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A NEW PARAMETRIC KERNEL FUNCTION WITH A TRIGONOMETRIC BARRIER TERM FOR $P_*(\kappa)$-LINEAR COMPLEMENTARITY PROBLEMS*

L. Li, J.Y. Tao, M. El Ghami, X.Z. Cai and G.Q. Wang†

Abstract: In this paper, we consider a new parametric kernel function with a trigonometric barrier term. The properties of the proposed parametric kernel function and the corresponding barrier functions are investigated. Furthermore, we present a class of large- and small-update interior-point methods for $P_*(\kappa)$-linear complementarity problems based on the new parametric kernel function. By utilizing the feature of the parametric kernel function, we obtain the iteration bounds for large- and small-update methods, namely, $O((1+2\kappa)n^{\frac{2}{3}}\log\frac{n}{\varepsilon})$ and $O((1+2\kappa)\sqrt{n}\log\frac{n}{\varepsilon})$, respectively.

Key words: *interior-point methods, $P_*(\kappa)$-linear complementarity problems, Kernel function, large- and small-update methods, polynomial complexity*

Mathematics Subject Classification: *90C33, 90C51*

1 Introduction

Given a matrix $M \in \mathbf{R}^{n\times n}$ and a vector $q \in \mathbf{R}^n$, the standard linear complementarity problem (LCP) is to find a pair of vectors $(x,s) \in \mathbf{R}^{2n}$ such that

$$s = Mx + q, \quad xs = 0, \quad (x,s) \geq 0,$$

where xs denotes the coordinatewise product of the vectors x and s.

For years, LCPs have been one of the most active research areas in mathematical programming due to its many applications, algorithms, and theoretical existence results. In fact, the Karush-Kuhn-Tucker (KKT) optimality conditions for linear optimization (LO) and convex quadratic optimization (CQO) can be formulated as LCPs. For a comprehensive treatment of LCPs theory and applications, we refer the reader to the monograph of Kojima et al. [16].

The class of P_*-matrices, introduced by Kojima et al. [16], includes many types of matrices in practical applications. Let κ be a nonnegative number. A matrix M is called a $P_*(\kappa)$-matrix if

$$(1+4\kappa)\sum_{i\in I_+(x)} x_i(Mx)_i + \sum_{i\in I_-(x)} x_i(Mx)_i \geq 0, \quad \forall x \in \mathbf{R}^n,$$

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†*Corresponding author

where $I_+(x) = \{i : x_i(Mx)_i > 0\}$ and $I_-(x) = \{i : x_i(Mx)_i < 0\}$ are two index sets. It is equivalent to the following inequality

$$x^T Mx \geq -4\kappa \sum_{i \in I_+(x)} x_i(Mx)_i, \quad \forall x \in \mathbf{R}^n.$$

The union of all the $P_*(\kappa)$-matrices is defined by

$$P_* = \bigcup_{\kappa \geq 0} P_*(\kappa),$$

i.e., M is a P_*-matrix if $M \in P_*(\kappa)$ for some $\kappa \geq 0$.

In this paper, we consider LCPs with M being a $P_*(\kappa)$-matrix, namely $P_*(\kappa)$-LCPs, which contains monotone LCPs (i.e., $P_*(0)$-LCPs) as a special case when $\kappa = 0$. There are many approaches for $P_*(\kappa)$-LCPs. Among them, interior-point methods (IPMs) gain much more attention. Due to the fact that LCPs is closely related to LO, several IPMs designed for LO have been extended to $P_*(\kappa)$-LCPs. Kojima et al. [16] first proved the existence of the central path for $P_*(\kappa)$-LCPs and generalized the primal-dual IPMs for LO to $P_*(\kappa)$-LCPs. Consequently, several efficient IPMs have been proposed for $P_*(\kappa)$-LCPs [1] [22] [26] [30] [32] and nonlinear P_* complementarity problems [38]. In the meantime, many researchers considered some other effective methods for symmetric cone complementarity problems [13] [14], which includes second-order cone complementarity problems [6] [7] and semidefinite complementarity problems [12] as special cases.

Peng et al. [23] presented primal-dual IPMs for LO, second-order cone optimization (SOCO), semidefinite optimization (SDO), and also extended to $P_*(\kappa)$-LCPs based on the self-regular proximities. Later on, Bai et al. [3], Cho and Kim [8], Wang et al. [27], Peyghami and Amini [24], and Lee et al. [17] analyzed IPMs for $P_*(\kappa)$-LCPs based on some special eligible kernel functions, which are not necessarily self-regular. Lesaja and Roos [18] and El Ghami and Steihaug [10] provided a unified approach and comprehensive treatment of kernel-based IPMs for $P_*(\kappa)$-LCPs. All these kernel-based IPMs for $P_*(\kappa)$-LCPs depend explicitly on the handicap κ of the problem. For some other related kernel-function based IPMs we refer to the references [19] [25] [34] [37].

Recently, El Ghami et al. [9] considered a trigonometric kernel function for primal-dual IPMs for LO. They established the worst case iteration bounds for large- and small-update methods, namely, $O(n^{\frac{3}{4}} \log \frac{n}{\varepsilon})$ and $O(\sqrt{n} \log \frac{n}{\varepsilon})$, respectively. Subsequently, Peyghamia et al. [25] considered a new kernel function with a trigonometric barrier term. Based on this kernel function, they proved that large-update method for LO has the worst case iteration bound, namely, $O(n^{\frac{2}{3}} \log \frac{n}{\varepsilon})$, which improves the obtained iteration bound for large-update methods based on the trigonometric kernel function proposed in [9]. Some well known trigonometric kernel function and the corresponding iteration bounds for large-update methods are collected in Table 1. In most cases, the complexity results for small-update IPMs are essentially the same small-update methods based on the classic logarithmic barrier function, which is $O(\sqrt{n} \log \frac{n}{\varepsilon})$.

Motivated by their work, the aim of this paper is to propose a class of primal-dual IPMs based on the following new kind of parametric kernel function with a trigonometric barrier term, i.e.,

$$\psi(t) = \frac{t^2 - 1}{2} - \log t + \lambda \tan^2(h(t)), \quad t > 0, \tag{1.1}$$

where

$$h(t) = \frac{\pi u(1-t)}{t + 2u}, \quad 0 < u \leq \frac{1}{3}, \tag{1.2}$$

i	The kernel functions $\psi_i(t)$	Large-update methods	Ref.
1	$\frac{t^2-1}{2}+\frac{6}{\pi}\tan\left(\frac{\pi(1-t)}{2+4t}\right)$	$O\left(n^{\frac{3}{4}}\log\frac{n}{\varepsilon}\right)$	[9]
2	$\frac{t^2-1}{2}+\frac{4}{\pi}\cot\left(\frac{\pi t}{1+t}\right)$	$O\left(n^{\frac{3}{4}}\log\frac{n}{\varepsilon}\right)$	[15]
3	$\frac{t^2-1}{2}-\log t+\frac{1}{8}\tan^2\left(\frac{\pi(1-t)}{2+4t}\right)$	$O\left(n^{\frac{2}{3}}\log\frac{n}{\varepsilon}\right)$	[25]
4	$\frac{(t-1)^2}{2}+\frac{(t-1)^2}{2t}+\frac{1}{8}\left(\tan^2\left(\frac{\pi(1-t)}{2+4t}\right)\right)$	$O\left(n^{\frac{2}{3}}\log\frac{n}{\varepsilon}\right)$	[19]
5	$\frac{t^2-1}{2}-\log t+\lambda\tan^2\left(\frac{\pi(1-t)}{2+3t}\right), 0<\lambda\leq\frac{8}{25\pi}$	$O\left(n^{\frac{3}{4}}\log\frac{n}{\varepsilon}\right)$	[5]
6	$\frac{t^2-1}{2}-\frac{4}{\pi p}\left(\tan^p\left(\frac{\pi}{2+2t}\right)-1\right), p\geq 2$	$O\left(pn^{\frac{p+2}{2(p+1)}}\log\frac{n}{\varepsilon}\right)$	[4]

