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ON THE STRUCTURE OF THE SET OF Z-TRANSFORMATIONS ON PROPER CONES

Jun Fan*, J. Tao† and G. Ravindran

Abstract: In this paper, we study the structure of the set $\mathbf{Z}(K)$ of all $\mathbf{Z}$-transformations on a proper cone in a finite dimensional real Hilbert space and describe some properties of individual elements of $\mathbf{Z}(K)$.

Key words: *proper cone,* **Z** *and strong* **Z** *transformations, Lyapunov-like and Stein-like transformations, Euclidean Jordan algebras*

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1 Introduction

Let H be a finite dimensional real Hilbert space and $K \subseteq H$ be a proper cone in H. A linear transformation $L : H \to H$ is said to be a $\mathbf{Z}$-transformation on K if

$$x \in K,\ y \in K^*,\ \text{and}\ \langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle \leq 0,$$

where K^* denotes the dual of K. This is a generalization of the concept of a $\mathbf{Z}$-matrix (which is a square matrix with non-positive off-diagonal entries). Schneider and Vidyasagar, in [4], called the negatives of these transformations cross-positive matrices and established many basic properties. Such transformations appear in various areas including dynamical systems, optimization, and game theory. The main objective of this paper is to study the structure of the set $\mathbf{Z}(K)$ of all $\mathbf{Z}$-transformations on K and describe some properties of individual elements of $\mathbf{Z}(K)$.

We recall some known properties of $\mathbf{Z}$-transformations.

Theorem 1.1. *For a linear transformation L, the following are equivalent ([4]):*

(i) *L is a $\mathbf{Z}$-transformation on K.*

(ii) *$e^{-tL}(K) \subseteq K$ for all $t \geq 0$ in R.*

(iii) *The trajectory of the differential equation $\frac{dx}{dt} + L(x) = 0$ which starts in K, stays in K for all $t \geq 0$.*

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Theorem 1.2. *Suppose L is a $\mathbf{Z}$-transformation on K. Then the following are equivalent ([3]):*

(1) *L is positive stable (which means that every eigenvalue of L has positive real part).*

(2) *L is real positive stable (which means that all real eigenvalues are positive).*

(3) *There exists a $d \in int(K)$ such that $L(d) \in int(K)$.*

(4) *For all $t \geq 0$, $L + tI$ is invertible.*

(5) *There exists a $\delta > 0$ such that $||(L + tI)x|| \geq \delta||x||$ for all $x \in H$ and $t \geq 0$ in R.*

(6) *L is invertible and $L^{-1}(K) \subseteq K$.*

(7) *For every $q \in H$, the linear complementrity problem, $LCP(L, K, q)$, has a solution.*

Moreover, when K is self-dual, the above are further equivalent to

(8) *The game-theoretic value of L with respect to an $e \in int(K)$ and the strategy set $\Delta := \{x \in H : \langle x, e\rangle = 1\}$ is positive.*

$\boxed{2}$ Preliminaries

Throughout, we fix a finite dimensional real inner product space H, let $\mathcal{B}(H)$ denote the space of all continuous linear transformations on H under the operator norm. Let K denote a proper cone in H.

Definition 2.1. We say that L

(1) is a $\mathbf{Z}$-transformation (and write $L \in \mathbf{Z}(K)$) if

$$x \in K,\ y \in K^*,\ \text{and}\ \langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle \leq 0.$$

(2) is a Lyapunov-like transformation (and write $L \in \mathbf{LL}(K)$) if

$$x \in K,\ y \in K^*,\ \text{and}\ \langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle = 0.$$

(3) is a Strict $\mathbf{Z}$-transformation (and write $L \in \mathbf{Str}\,\mathbf{Z}(K)$) if

$$0 \neq x \in K,\ 0 \neq y \in K^*,\ \text{and}\ \langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle < 0.$$

(4) $L \in \Pi(K)$ if $L(K) \subseteq K$.

