k = T^{-1}F(T^{-1}x^k) + \mu x^k - g(x^k, s^k).$$

Let $\{a^k\}$ be an arbitrary bounded sequence and

$$b^k := T^{-1}F(T^{-1}a^k) + \mu a^k - g(x^k, s^k).$$

Thus $\{F(T^{-1}a^k)\}$ is bounded and therefore $\{b^k\}$ is bounded. Since F is monotone, we have

$$\begin{aligned}\langle x^k - a^k, s^k - b^k\rangle &= \langle x^k - a^k, [T^{-1}F(T^{-1}x^k) + \mu x^k] - [T^{-1}F(T^{-1}a^k) + \mu a^k]\rangle \\ &= \langle T^{-1}x^k - T^{-1}a^k, F(T^{-1}x^k) - F(T^{-1}a^k)\rangle + \mu\|x^k - a^k\|^2 \\ &\geq \mu\|x^k - a^k\|^2 \\ &\geq 0.\end{aligned}$$

Thus, it follows from Theorem 3.4 that $\lim_{\|(x^k,s^k)\|\to\infty} \|\phi(\mu, x^k, s^k)\| = +\infty$, which contradicts the boundedness of $\{\phi(\mu, x^k, s^k)\}$. Therefore, $\Phi_\mu(x, s)$ is coercive in (x, s) for each $\mu > 0$. This completes the proof. □

For any $\mu > 0$ and $c > 0$, let the level set

$$L_\mu(c) = \left\{(x, s) \in R^{2n} : \|H(\mu, x, s)\| \leq c\right\},$$

and for any $0 < \underline{\mu} \leq \overline{\mu}$ and $c > 0$, let

$$L(c) = \bigcup_{\underline{\mu}\leq\mu\leq\overline{\mu}} L_\mu(c).$$

Theorem 3.6. *Suppose that F is a continuously differentiable monotone function. Then for each $\mu > 0$, the SOCCP (F_μ) (3.1) has a unique bounded solution $x(\mu)$.*

Proof. We first show the existence of a solution. For any $\mu > 0$, let $z^0 := (\mu, x^0, s^0)$ be an arbitrary point and define $c_0 := \sqrt{\Psi(z^0)}$. By Theorem 3.5, the level set $L_\mu(c_0)$ is nonempty and compact. Thus, the continuous function $\|H(\mu, x, s)\|$ attains a bounded global minimum $x(\mu)$ on $L_\mu(c_0)$, which is also a global minimum of $\|H(\mu, x, s)\|$ on R^n. Therefore, $x(\mu)$ is a stationary point of $\|H(\mu, x, s)\|$. Since F is a monotone function, it follows from Definition 2.2 and Lemma 3.1 that F_μ has the Cartesian $P-$property. Thus, we obtain from Propostion 3.1 (c) [25] that $x(\mu)$ is a bounded solution of the regularized problem SOCCP (F_μ) (3.1).

Suppose that $x(\mu)$ and $\hat{x}(\mu)$ are two different solutions of the SOCCP (F_μ). Since F is monotone, we have

$$\begin{aligned}&0 < \mu\|x(\mu) - \hat{x}(\mu)\|^2 \\ &\leq \langle T^{-1}x(\mu) - T^{-1}\hat{x}(\mu), F(T^{-1}x(\mu)) - F(T^{-1}\hat{x}(\mu))\rangle + \mu\|x(\mu) - \hat{x}(\mu)\|^2 \\ &= \langle x(\mu) - \hat{x}(\mu), F_\mu(x(\mu)) - F_\mu(\hat{x}(\mu))\rangle \\ &= -\langle x(\mu), F_\mu(\hat{x}(\mu))\rangle - \langle \hat{x}(\mu), F_\mu(x(\mu))\rangle,\end{aligned}$$

where the last equality is due to $\langle x(\mu), F_\mu(x(\mu))\rangle = 0$ and $\langle \hat{x}(\mu), F_\mu(\hat{x}(\mu))\rangle = 0$. Thus we obtain a contradiction with $\langle x(\mu), F_\mu(\hat{x}(\mu))\rangle > 0$ and $\langle \hat{x}(\mu), F_\mu(x(\mu))\rangle > 0$. Therefore, the SOCCP ($F_\mu$) (3.1) has a unique bounded solution $x(\mu)$. This completes the proof. □