Table 1. Iteration bounds for the kernel functions with trigonometric barrier terms

and $0<\lambda\leq\lambda(u)$, here $\lambda(u)$ is a function of u given by

$$\lambda(u)=\frac{(1+2u)^3}{4\pi u(10\pi^2u^2-3\pi u+3)}. \tag{1.3}$$

It should be noted that if $u=0$, then $\psi(t)=\frac{t^2-1}{2}-\log t$, this is the kernel function of the classic barrier function. Particularly, if one takes $u=\frac{1}{4}$, then

$$\lambda\left(\frac{1}{4}\right)=\frac{27}{\pi(5\pi^2-6\pi+24)}\approx 0.1577.$$

This means that our parametric kernel function includes the kernel function considered in [25] as a special case, where $\lambda=\frac{1}{8}$. The properties of the proposed kernel functions and the corresponding barrier functions are investigated. By utilizing the feature of the parametric kernel function, we derive the iteration bound for large-update methods, namely, $O((1+2\kappa)n^{\frac{2}{3}}\log\frac{n}{\varepsilon})$, which improves the classical iteration complexity with a factor $n^{\frac{1}{3}}$, and for small-update methods, we obtain the iteration bound, namely $O((1+2\kappa)\sqrt{n}\log\frac{n}{\varepsilon})$, which matches the currently best known iteration bound for small-update methods.

The remainder of this paper is organized as follows: In Section 2, we introduce the new parametric kernel function with a trigonometric barrier term and develop some useful properties of the new kernel function, as well as the corresponding barrier function. In Section 3, we present the framework of kernel-based IPMs for $P_*(\kappa)$-LCPs. The analysis and the complexity of the algorithms for large- and small-update methods are presented in Section 4. Finally, the paper will end with some concluding remarks follow in Section 5.

Some notations used throughout the paper are as follows. $\mathbf{R}^n$, $\mathbf{R}^n_+$ and $\mathbf{R}^n_{++}$ denote the set of vectors with n components, the set of nonnegative vectors and the set of positive vectors, respectively. $\|x\|$ denotes the 2-norm of the vector x. e denotes the identity vector, that is, a vector whose entries take value 1. For any $x\in\mathbf{R}^n$, x_{min} and x_{max} denote the smallest and the largest value of the components of x, respectively. Finally, if $g(x)\geq 0$ is a real valued function of a real nonnegative variable, the notation $g(x) = O(x)$ means that $g(x) \leq \bar{c}x$ for some positive constant $\bar{c}$ and $g(x) = \Theta(x)$ that $c_1 x \leq g(x) \leq c_2 x$ for two positive constants c_1 and c_2.

$\boxed{2}$ The new parametric kernel function

In this section, we consider the new parametric kernel function $\psi(t)$ given by (1.1). Some useful properties of the new kernel function and the corresponding barrier function are provided.

The first three derivatives of $\psi(t)$ are listed as follows:

$$\psi'(t) = t - \frac{1}{t} + 2\lambda h'(t)\tan(h(t))\sec^2(h(t)), \tag{2.1}$$

$$\psi''(t) = 1 + \frac{1}{t^2} + 2\lambda \sec^2(h(t))\varphi_1(t), \tag{2.2}$$

$$\psi'''(t) = -\frac{2}{t^3} + 2\lambda \sec^2(h(t))\varphi_2(t). \tag{2.3}$$

where

$$\varphi_1(t) = h''(t)\tan(h(t)) + h'(t)^2(3\tan^2(h(t)) + 1),$$
$$\varphi_2(t) = 3h'(t)h''(t)(3\tan^2(h(t)) + 1) + 4h'(t)^3\tan(h(t))(3\tan^2(h(t)) + 2) + h'''(t)\tan(h(t)),$$

and

$$h'(t) = -\frac{\pi u(1+2u)}{(t+2u)^2}, \quad h''(t) = \frac{2\pi u(1+2u)}{(t+2u)^3}, \quad h'''(t) = -\frac{6\pi u(1+2u)}{(t+2u)^4}.$$

It is obvious that

$$\psi(1) = \psi'(1) = 0. \tag{2.4}$$

Furthermore, we have

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta)\,d\zeta\,d\xi. \tag{2.5}$$

Some technical lemmas related to the proposed parametric kernel function $\psi(t)$ given by (1.1) are presented.

Lemma 2.1. *One has*

$$g(u) := \tan\left(\frac{\pi(1-2u)}{4}\right) - \frac{2}{3\pi(1+2u)} > 0, \quad 0 < u \leq \frac{1}{3}.$$

Proof. It follows from $0 \leq x < \frac{\pi}{2}$ that $\cos(x) = \sin(\frac{\pi}{2} - x) \leq \frac{\pi}{2} - x$. We have

$$\begin{aligned}
g'(u) &= -\frac{\pi}{2}\sec^2\left(\frac{\pi(1-2u)}{4}\right) + \frac{4}{3\pi(1+2u)^2} \\
&= \sec^2\left(\frac{\pi(1-2u)}{4}\right)\left(-\frac{\pi}{2} + \frac{4}{3\pi(1+2u)^2}\cos^2\left(\frac{\pi(1-2u)}{4}\right)\right) \\
&\leq \sec^2\left(\frac{\pi(1-2u)}{4}\right)\left(-\frac{\pi}{2} + \frac{4}{3\pi(1+2u)^2}\frac{\pi^2(1+2u)^2}{16}\right) \\
&= -\frac{5\pi}{12}\sec^2\left(\frac{\pi(1-2u)}{4}\right) \\
&< 0.
\end{aligned}$$

Thus $g(u)$ is decreasing in $\left(0,\frac{1}{3}\right]$, and from $g(\frac{1}{3})=\tan(\frac{\pi}{12})-\frac{2}{5\pi}\approx 0.1406>0$, this implies that $g(u)>0$ for $0<u\le\frac{1}{3}$. Hence, the proof of the lemma is finished. $\square$

Lemma 2.2. *Let $h(t)$ be given by (1.2). Then*

$$f(t,u):=\tan(h(t))-\frac{4u}{3\pi(1+2u)t}>0,\qquad 0<t\le 2u\ ,\ 0<u\le\frac{1}{3}.$$

Proof. For $0<t\le 1$, one has $0\le h(t)<\frac{\pi}{2}$, therefore $\cos(h(t))\le\frac{\pi}{2}-h(t)$. Differentiating the function $f(t,u)$ with respect to t, we have

$$\begin{aligned}\frac{\partial f(t,u)}{\partial t}&=\frac{1}{\cos^2(h(t))}h'(t)+\frac{4u}{3\pi(1+2u)t^2}\\&=\frac{1}{3\pi t^2\cos^2(h(t))}\left(3\pi t^2h'(t)+\frac{4u}{1+2u}\cos^2h(t)\right)\\&\le\frac{1}{3\pi t^2\cos^2(h(t))}\left(3\pi t^2h'(t)+\frac{4u}{1+2u}\left(\frac{\pi}{2}-h(t)\right)^2\right)\\&=\frac{1}{3\pi t^2\cos^2(h(t))}\left(-3\pi t^2\frac{\pi u(1+2u)}{(t+2u)^2}+\frac{4u}{1+2u}\frac{\pi^2(1+2u)^2t^2}{4(t+2u)^2}\right)\\&=-\frac{2\pi u(1+2u)}{3(t+2u)^2\cos^2(h(t))}\\&<0.\end{aligned}$$