(5) an $\mathbf{M}$-transformation (and write $L \in \mathbf{M}(K)$) if $L = rI - S$, where $r \in R$, $S \in \Pi(K)$, and $r \geq \rho(S)$.

(6) is a Stein-like transformation (and write $L \in \mathbf{SL}(K)$) if $L = I - S$,where $S \in \overline{Aut(K)}$ ($Aut(K)$ is the set of all invertible linear transformations $L : H \to H$ such that $L(K) = K$ and $\overline{Aut(K)}$ is the closure of $Aut(K)$) .

Given elements $u, v \in H$, we define the operator $uv^T \in \mathcal{B}(H)$ by

$$uv^T(x) := \langle v, x \rangle \, u.$$

For illustration purposes, we consider Euclidean Jordan algebras [1]. In that setting, we assume that the algebra (denoted by V) carries the trace inner product and K is the corresponding symmetric cone. If V is a Euclidean Jordan algebra of rank r, for any element $x \in V$, we have the spectral decomposition

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_r e_r,$$

where $\{e_1, e_2, \ldots, e_r\}$ is a Jordan frame and $x_1, x_2, \ldots, x_r$ are the eigenvalues of x. When $x \in K$, these eigenvalues are nonnegative. Note that $trace(x) = x_1 + x_2 + \cdots + x_r$ and $||x||^2 = x_1^2 + x_2^2 + \cdots + x_r^2$. Thus,

$$||x|| \leq trace(x) \leq ||x||\sqrt{k} \quad \forall\, x \in K,$$

where k is the number of nonzero eigenvalues of x.

$\boxed{3}$ Numerical Description of Elements of Z(K)

Our objective in this section is to describe the **Z**-property by means of numerical quantity. Let K be a proper cone in H. Define

$$\Omega = \{(x, y) : ||x|| = ||y|| = 1,\, 0 \neq x \in K,\, 0 \neq y \in K^*, \text{ and } \langle x, y \rangle = 0\}. \tag{3.1}$$

We observe that Ω is compact in $H \times H$. For any $L \in \mathcal{B}(H)$, we define

$$\alpha(L) := \min\{\langle L(x), y \rangle \ : (x, y) \in \Omega\}$$

and

$$\gamma(L) := \max\{\langle L(x), y \rangle \ : (x, y) \in \Omega\}.$$

When the context is clear, we simply write α and γ in place of $\alpha(L)$ and $\gamma(L)$. Here are some elementarity properties:

- $\alpha(L + \epsilon I) = \alpha(L)$ and $\gamma(L + \epsilon I) = \gamma(L)$ for any $\epsilon \in R$.
- $-||L|| \leq \alpha(L) \leq \gamma(L) \leq ||L||$.
- $\gamma(-L) = -\alpha(L)$.
- On $\mathcal{B}(H)$, $\alpha(L)$ and $\gamma(L)$ are Lipschitz continuous with constant one.

The last statement follows from the inequality $\gamma(L_1) \leq \gamma(L_2) + ||L_1 - L_2||$.

Theorem 3.1. *Suppose K is a proper cone in H and $L \in \mathcal{B}(H)$. Then*

(i) $L \in \mathbf{Z}(K)$ *if and only if* $\gamma(L) \leq 0$.

(ii) $L \in Str\mathbf{Z}(K)$ *if and only if* $\gamma(L) < 0$. *Hence,* $int(\mathbf{Z}(\mathbf{K})) = \mathbf{Str}\,\mathbf{Z}(\mathbf{K})$.

(iii) $L \in bdy(\mathbf{Z}(K))$ *if and only if* $\gamma(L) = 0$.

(iv) $L \in \mathbf{LL}(K)$ *if and only if* $\alpha(L) = \gamma(L) = 0$.

(v) $L \in bdy(\mathbf{Z}(K))\backslash\mathbf{LL}(K)$ *if and only if* $\alpha(L) < 0 = \gamma(L)$.