Let $\gamma \in (0,1)$ and $\mu_0 > 0$. Define the function $\beta : R_+ \times R^n \times R^n \to R_+$ by

$$\beta(z) := \gamma \min\{1, \Psi(z)\}. \tag{3.23}$$

Algorithm 3.1 (A regularized inexact smoothing Newton method)**. Step 0** Choose constants $\delta \in (0,1)$, $\sigma \in (0,1/2)$ and $\mu_0 > 0$. Let $\overline{z} := (\mu_0, 0, 0) \in R_{++} \times R^n \times R^n$, and $z^0 := (\mu_0, x^0, s^0) \in R_{++} \times R^n \times R^n$ be an arbitrary point. Choose $\varepsilon, \gamma \in (0,1)$ such that $\varepsilon\mu_0 + \gamma\mu_0 < 1$ and $\gamma\Psi(z^0) < 1$. Set $k := 0$.
Step 1 If $\|H(z^k)\| = 0$, stop. Otherwise, let $\beta_k := \beta(z^k)$.
Step 2 Compute $\Delta z^k := (\Delta\mu_k, \Delta x^k, \Delta s^k) \in R \times R^n \times R^n$ by

$$H(z^k) + H'(z^k)\Delta z^k = \beta_k \overline{z} + \mathbf{r}^k, \tag{3.24}$$

where $\mathbf{r}^k := (0, 0, r^k) \in R \times R^n \times R^n$ is a residual satisfying $\|\mathbf{r}^k\| \le \varepsilon\mu_0 \min\{1, \Psi(z^k)\}$.
Step 3 Let $\lambda_k = \max\{\delta^l |\ l = 0, 1, 2, \ldots\}$ such that

$$\Psi(z^k + \lambda_k \Delta z^k) \le [1 - 2\sigma(1 - \gamma\mu_0 - \varepsilon\mu_0)\lambda_k]\Psi(z^k). \tag{3.25}$$

Step 4 Set $z^{k+1} := z^k + \lambda_k \Delta z^k$, and $k := k + 1$. Go to Step 1.

Theorem 3.7. *If F has the Cartesian P_0-property, Algorithm 3.1 is well-defined.*

Proof. Suppose that $\mu_k > 0$. Then by (3.23) and (3.24), we have

$$\mu_{k+1} = (1 - \lambda_k)\mu_k + \lambda_k \beta(z^k)\mu_0 > 0. \tag{3.26}$$

The inequality (3.26) together with $\mu_0 > 0$, implies that $\mu_k \in R_{++}$ for any $k \ge 0$. Since F has the Cartesian P_0-property, it follows from Theorem 3.2 that the matrix $H'(z^k)$ is nonsingular. Hence Step 2 is well-defined.

Next we show that Step 3 is well-defined. For any $\alpha \in (0,1]$, define

$$r(\alpha) := \Psi(z^k + \alpha\Delta z^k) - \Psi(z^k) - \alpha\Psi'(z^k)\Delta z^k. \tag{3.27}$$

Taking into account the fact $\mu_k > 0$ and using Theorem 3.2 (i), we obtain that $H(\cdot)$ is continuously differentiable around z^k. By (3.27), we have

$$|r(\alpha)| = o(\alpha). \tag{3.28}$$

It follows from (3.23), (3.24), (3.27) and (3.28) that for any $\alpha \in (0,1]$

$$\begin{aligned}
\Psi(z^k + \alpha\Delta z^k) &= \Psi(z^k) + \alpha\Psi'(z^k)\Delta z^k + r(\alpha) \\
&\le (1 - 2\alpha)\Psi(z^k) + 2\alpha H(z^k)^T(\beta(z^k)\overline{z} + \mathbf{r}^k) + o(\alpha) \\
&\le (1 - 2\alpha)\Psi(z^k) + 2\alpha\mu_0\beta(z^k)\|H(z^k)\| \\
&\quad + 2\alpha\|\mathbf{r}^k\| \cdot \|H(z^k)\| + o(\alpha) \\
&\le (1 - 2\alpha)\Psi(z^k) + 2\alpha(\gamma + \varepsilon)\mu_0\Psi(z^k) + o(\alpha) \\
&= [1 - 2(1 - \gamma\mu_0 - \varepsilon\mu_0)\alpha]\Psi(z^k) + o(\alpha).
\end{aligned} \tag{3.29}$$

Here, if $\|H(z^k)\| \le 1$, we have $\beta(z^k) = \gamma\Psi(z^k)$; and if $\|H(z^k)\| > 1$, we have $\beta(z^k) = \gamma$ and $\|H(z^k)\| \le \Psi(z^k)$. Therefore, $\beta(z^k)\|H(z^k)\| \le \gamma\Psi(z^k)$. Similarly, we can show that $\|\mathbf{r}^k\| \cdot \|H(z^k)\| \le \varepsilon\mu_0\Psi(z^k)$, since $\|\mathbf{r}^k\| \le \varepsilon\mu_0 \min\{1, \Psi(z^k)\}$. Thus, the third inequality holds. Then the inequality (3.29) implies that there exists a constant $\overline{\alpha} \in (0,1]$ such that

$$\Psi(z^k + \alpha\Delta z^k) \le [1 - 2\sigma(1 - \gamma\mu_0 - \varepsilon\mu_0)\alpha]\Psi(z^k)$$

holds for any $\alpha \in (0, \overline{\alpha}]$. This demonstrates that Step 3 is well-defined, which completes the proof. □

4 Convergence Analysis

From Theorem 3.7, Algorithm 3.1 generates an infinite sequence $\{z^k\} := \{(\mu_k, x^k, s^k)\}$ under suitable assumptions. In this section, we show that the sequence $\{z^k\}$ is bounded based on the coerciveness of the function ϕ defined by (3.3). Moreover, we prove that if an accumulation point of the iteration sequence $\{z^k\}$ satisfies a nonsingularity assumption, then the iteration sequence converges to the accumulation point globally and locally quadratically without strict complementarity. To show the global convergence of Algorithm 3.1, we need the following two lemmas.