This implies that $f(t,u)$ with respect to t is decreasing in $(0,2u]$. From Lemma 2.1, we have

$$f(2u,u)=\tan\left(\frac{(1-2u)\pi}{4}\right)-\frac{2}{3\pi(1+2u)}>0,\quad 0<u\le\frac{1}{3}.$$

This means that $f(t,u)>0$. Hence, the proof of the lemma is finished. $\square$

Lemma 2.3. *Let c be a constant, and*

$$w(t;u,\lambda)=L_n(u,\lambda)t^n+L_{n-1}(u,\lambda)t^{n-1}+\cdots+L_1(u,\lambda)t+L_0(u,\lambda),$$

where the parameters u and λ are in $\mathbf{R}$, and $L_i(u,\lambda)$ are the functions of u and λ for $i=0,1,\ldots,n$. If $L_n(u,\lambda)>0$, $w(c;u,\lambda)>0$ and $\frac{\partial^i w(t;u,\lambda)}{\partial t^i}|_{t=c}>0$ for $i=1,\ldots,n-1$, then we have $w(t;u,\lambda)>0$ for all $t>c$.

Proof. It is obvious that $\frac{\partial^n w(t;u,\lambda)}{\partial t^n}|_{t=c}=n!L_n(u,\lambda)>0$, for all $t\in\mathbf{R}$. This implies that $\frac{\partial^{n-1}w(t;u,\lambda)}{\partial t^{n-1}}$ is monotone increasing. Since $\frac{\partial^{n-1}w(t;u,\lambda)}{\partial t^{n-1}}|_{t=c}>0$, we have $w^{(n-1)}(t;u,\lambda)>0$ for all $t>c$. And so on, we can conclude that $w(t;u,\lambda)$ for all $t>c$. This finishes the proof of the lemma. $\square$

Lemma 2.4. *Let $t>0$. Then*

$$\psi''(t)>1; \tag{2.6}$$
$$t\psi''(t)+\psi'(t)>0; \tag{2.7}$$
$$t\psi''(t)-\psi'(t)>0; \tag{2.8}$$
$$\psi'''(t)<0. \tag{2.9}$$

Proof. See Appendix A.

The barrier function $\Psi(v) : \mathbf{R}^n_{++} \to \mathbf{R}_+$ based on the parametric kernel function given by (1.1) is defined by

$$\Psi(v) := \sum_{i=1}^{n} \psi(v_i). \tag{2.10}$$

Furthermore, we define the norm-based proximity measure $\delta(v) : \mathbf{R}^n_{++} \to \mathbf{R}_+$ as follows:

$$\delta(v) := \frac{1}{2} \parallel \bigtriangledown\Psi(v) \parallel . \tag{2.11}$$

Due to the properties of the parametric kernel function $\psi(t)$, we can conclude that $\Psi(v)$ is a strictly convex function and attains minimal value at $v = e$ and $\Psi(e) = 0$, i.e.,

$$\nabla\Psi(v) = 0 \Leftrightarrow \Psi(v) = 0 \Leftrightarrow v = e. \tag{2.12}$$

The property described below is exponential convexity, which has been proven to be very useful in the analysis of primal-dual IPMs based on the kernel functions [2] [23].

Lemma 2.5. *Let $t_1, t_2 > 0$. Then*

$$\psi(\sqrt{t_1 t_2}) \leq \frac{1}{2}(\psi(t_1) + \psi(t_2)).$$

Proof. The result of the lemma follows immediately from Lemma 1 in [23], which states that the above inequality holds if and only if $t\psi''(t) + \psi'(t) > 0$ for all $t > 0$. Hence, from (2.7) in Lemma 2.4, the proof of the lemma is finished. □

From (i) of Lemma 2.4 (i.e., $\psi''(t) > 1$), we say that $\psi(t)$ is strongly convex. The following lemma provides an important consequence of this property. These results can be directly obtained from the corresponding results in the LO case [2].

Lemma 2.6. *Let $t > 0$. Then*

$$\frac{1}{2}(t-1)^2 \leq \psi(t) \leq \frac{1}{2}\psi'(t)^2.$$

As the consequences of Lemma 2.6, we have the following two important corollaries.

Corollary 2.7. *Let $\Psi(v) \geq 1$. Then*

$$\delta(v) \geq \frac{1}{2}\sqrt{\Psi(v)}.$$

Corollary 2.8. *Let $\Psi(v) \geq 1$. Then*

$$\|v\| \leq \sqrt{n} + 2\delta(v).$$

Lemma 2.9. *Let $\beta \geq 1$. Then*

$$\psi(\beta t) \leq \psi(t) + \frac{1}{2}(\beta^2 - 1)t^2.$$

Proof. See Appendix B.

Theorem 2.10. *Let $0 < \theta < 1$ and $v_+ = \frac{v}{\sqrt{1-\theta}}$. Then*

$$\Psi(v_+) \le \Psi(v) + \frac{\theta}{2(1-\theta)} \left(2\Psi(v) + 2\sqrt{2n\Psi(v)} + n\right).$$

Proof. Let $\beta = \frac{1}{\sqrt{1-\theta}}$. We have, by Lemma 2.9,

$$\Psi(\beta v) \le \Psi(v) + \frac{1}{2}\sum_{i=1}^{n}(\beta^2 - 1)v_i^2 = \Psi(v) + \frac{\theta\|v\|^2}{2(1-\theta)}.$$

It follows from Corollary 2.8 that

$$\Psi(v_+) \le \Psi(v) + \frac{\theta}{2(1-\theta)} \left(2\Psi(v) + 2\sqrt{2n\Psi(v)} + n\right).$$

The proof of the theorem is finished. □

3 The kernel-based IPMs for $P_*(\kappa)$-LCPs

In this section, we briefly recall the outline of primal-dual IPMs for $P_*(\kappa)$-LCPs, which includes the central path, the new search directions and the generic primal-dual IPMs for $P_*(\kappa)$-LCPs.

3.1 The central path for $P_*(\kappa)$-LCPs

Throughout the paper, we assume that $P_*(\kappa)$-LCPs satisfy the interior-point condition (IPC), i.e., there exists a pair $(x^0, s^0) > 0$ such that $s^0 = Mx^0 + q$, which implies the existence of a solution for $P_*(\kappa)$-LCPs. In fact, the IPC can be assumed without loss of generality. For this and some other properties mentioned below, we refer to Kojima et al. [16].

Finding an approximate solution of $P_*(\kappa)$-LCPs is equivalent to solving the following system

$$\begin{pmatrix} -Mx + s \\ \\ xs \end{pmatrix} = \begin{pmatrix} q \\ \\ 0 \end{pmatrix}, \quad x, s \ge 0. \tag{3.1}$$

The standard approach is to replace the second equation in (3.1), the so-called complementarity condition for $P_*(\kappa)$-LCPs, by the parameterized equation $xs = \mu e$, with $\mu > 0$. This leads to the following system

$$\begin{pmatrix} -Mx + s \\ \\ xs \end{pmatrix} = \begin{pmatrix} q \\ \\ \mu e \end{pmatrix}, \quad x, s \ge 0. \tag{3.2}$$

From Lemma 4.3 in [16], the parameterized system (3.2) has a unique solution for each $\mu > 0$ due to the fact that M is a $P_*(\kappa)$-matrix and the IPC holds. We denote this solution as

$(x(\mu), s(\mu))$ and call it the μ-center of $P_*(\kappa)$-LCPs. The set of μ-centers (with μ running through all positive real numbers) gives a homotopy path, which is called the central path of $P_*(\kappa)$-LCPs. If $\mu \to 0$, then the limit of the central path exists and since the limit points satisfy the complementarity condition, i.e., $xs = 0$, the limit yields a solution for $P_*(\kappa)$-LCPs (cf. Theorem 4.4 in [16]).