Proof. Item (i) and the first part of (ii) follow directly from the definitions. Now, $Str\,\mathbf{Z}(K) = \{L : \gamma(L) < 0\}$. By continuity of γ, we see that $Str\,\mathbf{Z}(K)$ is an open set and hence must be contained in $int(\mathbf{Z}(K))$. To prove the reverse inclusioin, let $L \in int(\mathbf{Z}(K))$ and suppose, if possible, $\gamma(L) = 0$. Then there exists $(u, v) \in \Omega$ such that $\langle L(u), v\rangle = \gamma(L) = 0$. It follows that for small positive ε, $T := L + \varepsilon\, vu^T \in \mathbf{Z}(K)$. Hence, $0 \geq \langle Tu, v\rangle = \langle Lu, v\rangle + \varepsilon\langle(vu^T)(u), v\rangle = \varepsilon > 0$. This contradiction shows that $\gamma(L) < 0$ whenever $L \in int(\mathbf{Z}(K))$. We thus have $int(\mathbf{Z}(K)) = Str\,\mathbf{Z}(K)$.
As $\mathbf{Z}(K)$ is the disjoint union of $int(\mathbf{Z}(K))$ and $bdy(\mathbf{Z}(K)$, (iii) follows from (i) and (ii). Item (iv) follows from the definition of $\mathbf{LL}(K)$. Finally (v) follows from (iii) and (iv). $\square$

In what follows, we provide some examples.

Example 3.2. For $u, v \in H$ and $\rho \in R$, let $L = \rho I - uv^T$.

(a) If $u \in K$ and $v \in K^*$, then $L \in \mathbf{Z}(K)$ with $\gamma(L) = -\min\{\langle v, x\rangle\,\langle u, x\rangle : (x, y) \in \Omega\}$.

(b) If $u \in int(K)$ and $v \in int(K^*)$, then $L \in Str\,\mathbf{Z}(K)$.

Example 3.3. Let $u \in int(K)$ and $v \in int(K^*)$. Then $L = (I + uv^T)^{-1} \in Str\mathbf{Z}(K)$ with

$$\gamma(L) = \frac{-min_{(x,y)\in\Omega}\langle v, x\rangle\langle u, y\rangle}{1 + \langle v, u\rangle}.$$

Proof. Let $(x, y) \in \Omega$ and $z := L(x)$. Then

$$(I + uv^T)z = x \Rightarrow z + \langle v, z\rangle u = x \Rightarrow \langle v, z\rangle + \langle v, z\rangle\langle v, u\rangle = \langle v, x\rangle \Rightarrow \langle v, z\rangle = \frac{\langle v, x\rangle}{1 + \langle v, u\rangle}.$$

Thus, $z = x - \frac{\langle v,x\rangle}{1+\langle v,u\rangle}u$. Hence, $\langle L(x), y\rangle = \langle z, y\rangle = -\frac{\langle v,x\rangle}{1+\langle v,u\rangle}\langle u, y\rangle < 0$. This yileds the stated expression for $\gamma(L)$. $\square$

Example 3.4. Suppose V is a Euclidean Jordan aglebra of rank r and K is the corresponding symmetric cone. Let $L \in \mathcal{B}(V)$ and

$$\theta := \max\{\langle L(e_i), e_j\rangle\ : i \neq j,\ \{e_1, e_2, \ldots, e_r\} \text{ is a Jordan frame}\}.$$

(i) If $L \in \mathbf{Z}(K)$, then $\gamma(L) = \theta \leq 0$.

(ii) If $L \notin \mathbf{Z}(K)$, then $\theta > 0$,

$$\begin{array}{ll} \theta \leq \gamma(L) \leq \frac{r}{2}\theta & \text{when } r \text{ is even}, \\ \theta \leq \gamma(L) \leq \frac{\sqrt{r^2-1}}{2}\theta & \text{when } r \text{ is odd, and} \\ \gamma(L) = \theta & \text{when } r = 2. \end{array}$$