Lemma 4.1. *Suppose that F has the Cartesian P_0-property, and $\{z^k\}$ is the infinite sequence generated by Algorithm 3.1. Then $\mu_k \in R_{++}$ and $z^k \in \Omega$ for any $k \geq 0$, where*

$$\Omega = \{z = (\mu, x, s) \in R_{++} \times R^n \times R^n : \ \mu \geq \beta(z)\mu_0\}.$$

Proof. By Theorem 3.7, we have $\mu_k \in R_{++}$ for any $k \geq 0$. Now, we prove $\mu_k \geq \beta(z^k)\mu_0$ for any $k \geq 0$ by mathematical induction on k. It is obvious that $\mu_0 \geq \beta(z^0)\mu_0$, because $\beta(z^0) \leq \gamma < 1$. Suppose that $\mu_k \geq \beta(z^k)\mu_0$. We prove $\mu_{k+1} \geq \beta(z^{k+1})\mu_0$ by considering the following two cases:

Case (i) If $\|H(z^k)\| > 1$, by (3.23) we have

$$\beta(z^k) = \gamma. \tag{4.1}$$

Since $\mu_k \geq \beta(z^k)\mu_0$, it follows from (3.26) and (4.1) that

$$\begin{aligned} \mu_{k+1} - \beta(z^{k+1})\mu_0 &= (1 - \lambda_k)\mu_k + \lambda_k\beta(z^k)\mu_0 - \beta(z^{k+1})\mu_0 \\ &\geq (1 - \lambda_k)\beta(z^k)\mu_0 + \lambda_k\gamma\mu_0 - \gamma\mu_0 \\ &= 0. \end{aligned} \tag{4.2}$$

Case (ii) If $\|H(z^k)\| \leq 1$, we obtain from (3.23)

$$\beta(z^k) = \gamma\Psi(z^k). \tag{4.3}$$

By (3.23) and (3.25), we have $\Psi(z^{k+1}) \leq \Psi(z^k) \leq 1$ and hence

$$\beta(z^{k+1}) = \gamma\Psi(z^{k+1}). \tag{4.4}$$

Since $\mu_k \geq \beta(z^k)\mu_0$, it follows from (3.26), (4.3) and (4.4) that

$$\begin{aligned} \mu_{k+1} - \beta(z^{k+1})\mu_0 &= (1 - \lambda_k)\mu_k + \lambda_k\beta(z^k)\mu_0 - \gamma\mu_0\Psi(z^{k+1}) \\ &\geq (1 - \lambda_k)\beta(z^k)\mu_0 + \lambda_k\gamma\mu_0\Psi(z^k) - \gamma\mu_0\Psi(z^k) \\ &= 0. \end{aligned} \tag{4.5}$$

Combining (4.2) and (4.5) yields $\mu_k \geq \beta(z^k)\mu_0$ for any $k \geq 0$. This completes the proof. □

Lemma 4.2. *Suppose that F is a continuously differentiable monotone function. Then the sequence $\{\Psi(z^k)\}$ generated by Algorithm 3.1 is convergent. If it does not converge to zero, then the sequence $\{z^k\}$ is bounded.*

Proof. From Step 3 of Algorithm 3.1, the sequence $\{\Psi(z^k)\}$ is monotonically decreasing and thus it is convergent. Then there exists Ψ^* such that $\Psi(z^k) \to \Psi^*$. If $\{\Psi(z^k)\}$ does not converge to zero, then $\Psi^* > 0$. It follows from (3.26) and Lemma 4.1 that $z^k \in \Omega$ and

$$0 < \mu_{k+1} = (1-\lambda_k)\mu_k + \lambda_k \beta(z^k)\mu_0 \le \mu_k,$$

which imply $\{\mu_k\}$ is bounded. Then there exist $\underline{\mu}, \overline{\mu} > 0$ such that

$$0 < \underline{\mu} \le \mu_k \le \overline{\mu}$$

for all $k \ge 0$. Let $c_0 := \sqrt{\Psi(z^0)}$ and

$$L(c_0) = \bigcup_{\underline{\mu} \le \mu_k \le \overline{\mu}} L_{\mu_k}(c_0).$$

It is not difficult to see that $(x^k, s^k) \in L(c_0)$, because of $(x^k, s^k) \in L_{\mu_k}(c_0)$. It follows from Theorem 3.5 that the set $L(c_0)$ is bounded and therefore $\{(x^k, s^k)\}$ is bounded. Hence, the sequence $\{z_k\} := \{(\mu_k, x^k, s^k)\}$ is bounded. This completes the proof. □