3.2 The new search directions for $P_*(\kappa)$-LCPs

IPMs follow the central path approximately and find an approximate solution of $P_*(\kappa)$-LCPs as μ goes to zero. A natural way to define a search direction is to follow Newton's approach and linearize the second equation in (3.2). This yields to the following system

$$\begin{pmatrix} -M\Delta x + \Delta s \\ \\ s\Delta x + x\Delta s \end{pmatrix} = \begin{pmatrix} 0 \\ \\ \mu e - xs \end{pmatrix}. \tag{3.3}$$

It follows from Lemma 4.1 in [16] that the modified Newton-system (3.3) has a unique solution.

Let

$$v := \sqrt{\frac{xs}{\mu}}, \tag{3.4}$$

and

$$d_x := \frac{v\Delta x}{x}, \quad d_s := \frac{v\Delta s}{s}. \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\begin{pmatrix} -\overline{M}d_x + d_s \\ \\ d_x + d_s \end{pmatrix} = \begin{pmatrix} 0 \\ \\ v^{-1} - v \end{pmatrix}, \tag{3.6}$$

where $\overline{M} := DMD$ with $D := X^{\frac{1}{2}}S^{-\frac{1}{2}}, X := \mathrm{diag}\,(x)$ and $S := \mathrm{diag}\,(s)$. It is obvious that the right-hand side $v^{-1} - v$ in the second equation of the system (3.6) equals minus the derivative of the classic barrier function $\Psi_c(v)$, i.e.,

$$\Psi_c(v) := \sum_{i=1}^{n} \psi_c(v_i), \quad v \in \mathbf{R}^n_{++}, \tag{3.7}$$

where

$$\psi_c(t) := \frac{t^2 - 1}{2} - \log t$$

is the kernel function of the classic barrier function $\Psi_c(v)$. Thus, the system (3.6) can be rewritten as the following system.

$$\begin{pmatrix} -\overline{M}d_x + d_s \\ \\ d_x + d_s \end{pmatrix} = \begin{pmatrix} 0 \\ \\ -\nabla\Psi_c(v) \end{pmatrix}. \tag{3.8}$$

By replacing the right-hand side of the second equation in (3.8) by $-\nabla\Psi(v)$, we have

$$\begin{pmatrix} -\overline{M}d_x + d_s \\ \\ d_x + d_s \end{pmatrix} = \begin{pmatrix} 0 \\ \\ -\nabla\Psi(v) \end{pmatrix}. \tag{3.9}$$

This system also has a unique search direction. Furthermore, we can conclude that Δx and Δs both vanish if and only if $v = e$, i.e., if and only if $x = x(\mu), s = s(\mu)$. Otherwise, we will use $(\Delta x, \Delta s)$ as the new search direction. Then, we have

$$x_+ := x + \alpha\triangle x, \quad \text{and } s_+ := s + \alpha\triangle s, \tag{3.10}$$

where α is the default step size defined by some line search rules. Furthermore, we can easily verify that

$$d_x = d_s = 0 \Leftrightarrow \nabla\Psi(v) = 0 \Leftrightarrow \delta(v) = 0 \Leftrightarrow \Psi(v) = 0 \Leftrightarrow v = e. \tag{3.11}$$

This implies that the value of $\Psi(v)$ can be considered as a measure for the distance between the given pair (x, s) and the corresponding μ-center $(x(\mu), s(\mu))$.

3.3 The generic IPMs for $P_*(\kappa)$-LCPs

Now we can outline the generic IPMs that uses the barrier function defined by (2.10). Suppose that $(x(\mu), s(\mu))$ is known for some positive μ and is in the τ-neighborhood of the corresponding μ-center, i.e., $\Psi(v) \leq \tau$. For example, due to the above assumption we may assume this for $\mu = 1$, with $x(1) = s(1) = e$. Then, we decrease μ to $\mu := (1-\theta)\mu$ with $\theta \in (0, 1)$, which changes the value of v according to (3.4) and defines a new μ-center $(x(\mu), s(\mu))$. This may cause the increase of the value of the barrier function above the threshold value of τ, i.e., $\Psi(v) > \tau$. Now we start the inner iteration by solving the scaled Newton system (3.9) and through (3.5) to get the new search direction $(\Delta x, \Delta s)$. The new iterate (x_+, s_+) is calculated by (3.10). If necessary, we repeat the procedure until we find iterates that are in the neighborhood of $(x(\mu), s(\mu))$. During the inner iteration the value of μ is kept constant. At this point we start a new outer iteration by reducing the value of μ again. Then we apply Newton's method targeting at the new μ-centers, and so on. This process is repeated until μ is small enough, say until $n\mu < \varepsilon$, at this stage we have found an ε-solution of $P_*(\kappa)$-LCPs. The generic form of this algorithm is shown in Fig. 1.

4 The analysis and the complexity of the algorithms

In this section, we first choose a default step size. Then, we derive an upper bound for the decrease of the barrier function during an inner iteration. Finally, the iteration bounds for large- and small-update methods are established.

4.1 The default step size

In each inner iteration, we first compute the search direction (d_x, d_s) from the system (3.9). Then through (3.5), we obtain the search direction $(\Delta x, \Delta s)$. After a step with size α the new iteration is given by (3.10). Note that during an inner iteration the parameter μ is fixed. Hence, after the step the new v-vector is

$$v_+ = \sqrt{\frac{x_+ s_+}{\mu}}.$$

Generic IPMs for $P_*(\kappa)$-LCPs

Input:
A threshold parameter $\tau > 0$;
an accuracy parameter $\varepsilon > 0$;
a fixed barrier update parameter θ, $0 < \theta < 1$;
begin
$x := e$; $s := e$; $\mu := 1$;
while $n\mu \geq \varepsilon$ **do**
begin
$\mu := (1-\theta)\mu$;
while $\Psi(v) > \tau$ **do**
begin
calculate the search direction $(\Delta x, \Delta s)$;
determine the default step size α;
update $(x, s) := (x, s) + \alpha(\Delta x, \Delta s)$.
end
end
end

Figure 1: Algorithm

Since

$$x_+ = x\left(e + \alpha\frac{\Delta x}{x}\right) = \frac{x}{v}\left(v + \alpha d_x\right), \quad s_+ = s\left(e + \alpha\frac{\Delta s}{s}\right) = \frac{s}{v}\left(v + \alpha d_s\right),$$

we have, by $xs = \mu v^2$,

$$v_+ = \sqrt{(v + \alpha d_x)(v + \alpha d_s)}.$$

In what follows, we consider the decrease in Ψ as a function of α and define

$$f(\alpha) := \Psi(v_+) - \Psi(v). \tag{4.1}$$

However, working with $f(\alpha)$ may not be easy because in general $f(\alpha)$ is not convex. Thus, we are searching for the convex function $f_1(\alpha)$ that is an upper bound of $f(\alpha)$ and whose derivatives are easier to calculate than those of $f(\alpha)$.