Proof. (i) It is clear that $\gamma(L) \geq \theta$. Now assume that $L \in \mathbf{Z}(K)$ so that $\theta \leq 0$. Let $(x, y) \in \Omega$. We may write the spectral decompositions of x and y with respect to the same Jordan frame, and further write $x = \sum_1^k x_i e_i$ and $y = \sum_{k+1}^r y_i e_i$. Then using the inequalities $||x|| \leq trace(x) \leq ||x||\sqrt{k}$, where k is the number of nonzero x_is, and $\theta \leq 0$, we have

$$\langle L(x), y\rangle = \sum_{i,j} \langle L(e_i), e_j\rangle x_i y_j \leq \theta(\sum_1^k x_i)(\sum_{k+1}^r y_j) \leq \theta\, trace(x)\, trace(y) \leq \theta ||x||\, ||y|| = \theta.$$

Hence $\gamma(L) \leq \theta$. As $\gamma(L) \geq \theta$, we have $\gamma(L) = \theta$.
(ii) Suppose that $L \notin \mathbf{Z}(K)$. Then $\theta > 0$ and (as before)

$$\langle L(x), y\rangle \leq \theta(\sum_1^k x_i)(\sum_{k+1}^r y_j) \leq \theta\, trace(x)\, trace(y) \leq \theta ||x||\sqrt{k}\, ||y||\sqrt{r-k} = \theta\sqrt{k(r-k)}.$$

Note that this last expression is θ when $r = 2$ and $k = 1$. For general r, it is easy to verify that max $\sqrt{k(r-k)}$ is $\frac{r}{2}$ when r is even and $\frac{\sqrt{r^2-1}}{2}$ when r is odd. □

A special case: Let $V = R^n$ and $K = R^n_+$. If $A = [a_{ij}]$ is a **Z**-matrix, then

$$\gamma(A) = \max_{i \neq j} a_{ij}.$$

This is because, in R^n, there is only one Jordan frame, namely, the set of all standard co-ordinate vectors.

We now consider a special Euclidean Jordan algebra $V = \mathcal{S}^n$ with $K = \mathcal{S}^n_+$.

Example 3.5. For any $A \in R^{n \times n}$, consider $L = \rho I - M_A : \mathcal{S}^n \to \mathcal{S}^n$, where

$$M_A(X) := AXA^T.$$

(a) When $n \geq 3$, $\gamma(L) = 0$.

(b) When $n = 2$, $\gamma(L) = max\{-b^2, -c^2\}$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Proof. (a) Since $\gamma(L) = \gamma(-M_A) = -\alpha(M_A)$, it is enough to show that $\alpha(M_A) = 0$. As $M_A(X) \in \mathcal{S}^n_+$ when $X \in \mathcal{S}^n_+$, we see that $\langle M_A(X), Y\rangle \geq 0$ for all $X, Y \in \mathcal{S}^n_+$. Thus, $\alpha(M_A) \geq 0$. If $A = 0$, then $\alpha(M_A) = 0$. So assume that $A \neq 0$ and let $n \geq 3$. Take any vector $v \in R^n$ with $||v|| = 1$ and $Av \neq 0$. As the span of v and Av has dimension at most two and $n \geq 3$, there is a vector u in R^n of norm one perpendicular to this span. Thus,

$$u, v \in R^n,\ ||u|| = 1 = ||v||,\ \langle u, v\rangle = 0, \text{ and } u^T A v = 0.$$

Now, let $X = vv^T$ and $Y = uu^T$. It is easily seen that $(X, Y) \in \Omega$. Then

$$0 \leq \alpha(M_A) \leq \langle M_A(vv^T), uu^T\rangle = (u^T A v)^2 = 0.$$

Hence, $\alpha(M_A) = 0$.
(b) When $n = 2$ and $(X, Y) \in \Omega$, we see that X and Y are rank one matrices. Thus, $X = vv^T$, $Y = uu^T$, $||u|| = ||v|| = 1$, and $\langle u, v\rangle = 0$. Then

$$\alpha(M_A) = \min_{(X,Y) \in \Omega} \langle AXA^T, Y\rangle = \min_{\Delta} \langle Avv^T A^T, uu^T\rangle = \min_{\Delta} (U^T A v)^2,$$

where $\Delta = \{(u,v) \in R^2 \times R^2 : ||u|| = 1 = ||v||, \langle u, v\rangle = 0\}$. An easy computation shows that the last expression is $\min\{-b^2, -c^2\}$. Hence, $\gamma(L) = \max\{-b^2, -c^2\}$. $\square$