Theorem 4.3. *Suppose that F is a continuously differentiable monotone function, and $\{z^k\}$ is the iteration sequence generated by Algorithm 3.1. Then $\{\mu_k\}$ and $\{\|H(z^k)\|\}$ converge to zero as $k \to \infty$, and hence any accumulation point $z^* = (\mu^*, x^*, s^*)$ of $\{z^k\}$ is a solution of $H(z) = 0$.*

Proof. From Algorithm 3.1, we obtain that the sequence $\{\Psi(z^k)\}$ is monotonically decreasing, denoting its limit by Ψ^*. If $\Psi^* = 0$, we obtain the desired result. On the contrary, suppose $\Psi^* > 0$. Since $\{z_k\}$ is bounded by Lemma 4.2, we suppose that $\{z^k\} := \{(\mu_k, x^k, s^k)\}$ converges to $z^* = (\mu^*, x^*, s^*)$ as $k \to \infty$, by taking a subsequence if necessary. By (3.25), we get $\|H(z^{k+1})\| \le \|H(z^k)\|$. Then from the continuity of $H(\cdot)$ and (3.23), we have that $\{\|H(z^k)\|\}$ and $\{\beta_k\}$ converge to $\|H(z^*)\|$ and β^* respectively as $k \to \infty$, and

$$\Psi^* = \Psi(z^*) = \|H(z^*)\|^2, \quad \beta^* = \gamma \min\{1, \Psi^*\}.$$

It follows from Lemma 4.1 that $0 < \beta^* \mu_0 \le \mu^*$. Therefore, by Theorem 3.2, $H'(z^*)$ exists and is invertible. Hence, there exists a closed neighborhood $N(z^*)$ of z^* such that for any $z \in N(z^*)$, we have $\mu \in R_{++}$ and $H'(z)$ is invertible. For any $z \in N(z^*)$, let $\Delta z := (\Delta\mu, \Delta x, \Delta s) \in R \times R^n \times R^n$ be the unique solution of the following system of equations

$$H(z) + H'(z)\Delta z = \beta(z)\overline{z} + \mathbf{r}. \tag{4.6}$$

Then by (4.6) and the proof of Theorem 3.7, we can find a positive number $\overline{\alpha} \in (0, 1]$ such that

$$\Psi(z + \alpha\Delta z) \le [1 - 2\sigma(1 - \gamma\mu_0 - \varepsilon\mu_0)\alpha]\Psi(z)$$

holds for any $\alpha \in [0, \overline{\alpha}]$ and any $z \in N(z^*)$. Hence there exists a nonnegative integer $\bar{l}$ such that $\delta^{\bar{l}} \in (0, \overline{\alpha}]$ and for any $z \in N(z^*)$,

$$\Psi(z + \delta^{\bar{l}}\Delta z) \le [1 - 2\sigma(1 - \gamma\mu_0 - \varepsilon\mu_0)\delta^{\bar{l}}]\Psi(z). \tag{4.7}$$

For all sufficiently large k, since $\lambda_k = \delta^{l^k} \ge \delta^{\bar{l}}$, it follows from (3.25) and (4.7) that

$$\begin{aligned} \Psi(z^{k+1}) &= \Psi(z^k + \lambda_k \Delta z^k) \\ &\le [1 - 2\sigma(1 - \gamma\mu_0 - \varepsilon\mu_0)\lambda_k]\Psi(z^k) \\ &\le [1 - 2\sigma(1 - \gamma\mu_0 - \varepsilon\mu_0)\delta^{\bar{l}}]\Psi(z^k). \end{aligned}$$

This contradicts the fact that $\{\Psi(z^k)\}$ converges to $\Psi(z^*) > 0$. So, we complete our proof. □

Now we analyze the local quadratic convergence of our regularized inexact smoothing Newton method. To establish the rate of convergence for Algorithm 3.1, we assume that z^* satisfies the non-singularity condition but may not satisfy the strict complementarity.

Theorem 4.4. *Suppose that F is a continuously differentiable monotone function, and $z^* := (\mu^*, x^*, s^*)$ is an accumulation point of the iteration sequence $\{z^k\}$ generated by Algorithm 3.1. If $F'(\cdot)$ is Lipschitz continuous on R^n and all $V \in \partial H(z^*)$ are nonsingular, the sequence $\{z^k\}$ converges to z^* quadratically, i.e.,*

$$\|z^{k+1} - z^*\| = O(\|z^k - z^*\|^2) \text{ and } \mu_{k+1} = O((\mu_k)^2).$$