We have, by Lemma 2.5,

$$\Psi(v_+) = \Psi(\sqrt{(v + \alpha d_x)(v + \alpha d_s)}) \leq \frac{1}{2}(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)).$$

Let

$$f_1(\alpha) := \frac{1}{2}(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)) - \Psi(v). \tag{4.2}$$

Then $f(0) = f_1(0) = 0$ and $f(\alpha) \leq f_1(\alpha)$, which means that $f_1(\alpha)$ is an upper bound of $f(\alpha)$. Furthermore, we have

$$f_1'(\alpha) = \frac{1}{2}\sum_{i=1}^{n}\left(\psi'(v_i + \alpha(d_x)_i)(d_x)_i + \psi'(v_i + \alpha(d_s)_i)(d_s)_i\right), \tag{4.3}$$

and

$$f_1''(\alpha) = \frac{1}{2}\sum_{i=1}^{n}(\psi''(v_i + \alpha(d_x)_i)(d_x)_i^2 + \psi''(v_i + \alpha(d_s)_i)(d_s)_i^2). \tag{4.4}$$

It follows from the second expression of the system (3.8) that

$$f_1'(0) = \frac{1}{2}\nabla\Psi(v)^T(d_x + d_s) = -\frac{1}{2}\nabla\Psi(v)^T\nabla\Psi(v) = -2\delta(v)^2. \tag{4.5}$$

Below we use the shorthand notation: $\delta := \delta(v)$. The following lemma provides an upper bound of $f_1''(\alpha)$, which can be found in Lemma 3.3 in [27].

Lemma 4.1. *One has*

$$f_1''(\alpha) \le 2(1+2\kappa)\delta^2\psi''(v_{min} - 2\alpha\sqrt{1+2\kappa}\delta).$$

Following the strategy considered in [27], we briefly recall how to choose the default step size. Suppose that the step size α satisfies

$$-\psi'(v_{min} - 2\alpha\sqrt{1+2\kappa}\delta) + \psi'(v_{min}) \le \frac{2\delta}{\sqrt{1+2\kappa}}. \tag{4.6}$$

Then $f_1(\alpha) \le 0$. The largest possible value of the step size of α satisfying (4.6) is given by

$$\bar{\alpha} := \frac{1}{2\sqrt{1+2\kappa}\delta}\left(\rho(\delta) - \rho\left(\left(1 + \frac{1}{\sqrt{1+2\kappa}}\right)\delta\right)\right). \tag{4.7}$$

where $\rho(s) : [0,\infty) \to (0,1]$ is the inverse function of $-\frac{1}{2}\psi'(t)$ for $t \in (0,1]$. Furthermore, we can conclude that

$$\bar{\alpha} \ge \frac{1}{(1+2\kappa)\psi''\left(\rho\left(\left(1+\frac{1}{\sqrt{1+2\kappa}}\right)\delta\right)\right)}. \tag{4.8}$$

According to the properties of the kernel function $\psi(t)$ given by (1.1), we have the following result.

Lemma 4.2. *One has*

$$\bar{\alpha} \ge \frac{1}{(1+2\kappa)C(\lambda,u)\delta^{\frac{4}{3}}},$$

where $C(\lambda,u)$ is given below in (5.6)

Proof. See Appendix C.

In the sequel we use

$$\tilde{\alpha} := \frac{1}{(1+2\kappa)C(\lambda,u)\delta^{\frac{4}{3}}}, \tag{4.9}$$

as the default step size.

$\boxed{4.2}$ The decrease of the value of $\Psi(v)$ during an inner iteration

In what follows, we will show that the barrier function $\Psi(v)$ in each inner iteration with the default step size $\tilde{\alpha}$, as defined by (4.9), is decreasing. For this, we need the following technical result.

Lemma 4.3 (Lemma 12 in [23]). *Let $h(t)$ be a twice differentiable convex function with $h(0) = 0$, $h'(0) < 0$ and let $h(t)$ attain its (global) minimum at $t^* > 0$. If $h''(t)$ is increasing for $t \in [0, t^*]$, then*

$$h(t) \leq \frac{th'(0)}{2}, \quad 0 \leq t \leq t^*.$$

Lemma 4.4. *Let the step size α be such that $\alpha \leq \tilde{\alpha}$. Then*

$$f(\alpha) \leq -\alpha\delta^2.$$

Proof. Since $f_1(\alpha)$ is a twice differentiable convex function with $f_1(0) = 0$, and $f_1'(0) = -2\delta^2 < 0$, we have, by Lemma 4.3,

$$f(\alpha) \leq f_1(\alpha) \leq -\alpha\delta^2.$$

This finishes the proof of the lemma. $\square$

The following theorem shows that the default step size (4.9) yields the sufficient decrease of the barrier function value during each inner iteration.

Theorem 4.5. *Let $\tilde{\alpha}$ be the default step size given by (4.9). Then*

$$f(\tilde{\alpha}) \leq -\frac{1}{\sqrt[3]{2}(1+2\kappa)C(\lambda,u)}\Psi(v)^{\frac{1}{3}}.$$

Proof. From Lemma 4.4, (4.9) and Corollary 2.7, we have

$$f(\tilde{\alpha}) \leq -\tilde{\alpha}\delta^2 \leq -\frac{1}{(1+2\kappa)C(\lambda,u)}\delta^{\frac{2}{3}} \leq -\frac{1}{\sqrt[3]{2}(1+2\kappa)C(\lambda,u)}\Psi(v)^{\frac{1}{3}}.$$

This finishes the proof of the theorem. $\square$

$\boxed{4.3}$ The iteration bounds for large- and small-update methods

From Theorem 2.10, after decreasing the parameter μ to $(1-\theta)\mu$ with $0 < \theta < 1$, we have

$$\Psi(v_+) \leq \Psi(v) + \frac{\theta}{2(1-\theta)}\left(2\Psi(v) + 2\sqrt{2n\Psi(v)} + n\right). \tag{4.10}$$

At the start of an outer iteration and just before updating of the parameter μ, we have $\Psi(v) \leq \tau$. Due to (4.10), the value of $\Psi(v)$ exceeds from the threshold τ after updating of μ. Therefore, we need to count how many inner iterations are required to return to the situation where $\Psi(v) \leq \tau$. We denote the value of $\Psi(v)$ after the μ-update as Ψ_0, the subsequent values in the same outer iteration are denoted as Ψ_k, $k = 1, 2, \cdots, K$, where K denotes the total number of inner iterations in the outer iteration. Hence, we have

$$\Psi_0 \leq \tau + \frac{\theta}{2(1-\theta)}\left(2\tau + 2\sqrt{2n\tau} + n\right). \tag{4.11}$$

According to decrease of $f(\tilde{\alpha})$ in Lemma 4.5, we have

$$\Psi_{k+1} \leq \Psi_k - \beta(\Psi_k)^{1-\gamma}, \quad k = 0, 1, \ldots, K-1, \tag{4.12}$$

where $\beta = \frac{1}{\sqrt[3]{2}(1+2\kappa)C(\lambda,u)}$, and $\gamma = \frac{2}{3}$.

Lemma 4.6 (Lemma 14 in [23]). *Suppose $t_0, t_1, \ldots, t_K$ be a sequence of positive numbers such that*

$$t_{k+1} \leq t_k - \beta t_k^{1-\gamma}, \quad k = 0, 1, \ldots, K-1,$$

where $\beta > 0$ and $0 < \gamma \leq 1$. Then $K \leq \lceil \frac{t_0^\gamma}{\beta\gamma} \rceil$.

The following lemma provides an estimate for the number of inner iterations between two successive barrier parameter updates, in terms of Ψ_0 and $C(\lambda, u)$.