In the above example, we see that $\rho I - M_A$ can never be a strict **Z**-transformation for $n \geq 3$. However, for $n = 2$, there is a 2×2 matrix A for which $\rho I - M_A$ is a strict **Z**-transformation. In fact, Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Consider $0 \neq X$, $Y \in S^2_+$, $||X|| = ||Y|| = 1$, and $\langle X, Y\rangle = 0$. Then X and Y are rank one matrices. Let $X = vv^T$, $Y = uu^T$, $||u|| = ||v|| = 1$, and $\langle u, v\rangle = 0$. We have $\langle L(X), Y\rangle = -\langle AXA^T, Y\rangle = -(u^TAv)^2 = -||u||^2 = -1 < 0$, as $Av = u$. Thus $\rho I - M_A$ is a strict **Z**-transformation.

4 Strict Z-Transformations

Theorem 4.1. *A linear transformation $L : H \to H$ is a strict **Z**-transformation if and only if there exist $\rho > 0$ and a linear transformation S such that $L = \rho I - S$, with*

$$S(K \setminus \{0\}) \subseteq int(K).$$

Proof. The "If" part follows easily from the observation that $S(x) \in int(K), y \in K^* \Rightarrow \langle S(x), y\rangle > 0$. To see the "Only if" part, assume the contrary. Then for every natural number n, we have $(nI - L)(K \setminus \{0\}) \not\subseteq int(K)$. Thus we have a sequence x_n in K with $||x_n|| = 1$ and $(nI - L)x_n \notin int(K)$ for all n. By the well known Separation Theorem, there exists $0 \neq y_n \in H$ such that

$$\langle (nI - L)x_n, y_n\rangle \leq 0 \leq \langle z, y_n\rangle$$

for all $z \in int(K)$. From the second inequality, we have $y_n \in K^*$ for all n. Without loss of generality, we may assume that $||y_n|| = 1$ and let $x_n \to \overline{x}$ and $y_n \to \overline{x}$. From the first inequality, we have $\langle (I - \frac{1}{n}L)x_n, y_n\rangle \leq 0$. Thus, we have $\langle \overline{x}, \overline{y}\rangle \leq 0$. Since $\overline{x} \in K$ and $\overline{y} \in K^*$, $\langle \overline{x}, \overline{y}\rangle \geq 0$. Hence, $\langle \overline{x}, \overline{y}\rangle = 0$. This implies that $\langle L(\overline{x}), \overline{y}\rangle < 0$ and $\langle L(x_n), y_n\rangle < 0$ for large n. Then $\langle (nI - L)x_n, y_n\rangle > 0$, which is a contradiction. $\square$

5 Some Complementarity Results

Given a proper cone K, a linear transformation L on H and $q \in H$, the *linear complementarity problem*, LCP(L, K, q) is to find $x \in V$ such that

$$x \in K \;\; y := L(x) + q \in K^*, \text{ and } \langle x, y\rangle = 0.$$

We say that L is a **Q**-transformation on K if for every $q \in H$, LCP(L, K, q) has a solution. Now we recall the following theorem from [3] .

Theorem 5.1. *The following are equivalent for a **Z**-transformation (on K):*

(1) *L is positive stable (that is, real part of any eigenvalue of L is positive).*

(2) *L is invertible and $L^{-1}(K) \subseteq K$ (equivalently, $L^{-1}(int(K)) \subseteq int(K)$).*

(3) *There exists a $d \in int(K)$ such that $L(d) \in int(K)$.*

(4) *L is a **Q**-transformation.*

In the next result, we improve Item (2) for a strict **Z**-transformation.