Proof. By Theorem 4.3, we have that z^* is a solution of $H(z) = 0$. Since all $V \in \partial H(z^*)$ are nonsingular, it follows from Proposition 3.1 in [27] that for all z^k sufficiently close to z^*,

$$\|H'(z^k)^{-1}\| = O(1). \tag{4.8}$$

In view of Theorem 3.2, we know that $H(\cdot)$ is globally Lipschitz continuous and strongly semismooth at z^*, since $F'(\cdot)$ is Lipschitz continuous on R^n. Then, for all z^k sufficiently close to z^*,

$$\|H(z^k) - H(z^*)\| = O(\|z^k - z^*\|), \tag{4.9}$$

$$\|H(z^k) - H(z^*) - H'(z^k)(z^k - z^*)\| = O(\|z^k - z^*\|^2). \tag{4.10}$$

Thus from (3.23) and (4.9), we have for all z^k sufficiently close to z^*

$$\beta(z^k)\mu_0 = \gamma\mu_0\|H(z^k)\|^2 = O(\|z^k - z^*\|^2), \tag{4.11}$$

$$\|\mathbf{r}^k\| \leq \varepsilon\mu_0\|H(z^k)\|^2 = O(\|z^k - z^*\|^2). \tag{4.12}$$

It follows from (3.24), (4.8), (4.10), (4.11) and (4.12) that for all z^k sufficiently close to z^*

$$\begin{aligned}
\|z^k + \Delta z^k - z^*\| &= \|z^k + H'(z^k)^{-1}[-H(z^k) + \beta(z^k)\overline{z} + \mathbf{r}^k] - z^*\| \\
&\leq \|H'(z^k)^{-1}\|[\|H(z^k) - H(z^*) - H'(z^k)(z^k - z^*)\| + \beta(z^k)\mu_0 + \|\mathbf{r}^k\|] \\
&= O(\|z^k - z^*\|^2) + O(\|z^k - z^*\|^2) \\
&= O(\|z^k - z^*\|^2).
\end{aligned} \tag{4.13}$$

By following the proof of Theorem 3.1 in [24], for all z^k sufficiently close to z^*, we have

$$\|z^k - z^*\| = O(\|H(z^k) - H(z^*)\|). \tag{4.14}$$

From Theorem 3.2, $H(\cdot)$ is Lipschitz continuous in $R_+ \times R^n \times R^n$. Then by (4.13), (4.14) and Theorem 4.3, we obtain

$$\begin{aligned}
\Psi(z^k + \Delta z^k) &= \|H(z^k + \Delta z^k)\|^2 \\
&= O(\|z^k + \Delta z^k - z^*\|^2) = O(\|z^k - z^*\|^4) \\
&= O(\|H(z^k) - H(z^*)\|^4) = O(\|H(z^k)\|^4) \\
&= O(\Psi(z^k)^2).
\end{aligned}$$

Therefore, by Step 3 of Algorithm 3.1, we have for all z^k sufficiently close to z^*

$$z^{k+1} = z^k + \Delta z^k. \tag{4.15}$$

On account of (4.11) and (4.13), we get

$$\|z^{k+1} - z^*\| = O(\|z^k - z^*\|^2). \tag{4.16}$$

Next, it follows from Theorem 4.3 and (3.23) that for all sufficiently large k,

$$\beta(z^k)\mu_0 = \gamma\mu_0\|H(z^k)\|^2. \tag{4.17}$$

Due to (3.24), (4.15) and (4.17), we have for all k sufficiently large that

$$\mu_{k+1} = \mu_k + \Delta\mu_k = \beta(z^k)\mu_0 = \gamma\mu_0\|H(z^k)\|^2. \tag{4.18}$$

From (4.9), (4.14), (4.16) and (4.18), it follows that

$$\begin{aligned}
\mu_{k+1} &= O(\|H(z^k)\|^2) = O(\|H(z^k) - H(z^*)\|^2) \\
&= O(\|z^k - z^*\|^2) = O[(\|z^{k-1} - z^*\|^2)^2] \\
&= O[(\|H(z^{k-1}) - H(z^*)\|^2)^2] \\
&= O[(\|H(z^{k-1})\|^2)^2] \\
&= O((\mu_k)^2).
\end{aligned}$$

This completes the proof. □

5 Numerical examples

In order to evaluate the efficiency of Algorithm 3.1, we give some numerical examples to solve some linear and nonlinear CCCPs in this section. All the experiments were performed on a desktop computer with Intel(R) Pentium(R) Dual T2390 CPU 1.86 GHz and 1.00 GB memory. The operating system was Windows XP and the implementations were done in MATLAB 7.0.1. Throughout the numerical experiments, the parameters used in Algorithm 3.1 were as follows:

$$\theta \in (0, \frac{\pi}{2}),\ \mu_0 = 0.01,\ \sigma = 0.25,\ \delta = 0.75,\ \ \gamma = 0.5\min\{1, 1/\Psi(z^0)\}, \quad \text{and} \quad \varepsilon = 0.5.$$

We used $\Psi(z^k) \leq 10^{-16}$ as the stopping criterion.