Lemma 4.7. *One has*

$$K \leq \frac{3\sqrt[3]{2}(1+2\kappa)C(\lambda,u)}{2}(\Psi_0)^{\frac{2}{3}}.$$

Proof. Using (4.12), and also applying Lemma 4.6, the result of the lemma follows. This finishes the proof of the lemma. □

It is well known that the number of outer iterations is bounded above by $\frac{1}{\theta}\log\frac{n}{\varepsilon}$ (cf. [33] II.17, page 116). Then, we get an upper bound for the total number of iterations, namely,

$$\frac{3\sqrt[3]{2}(1+2\kappa)C(\lambda,u)}{2\theta}\left(\tau + \frac{\theta}{2(1-\theta)}\left(2\tau + 2\sqrt{2n\tau} + n\right)\right)^{\frac{2}{3}} \log\frac{n}{\varepsilon}. \tag{4.13}$$

Remark 4.1. The parameters λ and u will not improve the order of the theoretical complexity of the algorithm due to the properties of $C(\lambda, u)$, but it will affect the practical performance of the algorithm. See also the discussions in [29]. In theory, the bigger values of u and the smaller values of λ give the better complexity bound.

The following theorem provides the best iteration bound for large-update methods based on the parametric kernel function $\psi(t)$ is given by (1.1).

Theorem 4.8. *For large-update methods, one takes for θ a constant (independent on n), namely $\theta = \Theta(1)$, and $\tau = O(n)$. The iteration bound then becomes*

$$O\left((1+2\kappa)n^{\frac{2}{3}}\log\frac{n}{\varepsilon}\right),$$

which improves the classical iteration bound with a factor $n^{\frac{1}{3}}$.

Similar to the analysis in [2], the iteration bound for small-update methods is straight and we leave it for the interested readers.

Theorem 4.9. *For small-update methods, one takes for $\theta = \Theta(\frac{1}{\sqrt{n}})$ and $\tau = O(1)$. The iteration bound then becomes*

$$O\left((1+2\kappa)\sqrt{n}\log\frac{n}{\varepsilon}\right),$$

which matches the currently best known iteration bound for small-update methods.

5 Conclusions and remarks

In this paper, we presented a new parametric kernel function with a trigonometric barrier term for the development of large- and small-update IPMs for $P_*(\kappa)$-LCPs. By utilizing the feature of the parametric kernel function, we derived the iteration bounds for large-update methods, $O((1+2\kappa)n^{\frac{2}{3}}\log\frac{n}{\varepsilon})$ and small-update methods, $O((1+2\kappa)\sqrt{n}\log\frac{n}{\varepsilon})$, respectively.

Some interesting topics for further research remain. Firstly, the generalization of symmetric optimization [31] [35] [36], symmetric cone complementarity problems [13] [14] and the Cartesian $P_*(\kappa)$-LCPs over symmetric cones [20] [21] [28] deserve to be investigated. Secondly, numerical results may help us to compare the behavior of the algorithms proposed in this paper with the existing methods.

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Appendix A. Proof of Lemma 2.4

Proof. Firstly, we consider two cases to prove (2.6).

Case 2.6.1: Let $0 < t < 1$. In this case, $0 < h(t) < \frac{\pi}{2}$, $h''(t) > 0$. It follows from (2.2) that $\psi''(t) > 1$, for $\lambda > 0$ and $t \in (0, 1)$.

Case 2.6.2: Let $t \geq 1$. Define

$$\xi(t) := \frac{1}{t^2} + 2\lambda \sec^2(h(t))\varphi_1(t).$$

We need to prove that when $0 < u \leq \frac{1}{3}$ and $0 < \lambda \leq \lambda(u)$, $\xi(t) > 0$ holds. Since $h(t) \in (-\pi u\,, 0]$ for all $t \geq 1$, we have

$$\begin{aligned}\xi(t) &= 2\lambda\sec^2(h(t))\left(\frac{1}{2\lambda t^2\sec^2(h(t))} + h''(t)\tan(h(t)) + h'(t)^2(3\tan^2(h(t))+1)\right)\\ &\geq 2\lambda\sec^2(h(t))\left(\frac{1}{2\lambda t^2\sec^2(\pi u)} - \frac{2\pi(u+2u^2)\tan(\pi u)}{(t+2u)^3} + \frac{\pi^2(u+2u^2)^2}{(t+2u)^4}\right)\\ &= \frac{\sec^2(h(t))}{\sec^2(\pi u)t^2(t+2u)^4}\,\omega_1(t;u,\lambda),\end{aligned}$$

where

$$\begin{aligned}&\omega_1(t;u,\lambda)\\ &:= (t+2u)^4 - \lambda\pi(u+2u^2)\sec^2(\pi u)\left(4\tan(\pi u)t^2(t+2u) - 2\pi u(1+2u)t^2\right).\end{aligned}$$

By solving $\omega_1(1;u,\lambda) > 0$ and $\frac{\partial^i \omega_1(t;u,\lambda)}{\partial t^i}|_{t=1} > 0$ for $i = 1, 2, 3$, we can get the upper bounds of λ, denoted by $\lambda_{1i}(u)$ for $i = 0, 1, 2, 3$, respectively. Here

$$\lambda_{10}(u) := \frac{(1+2u)^2}{2\pi u\sec^2(\pi u)(2\tan(\pi u) - \pi u)}, \tag{5.1}$$

$$\lambda_{11}(u) := \frac{(1+2u)^2}{\pi u\sec^2(\pi u)((3+4u)\tan(\pi u) - \pi(u+2u^2))}, \tag{5.2}$$

$$\lambda_{12}(u) := \frac{3(1+2u)}{\pi u\sec^2(\pi u)((6+4u)\tan(\pi u) - \pi(u+2u^2))}, \tag{5.3}$$

$$\lambda_{13}(u) := \frac{1}{\pi u\sec^2(\pi u)\tan(\pi u)}. \tag{5.4}$$

From Lemma 2.3, we can infer that when $\lambda < \min\{\lambda_{11}(u), \lambda_{12}(u)\}$, $\xi(t) > 0$ holds.

Due to the fact that $\tan(x) > x$ for all $x > 0$, one can easily verify that $\lambda_{1i}(u) > 0$, for $i = 0, 1, 2, 3$ and $u > 0$. To prove (2.6), we need to prove that $\lambda(u) < \min\{\lambda_{11}(u), \lambda_{12}(u)\}$ for $0 < u \leq \frac{1}{3}$. We first prove that $\lambda(u) < \lambda_{11}(u)$ holds. One has

$$\lambda(u) - \lambda_{11}(u) = \frac{(1+2u)^2}{\pi u\sec^2(\pi u)\left((3+4u)\tan(\pi u) - \pi(u+2u^2)\right)(40\pi^2u^2 - 12\pi u + 12)}\,f_1(u),$$

where

$$f_1(u) := 40\pi^2u^2 - 12\pi u + 12 - (1+2u)\sec^2(\pi u)\left((3+4u)\tan(\pi u) - \pi(u+2u^2)\right).$$

We can easily verify that the minimum value of $f_1(u)$ in $(0, \frac{1}{3}]$ is

$$f_1\left(\frac{1}{3}\right) = \frac{40\pi^2}{9} - \frac{8\pi}{27} - \frac{260\sqrt{3}}{9} + 12 \approx 4.8970 > 0,$$

which implies that $\lambda(u) < \lambda_{11}(u)$ holds for all $0 < u \leq \frac{1}{3}$. Similarly, we can prove that $\lambda(u) < \lambda_{12}(u)$ holds for all $0 < u \leq \frac{1}{3}$. This implies that (2.6) holds in this case. The two cases together prove (2.6).

Then we consider three cases to prove (2.7). One has

$$t\psi''(t) + \psi'(t) = 2t + 2\lambda\sec^2(\pi u)\left((th''(t) + h'(t))\tan(h(t)) + th'(t)^2(3\tan^2(h(t))+1)\right).$$

Case 2.7.1: Let $t \geq 1$. Since $\lambda \geq 0$, we have $\psi'(t) > 0$ and $\psi''(t) > 0$, therefore $t\psi''(t) + \psi'(t) > 0$ holds for this case.