Theorem 5.2. *Suppose that L is a strict* **Z***-transformation. Then L has the Q-property if and only if $L^{-1}(K \setminus \{0\}) \subseteq int(K)$.*

Proof. "Only if" part: Let L be a **Q**-transformation. Then, by the above theorem, L is invertible and $L^{-1}(K) \subseteq K$. Suppose there exists $0 \neq x \in K$ such that $L^{-1}(x) \notin int(K)$. Then $L^{-1}(x) \in bdy(K)$. Thus, by the Separation Theorem, there exists $0 \neq d \in H$ such that

$$\langle L^{-1}(x), d\rangle \leq 0 \leq \langle u, d\rangle \quad \forall\, u \in K.$$

The second inequality implies that $d \in K^*$. Thus, $\langle L^{-1}(x), d\rangle \geq 0$. In view of the first inequality, we have $\langle L^{-1}(x), d\rangle = 0$. Since L is a strict **Z**-transformation, $\langle x, d\rangle = \langle L(L^{-1}(x)), d\rangle < 0$. However, $\langle x, d\rangle \geq 0$, as $x \in K$ and $d \in K^*$. This is a contradiction.

"If" part: First, we claim that L is invertible. Suppose not. Then for some $x \in K^\circ$, $L^{-1}(x) = u + Ker(L)$, where $L(u) = x$. This implies that $u + Ker(L) \subseteq K^\circ \subseteq K$. Then $\forall z \in Ker(L)$ and $\forall n \in N$, $\pm nz \in K$. This implies that $\pm z \in K$. Since K is a proper cone, we have $z = 0$. Thus, L is invertible and $L^{-1}(K) \subseteq K$. By Theorem 6, [3], L has the Q-property. ☐

Theorem 5.3. *Let $u, v \in K$ and $L = \rho I - uv^T$, where $\rho \in R$. If L has the Q-property, then L has the P-property.*

Proof. With $S = uv^T$, it is obvious that $S(K) \subseteq K$ and hence L is a **Z**-transformation. Since L has the Q-property, by Theorem 6, [3], there exists $d > 0$ such that $L(d) > 0$. This implies $\rho d - (uv^T)(d) \Rightarrow \rho > 0$. Now suppose there exists a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ such that $x = \sum x_i e_i$, $y = L(x) = \sum y_i e_i$ and $x_i y_i \leq 0$ for all i. We consider the following cases:
Case 1: $\langle x, v\rangle = 0$. Then $y = \rho x - (uv^T)(x) = \rho x - \langle x, v\rangle u = \rho x$. This implies that $x = 0$.
Case 2: Without loss of generality, $\langle x, v\rangle > 0$. Then $u = \frac{1}{\langle x,v\rangle}(-y + \rho x)$. Let $u = \sum u_i e_i$. Thus, $y_i = \rho x_i - \langle x, v\rangle u_i$, for all i. Hence, $x_i y_i \leq 0 \Rightarrow \rho x_i^2 - \langle x, v\rangle u_i x_i \leq 0$ for all i. If $u_i = 0$, then $x_i = 0$. If $u_i > 0$, then $x_i \geq 0$. Therefore, $x \geq 0$. If $x_i = 0$, then $y_i \leq 0$. If $x_i > 0$, then $y_i \leq 0$. Thus, $y \leq 0$. Since $y = L(x)$, we have $x = L^{-1}(y) \leq 0$. As $x \geq 0$, we must $x = 0$. Thus, L has the P-property. ☐

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Jun Fan
Department of Applied Mathematics
Beijing Jiaotong University, Beijing 100044, P. R. China
E–mail address: `fanjunmath@hotmail.com`

J. Tao
Department of Mathematics and Statistics
Loyola University Maryland, Baltimore, Maryland 21210, U.S.A.
E–mail address: `jtao@loyola.edu`

G. Ravindran
Indian Statistical Institute, Chennai, India
E–mail address: `gravi@hotmail.com`