In the tables of test results, n denote the size of problems; Iter denotes the (average) number of iterations; CPU(s) denotes the (average) CPU time in seconds; Gap denotes the (average) value of $|F(x^k)^T x^k|$; and FV denotes the final value of the merit function $\Psi(z^k)$ when the algorithm terminates.

Firstly, we solve the randomly generated linear CCCPs with size n from 50 to 800 and $m = 1$. In details, we generate a random matrix $N = rand(n, n)$ and a random vector $q = rand(n, 1)$, and then let $M := N^T N$. Since the matrix M is semidefinite positive, the generated problem (1.1) with $F(x) = Mx + q$ is the monotone CCCP, i.e, the generated problem (1.3) with $T^{-1}F(T^{-1}x) = T^{-1}(MT^{-1}x + q)$ is the monotone SOCCP. The random problems of each size are generated 10 times. The test results with initial points $x^0 = s^0 = e$, $x^0 = s^0 = 0.5e$ and $x^0 = s^0 = 0$ are listed in Table 5.1, Table 5.2 and Table 5.3 respectively, where e is the unit element in K^n.

Table 5.1 Numerical results for CCCPs with $x^0 = s^0 = e$.

n	Iter	CPU(s)	Gap	FV
50	6.1	0.031	9.6016e-010	5.3934e-019
100	6.8	0.109	1.7222e-012	3.6865e-024
150	7.0	0.328	2.5163e-015	1.0348e-027
200	7.5	0.766	1.5363e-014	2.3278e-026
250	7.8	1.625	5.4466e-015	1.6473e-026
300	7.9	2.907	5.4365e-013	8.0609e-025
350	8.0	4.469	1.5209e-011	7.8996e-022
400	8.0	6.391	6.8908e-015	4.2665e-026
450	7.9	8.781	1.5421e-014	1.3086e-025
500	8.0	11.687	2.7890e-012	4.3398e-023
550	9.1	17.313	1.3641e-013	7.2001e-025
600	8.9	21.812	5.1560e-013	2.5866e-024
650	9.1	27.313	1.3427e-012	1.4644e-023
700	9.2	33.765	9.0980e-012	6.1730e-022
750	8.9	40.890	9.1546e-012	5.7576e-022
800	9.1	49.437	1.8626e-011	2.4879e-021

Table 5.2 Numerical results for CCCPs with $x^0 = s^0 = 0.5e$.

n	Iter	CPU(s)	Gap	FV
50	6.2	0.032	5.7068e-012	1.3997e-023
100	7.0	0.125	2.0326e-012	5.9899e-024
150	6.9	0.316	5.6915e-015	5.1181e-027
200	7.2	0.750	5.2456e-011	7.4463e-021
250	7.6	1.610	3.5169e-015	5.7512e-027
300	7.9	2.896	1.1024e-013	5.2038e-026
350	7.9	4.437	2.1119e-013	2.7782e-025
400	7.9	6.375	5.0047e-014	6.8460e-026
450	8.0	8.797	7.7485e-012	2.4945e-022
500	8.1	11.688	5.6870e-011	1.4883e-020
550	8.2	15.234	3.9812e-009	4.7810e-017
600	8.8	21.781	8.6913e-014	5.3515e-025
650	8.9	27.282	7.7922e-012	5.6972e-022
700	9.0	33.594	6.0384e-011	3.9890e-020
750	8.9	40.781	8.8146e-011	7.5776e-020
800	9.0	49.297	1.8946e-010	3.5064e-019

Table 5.3 Numerical results for CCCPs with $x^0 = s^0 = 0$.

n	Iter	CPU(s)	Gap	FV
50	5.6	0.015	1.6424e-012	4.3335e-024
100	6.9	0.110	1.3442e-010	1.2050e-020
150	7.1	0.328	8.7321e-010	9.7362e-019
200	8.0	0.875	6.7762e-013	4.0285e-024
250	8.1	1.641	5.4950e-012	1.8651e-022
300	8.0	2.984	4.6720e-012	1.0733e-022
350	8.2	4.562	9.5269e-011	2.0080e-020
400	8.2	6.546	1.3604e-010	4.3035e-020
450	8.6	10.359	3.7250e-010	2.4952e-019
500	9.0	13.172	7.3178e-014	5.3020e-025
550	8.9	17.156	8.8283e-014	2.8849e-025
600	9.0	21.812	2.9380e-014	6.4902e-025
650	9.0	27.297	1.4048e-012	9.7702e-024
700	9.1	33.641	3.1555e-012	5.1778e-023
750	9.0	40.938	2.0388e-011	2.7301e-021
800	9.2	48.984	1.6508e-011	1.4478e-021

Secondly, we consider the nonlinear CCCP (1.1) with $\theta = \frac{\pi}{4}$, i.e., SOCCP on $K = K^3 \times K^2$, with F given by

$$F(x) = \begin{pmatrix} 24(2x_1 - x_2)^3 + e^{x_1 - x_3} - 4x_4 + x_5 \\ -12(2x_1 - x_2)^3 + 3(x_1 + 3x_2)/\sqrt{1 + (x_1 + 3x_2)^2} - 6x_4 - 7x_5 \\ -e^{x_1 - x_3} + (x_1 + 3x_2)/\sqrt{1 + (x_1 + 3x_2)^2} - 3x_4 + 5x_5 \\ 4x_1 + 6x_2 + 3x_3 - 1 \\ -x_1 + 7x_2 - 5x_3 + 2 \end{pmatrix}.$$

By Algorithm 3.1, we obtain the solution

$$x^* \approx (0.231996, 0.013019, 0.231630, 0.189174, -0.189174).$$

We test the problem by different initial points and the test results are listed in Table 5.4.