Case 2.7.2: Let $2u < t < 1$. Then $th''(t) + h'(t) = \frac{\pi(u+2u^2)(t^2-4u^2)}{(t+2u)^4} > 0$, therefore $t\psi''(t) + \psi'(t) > 0$ also holds for this case.

Case 2.7.3: Let $0 < t \leq 2u$. We have, by Lemma 2.2,

$$\begin{aligned}
&(th''(t) + h'(t))\tan(h(t)) + th'(t)^2(3\tan^2(h(t)) + 1)) \\
=&\frac{\pi(u+2u^2)(t^2-4u^2)}{(t+2u)^4}\tan(h(t)) + 3t\tan^2(h(t))\frac{(u+2u^2)^2(t^2-4u^2)}{(t+2u)^4}th'(t)^2 \\
\geq&\frac{\pi(u+2u^2)(t^2-4u^2)}{(t+2u)^4}\tan(h(t)) + 3t\tan(h(t))\frac{4u}{3(1+2u)\pi t}\frac{(u+2u^2)^2(t^2-4u^2)}{(t+2u)^4} + th'(t)^2 \\
=&\frac{\pi(u+2u^2)t^2\tan(h(t))}{(t+2u)^4} + th'(t)^2 \\
>&0.
\end{aligned}$$

This implies that $t\psi''(t) + \psi'(t) > 0$, for all $0 < t \leq 2u$.

From three cases above we conclude that when $0 < u \leq \frac{1}{3}$ and $0 < \lambda \leq \lambda(u)$, (2.7) holds.

Next, we discuss two cases to prove that (2.8) holds.

Case 2.8.1: Let $0 < t < 1$. We have $\psi'(t) < 0$ and $\psi''(t) > 0$, therefore $t\psi''(t) - \psi'(t) > 0$ holds for this case.

Case 2.8.2: Let $t \geq 1$. Due to the fact that $th''(t) - h'(t) > 0$ and $\tan(h(t)) \in (-\tan(\pi u), 0]$, we have

$$\begin{aligned}
&t\psi''(t) - \psi'(t) \\
=&\frac{2}{t} + 2\lambda\sec^2(h(t))\left((th''(t) - h'(t))\tan(h(t)) + th'(t)^2(3\tan^2(h(t)) + 1)\right) \\
\geq&\frac{2}{t} + 2\lambda\sec^2(h(t))\left(-t\tan(\pi u)h''(t) + \tan(\pi u)h'(t) + th'(t)^2\right) \\
=&2\lambda\sec^2(h(t))\left(\frac{1}{\lambda t\sec^2(h(t))} - \frac{2\pi(u+2u^2)t\tan(\pi u)}{(t+2u)^3} - \frac{\pi(u+2u^2)\tan(\pi u)}{(t+2u)^2} + \frac{\pi^2(u+2u^2)^2t)}{(t+2u)^4}\right) \\
\geq&2\lambda\sec^2(h(t))\left(\frac{1}{\lambda t\sec^2(\pi u)} - \frac{2\pi(u+2u^2)t\tan(\pi u)}{(t+2u)^3} - \frac{\pi(u+2u^2)\tan(\pi u)}{(t+2u)^2} + \frac{\pi^2(u+2u^2)^2t)}{(t+2u)^4}\right) \\
=&\frac{2\sec^2(h(t))}{\sec^2(\pi u)t(t+2u)^4}\,\omega_2(t\,;\,u,\lambda),
\end{aligned}$$

where

$$\begin{aligned}
&\omega_2(t\,;\,u,\lambda) := \\
&(t+2u)^4 - \lambda\pi(u+2u^2)\sec^2(\pi u)\left(2\tan(\pi u)t^2(t+2u) + \tan(\pi u)t(t+2u)^2 - \pi(u+2u^2)t^2\right).
\end{aligned}$$

By solving $\omega_2(1;u,\lambda) > 0$ and $\frac{\partial^i\omega_2(t;u,s)}{\partial t^i}|_{t=1} > 0$ for $i = 1,2,3$, we can get the upper bounds of λ, denoted by $\lambda_{2i}(u)$ for $i = 0,1,2,3$, respectively. From Lemma 2.3, we can infer that when $\lambda < \min\{\lambda_{20}(u), \lambda_{21}(u)\}$, $\xi(t) > 0$ holds, where

$$\begin{aligned}
\lambda_{20}(u) :=& \frac{(1+2u)^2}{\pi u\sec^2(\pi u)\left((3+2u)\tan(\pi u) - \pi u\right)}, \\
\lambda_{21}(u) :=& \frac{4(1+2u)^2}{\pi u\sec^2(\pi u)\left((9+16u+4u^2)\tan(\pi u) - 2\pi(u+2u^2)\right)}.
\end{aligned}$$

Also we can find that $\lambda_{20}(u) > 0$ and $\lambda_{21}(u) > 0$ for $u > 0$. Now we prove that for all $0 < u \leq \frac{1}{3}$, $\lambda(u) < \lambda_{20}(u)$ holds. One has

$$\lambda_{20}(u) - \lambda(u) = \frac{(1+2u)^2}{\pi u \sec^2(\pi u)\left((3+2u)\tan(\pi u) - \pi u\right)(40\pi^2 u^2 - 12\pi u + 12)} f_2(u),$$

where

$$f_2(u) := 40\pi^2 u^2 - 12\pi u + 12 - (1+2u)\sec^2(\pi u)\left((3+2u)\tan(\pi u) - \pi u\right).$$

We can easily verify that the minimum value of $f_2(u)$ in $(0, \frac{1}{3}]$ is

$$f_2\left(\frac{1}{3}\right) = \frac{4}{9}(10\pi^2 - 4\pi - 55\sqrt{3}) + 12 \approx 7.9408 > 0,$$

which implies that $\lambda(u) < \lambda_{20}(u)$ holds for all $0 < u \leq \frac{1}{3}$. Similarly, we can prove that $\lambda(u) < \lambda_{21}(u)$ holds for all $0 < u \leq \frac{1}{3}$. This implies that (2.8) holds in this case. The two cases together prove (2.8).

Next, we consider three cases to prove that (2.9) holds. One has

$$\psi'''(t) = -\frac{2}{t^3} + 2\lambda \sec^2(h(t))\varphi_2(t) = -2\sec^2(h(t))\zeta(t), \tag{5.5}$$

where

$$\zeta(t) := \frac{1}{t^3 \sec^2(h(t))} - \lambda\varphi_2(t).$$

Case 2.9.1: Let $0 < t \leq 1$. In this situation, we have $\tan(h(t)) > 0$, together with $h'(t) < 0$, $h''(t) > 0$ and $h'''(t) < 0$, one can easily prove that $\zeta(t) > 0$. Therefore (2.9) holds for this case.

We first assume that $u > \frac{1}{4}$ while discussing Case 2.9.2 and Case 2.9.3 (that is $t > 1$).