Now, we give several observations from the results of these tables.

(i) All the random problems have been solved in a small number of iterations and in very short CPU time.

(ii) The problem size only slightly affects the number of iterations.

(iii) For the same dimension of linear CCCPs, different initial points generally do not affect the number of iterations and the CPU time; However, even for the same nonlinear CCCP, the number of iterations and the CPU time vary with the initial points.

(iv) Our algorithm is effective for solving both linear and nonlinear CCCPs.

Table 5.4 Numerical results for the nonlinear SOCCP.

$x^0 = s^0$	Iter	CPU(s)	Gap	FV
(0,0,0,0,0)	9	0.031	5.3104e-010	5.2559e-018
(0.5,0,0,0.5,0)	11	0.031	1.0628e-010	2.1054e-019
(-0.5,0,0,-0.5,0)	10	0.031	4.5271e-011	3.8199e-020
(1,0,0,1,0)	15	0.047	5.2362e-011	5.1104e-020
(-1,0,0,-1,0)	12	0.032	2.0926e-010	8.1619e-019
(1,1,1,1,1)	12	0.031	3.6465e-011	2.4783e-020
(-1,-1,-1,-1,-1)	10	0.016	9.0490e-011	1.5262e-019
(10,10,10,10,10)	16	0.047	5.4691e-010	5.5749e-018
(-10,-10,-10,-10,-10)	14	0.032	4.4255e-010	3.6505e-018

6 Conclusions

In this paper, we have proved that the regularized Fischer-Burmeister smoothing function ϕ defined by (3.3) is coercive under suitable assumptions. Based on the function ϕ, we reformulate the CCCP (1.1) as a nonlinear system of equations $H(z) = 0$. Consequently, we show that the Jacobian $H'(z)$ is nonsingular if F has the Cartesian P_0-property, which is a weaker assumption than the monotonicity assumption usually used in the SOCCP. In addition, we develop a regularized inexact smoothing Newton method for the CCCP. In our algorithm, the regularized parameter is viewed as an independent variable. Hence, our algorithm is simpler and more easily implemented than many existing algorithms. Moreover, our algorithm solves only one linear system of equations and performs only one line search at each iteration. Also, our algorithm is shown to possess global convergence and local quadratic convergence without strict complementarity. Finally, some numerical results show that our algorithm is effective for solving both linear and nonlinear CCCPs.

Acknowledgements

The authors are grateful to the editor and the anonymous referees for their careful reading and valuable comments, which have greatly improved this paper.

References

[1] F. Alizadeh and D. Goldfarb, Second-order cone programming, *Math. Program.* 95 (2003) 3–51.

[2] B. Alzalg, Primal-dual path-following algorithms for circular programming, 2015, http:// www.optimization-online.org/DB_FILE/2015/07/4998.pdf.

[3] S. Boyd and B. Wegbreit, Fast Computation of Optimal Contact Forces, *IEEE T. Robot.* 23 (2007) 1117–1132.

[4] Y. Bai, X. Gao, and G. Wang, Primal-dual interior-point algorithms for convex quadratic circular cone optimization, *Numer. Algebra Control Optim.* 5 (2015) 211–231.

[5] Y. Bai, P. Ma, and J. Zhang, A polynomial-time interior-point method for circular cone programming based on kernel functions, *J. Ind. Manag. Optim.* 12 (2016) 739–756.

[6] J. Burke and S. Xu, A non-interior predictor-corrector path following algorithm for the monotone linear complementarity problem, *Math. Program.* 87 (2000) 113–130.

[7] X. Chi, S. Liu, A one-step smoothing Newton method for second-order cone programming, *J. Comput. Appl. Math.* 223 (2009) 114–123.

[8] H. Che, Y. Wang and M. Li, A smoothing inexact Newton method for P_0 nonlinear complementarity problem, *Front. Math. China* 7 (2012) 1043–1058.

[9] X. Chen, H. Qi, Cartesian P-property and its applications to the semidefinite linear complementarity problem, *Math. Program.* 106 (2006) 177–201.

[10] X. Chen, L. Qi and D. Sun, Global and superlinear convergence of the smoothing Newton method and its application to general box constrained variational inequalities, *Math. Comput.* 67 (1998) 519–540.

[11] B. Chen and N. Xiu, A global linear and local quadratic non-interior continuation method for nonlinear complementarity problems based on Chen-Mangasarian smoothing functions, *SIAM J. Optim.* 9 (1999) 605–623.

[12] F.H. Clarke, *Optimization and Nonsmooth Analysis*, John Wiley and Sons, New York, 1983.

[13] J. Dattorro, *Convex Optimization and Euclidean Distance Geometry*, California: Meboo Publishing, 2005.

[14] S. Engelke and C. Kanzow, Predictor-corrector smoothing methods for linear programs with a more flexible update of the smoothing parameter, *Comput. Optim. Appl.* 23 (2002) 299–320.

[15] F. Facchinei, F. Pang, *Finite Dimensional Variational Inequalities and Complementarity Problems*, Springer-Verlag, New York, 2003.

[16] J. Faraut, and A. Korányi, *Analysis on Symmetric Cones*, Oxford University Press, Oxford, UK, 1994.

2010) 557–579.

[17] M. Fukushima, Z.Q. Luo and P. Tseng, Smoothing functions for second-order-cone complementarity problems, *SIAM J. Optim.* 12 (2001) 436–460.

[18] Z. Huang and T. Nie, Smoothing algorithms for complementarity problems over symmetric cones, *Comput. Optim. Appl.* 45 (

[19] L. Kong, L. Tuncel, N. Xiu, Vector-valued implicit Lagrangian for symmetric cone complementarity problems, *Asia-Pac. J. Oper. Res.* 26 (2006) 199–233.

[20] L. Qi, D. Sun and G. Zhou, A new look at smoothing Newton methods for nonlinear complementarity problems and box constrained variational inequalities, *Math. Program.* 87 (2000) 1–35.

[21] Y. Li, X. Wang, and D. Wei, A new class of smoothing complementarity functions over symmetric cones, *Commun. Nonlinear Sci. Numer. Simul.* 15 (2010) 3299–3305.

[22] Z.J. Li, S.Z. Sam Ge and S.B. Liu, Contact-force distribution optimization and control for quadruped robots using both gradient and adaptive neural networks, *IEEE T. Neur. Net. Lear.* 8 (2014) 1460–1473.

[23] X. Miao, S. Guo, N. Qi, and J.-S. Chen, Constructions of complementarity functions and merit functions for circular cone complementarity problem, *Comput. Optim. Appl.* 63 (2016) 495–522.

[24] R. Mifflin, Semismooth and semiconvex functions in constrained optimization, *SIAM J. Control Optim.* 15 (1977) 957–972.

[25] S. Pan, J.-S. Chen, A regularization method for the second-order cone complementarity problem with the Cartesian P_0-property, *Nonlinear Anal.* 70 (2009) 1475–1491.

[26] S. Pan and J.-S. Chen, A damped Gauss-Newton method for the second-order cone complementarity problem, *Appl. Math. Optim.* 59 (2009) 293–318.

[27] L. Qi, and J. Sun, A nonsmooth version of Newton's method, *Math. Program.* 58 (1993) 353—367.

[28] S. H. Schmieta, F. Alizadeh, Extension of primal-dual interior-point algorithms to symmetric cones, *Math. Program.* 96 (2003) 409–438.

[29] D. Sun, and J. Sun, Strong semismoothness of Fischer-Burmeister SDC and SOC complementarity functions, *Math. Program.* 103 (2005) 575–581.

[30] G. Wang, Y. Bai, A class of polynomial interior point algorithms for the Cartesian P-Matrix linear complementarity problem over symmetric cones, *J. Optim. Theory Appl.* 152 (2012) 739–772.

[31] J. Zhou and J.-S. Chen, Properties of circular cone and spectral factorization associated with circular cone, *J. Nonlinear Convex Anal.* 14 (2013) 1504–1509.

[32] J. Zhou, J.-S. Chen, and B.S. Mordukhovich, Variational analysis of circular cone programs, *Optim.* 64 (2014) 113–147.

Manuscript received
revised
accepted for publication

Xiaoni Chi
School of Mathematics and Computing Science
Guangxi Key Laboratory of Cryptography and Information Security
Guilin University of Electronic Technology
Guilin 541004, Guangxi, P. R. China
E–mail address: chixiaoni@126.com

Jiyuan Tao
Department of Mathematics and Statistics
Loyola University Maryland
Baltimore, MD 21210, USA
E–mail address: jtao@loyola.edu

Zhibin Zhu
School of Mathematics and Computing Science
Guangxi Key Laboratory of Automatic Detecting Technology and Instruments
Guilin University of Electronic Technology
Guilin 541004, Guangxi, P. R. China
E–mail address: zhuzbma@hotmail.com

Fujian Duan
School of Mathematics and Computing Science
Guangxi Colleges and Universities Key Laboratory of Data Analysis and Computation
Guilin University of Electronic Technology
Guilin 541004, Guangxi, P. R. China
E–mail address: duanfj@guet.edu.cn