Case 2.9.2: Let $1 < t \leq \frac{6u}{4u-1}$. Then $-\frac{\pi}{4} \leq h(t) < 0$, $-1 \leq \tan(h(t)) < 0$. We have

$$\begin{aligned}
\zeta(t) &\geq \frac{1}{2t^3} + \lambda(-3h'(t)h''(t) + 20h'(t)^3 + h'''(t)) \\
&= \frac{1}{2t^3} - \lambda\left(\frac{20\pi^3(u+2u^2)^3}{(t+2u)^6} + \frac{6\pi u(1+2u)}{(t+2u)^4} - \frac{6\pi^2 u^2(1+2u)^2}{(t+2u)^5}\right) \\
&= \frac{1}{2t^3(t+2u)^6}\omega_3(t; u, \lambda),
\end{aligned}$$

where

$$\begin{aligned}
&\omega_3(t; u, \lambda) \\
&:= (t+2u)^6 - 4\lambda\pi(u+2u^2)\left(10\pi^2(u+2u^2)^2 t^3 + 3t^3(t+2u)^2 - 3\pi(u+2u^2)t^3(t+2u)\right).
\end{aligned}$$

By solving $\omega_3(1; u, \lambda) > 0$ and $\frac{\partial^i \omega_3(t; u, \lambda)}{\partial t^i}|_{t=1} > 0$ for $i = 1, \ldots, 5$, we can get the upper bounds of λ, denoted by $\lambda_{3i}(u)$ for $i = 0, 1, \ldots, 5$, respectively. From Lemma 2.3, we can infer that when $\lambda < \lambda_{30}(u)$, $\zeta(t) > 0$ holds, where

$$\lambda_{30}(u) := \frac{(1+2u)^3}{40\pi^3 u^3 - 12\pi^2 u^2 + 12\pi u}.$$

It should be pointed out that $\lambda_{30}(u)$ actually is $\lambda(u)$. Therefore (2.9) holds for this case.

Case 2.9.3: Let $t > \frac{6u}{4u-1}$. Then $-\tan(\pi u) < \tan(h(t)) < -1$. We have

$$\begin{aligned}
\zeta(t) \geq& \frac{1}{t^3\sec^2(\pi u)} - \lambda\left(12h'(t)h''(t) - 4\tan(\pi u)(3\tan^2(\pi u)+2)h'(t)^3 - h'''(t)\tan(\pi u)\right)\\
=& \frac{1}{t^3\sec^2(\pi u)} - \lambda\left(\frac{4\pi^3(u+2u^2)^3\tan(\pi u)(3\tan^2(\pi u)+2)}{(t+2u)^6}\right.\\
&\left. + \frac{6\pi u(1+2u)\tan(\pi u)}{(t+2u)^4} - \frac{24\pi^2u^2(1+2u)^2}{(t+2u)^5}\right)\\
=& \frac{1}{t^3\sec^2(\pi u)(t+2u)^6}\omega_4(t;u,\lambda),
\end{aligned}$$

where

$$\begin{aligned}
\omega_4(t;u,\lambda) := (t+2u)^6 - \lambda\pi\sec^2(\pi u)&\left(4\pi^2\tan(\pi u)(3\tan^2(\pi u)+2)(u+2u^2)^3t^3\right.\\
&\left. + 6\tan(\pi u)(u+2u^2)t^3(t+2u)^2 - 24\pi(u+2u^2)^2t^3(t+2u)\right).
\end{aligned}$$

By solving $\omega_4(\frac{6u}{4u-1};u,\lambda) > 0$ and $\frac{\partial^i\omega_4(t;u,\lambda)}{\partial t^i}|_{\frac{6u}{4u-1}} > 0$ for $i = 1,\ldots,5$, we can get the upper bounds of λ, denoted by $\lambda_{4i}(u)$ for $i = 0,1,\ldots,5$, respectively. From Lemma 2.3, we can infer that $\lambda < \lambda_{40}(u)$, $\zeta(t) > 0$ holds, where

$$\begin{aligned}
&\lambda_{40}(u):\\
=& \frac{128(1+2u)^3}{27\pi\sec^2(\pi u)\left(\pi^2(4u-1)^3(3\tan^2(\pi u)+2)\tan(\pi u) + 24(4u-1)\tan(\pi u) - 24\pi(4u-1)^2\right)}.
\end{aligned}$$

It follows from (5.5) that $\psi'''(t) < 0$. Also we can find that $\lambda_{40}(u) > 0$ for all $\frac{1}{4} < u \leq \frac{1}{3}$.

For any $u \in (\frac{1}{4}, \frac{1}{3}]$, we have $\tan(\pi u) \in (1, \sqrt{3}]$. Then

$$\begin{aligned}
\lambda_{40}(u) \geq& \frac{32(1+2u)^3}{27\pi(4u-1)\left(11\sqrt{3}\pi^2(4u-1)^2 + 24\sqrt{3} - 24\pi(4u-1)\right)}\\
\geq& \frac{(1+2u)^3}{\pi(4u-1)\left(11\sqrt{3}\pi^2(4u-1)^2 + 24\sqrt{3} - 24\pi(4u-1)\right)}.
\end{aligned}$$

Let

$$f_3(u) := 4u(10\pi^2u^2 - 3\pi u + 3) - (4u-1)\left(11\sqrt{3}\pi^2(4u-1)^2 + 24\sqrt{3} - 24\pi(4u-1)\right).$$

We can easily verify that the minimum value of $f_3(u)$ in $(\frac{1}{4}, \frac{1}{3}]$ is

$$f_3\left(\frac{1}{3}\right) = \frac{(40-11\sqrt{3})\pi^2}{27} + \frac{4\pi}{3} + 4 - 8\sqrt{3} \approx 1.9895 > 0.$$

This implies that $f_3(u) > 0$ for $u \in (\frac{1}{4}, \frac{1}{3}]$. Hence, we have

$$4\pi u(10\pi^2u^2 - 3\pi u + 3) > \pi(4u-1)\left(11\sqrt{3}\pi^2(4u-1)^2 + 24\sqrt{3} - 24\pi(4u-1)\right).$$

Furthermore, we can conclude that $\lambda_{40}(u) > \lambda(u)$, which implies that (2.9) holds in this case.

Now we come to the situation that $0 < u \le \frac{1}{4}$. For any $t > 1$, we have

$$-\frac{\pi}{4} \le -\pi u < h(t) < 0, \quad -1 < \tan(h(t)) < 0,$$

the proof of (2.9) in this situation is similar to the proof in Case 2.9.2.

From the above discussions, the proof of the lemma is finished. □

Appendix B. Proof of Lemma 2.9

Proof. Let

$$v(t) := -\log(t) + \lambda \tan^2(h(t)), \quad 0 < u \le \frac{1}{3}, \quad 0 < \lambda \le \lambda(u).$$

Then

$$\psi(t) = \frac{t^2-1}{2} + v(t),$$

and

$$\psi(\beta t) - \psi(t) = \frac{1}{2}(\beta^2-1)t^2 + v(\beta t) - v(t).$$

As $\beta \ge 1$, to prove the lemma, it is sufficient to show that the function $v(t)$ is a decreasing function. For this purpose, we have

$$\begin{aligned} v'(t) &= -\frac{1}{t} + 2\lambda h'(t)\tan(h(t))\sec^2(h(t)) \\ &= -\frac{1}{t} - \frac{2\lambda\pi(u+2u^2)}{(t+2u)^2}\tan(h(t))\sec^2(h(t)). \end{aligned}$$

If $0 < t \le 1$, then $\tan(h(t)) \ge 0$, so $v'(t) < 0$.

If $t > 1$, using the fact that $-\tan(\pi u) < \tan(h(t)) < 0$, we have

$$\begin{aligned} v'(t) &\le -\frac{1}{t} + \frac{2\lambda\pi(u+2u^2)}{(t+2u)^2}\tan(\pi u)\sec^2(\pi u) \\ &= -\frac{1}{t(t+2u)^2}\omega_5(t;u,\lambda), \end{aligned}$$

where

$$\omega_5(t;u,\lambda) := (t+2u)^2 - 2\lambda(u+2u^2)\tan(\pi u)\sec^2(\pi u)t.$$

Since $0 < \lambda \le \lambda(u)$, we have

$$\begin{aligned} \omega_5(1;u,\lambda) &\ge (1+2u)^2 - 2\lambda(u)(u+2u^2)\tan(\pi u)\sec^2(\pi u) \\ &= \frac{(1+2u)^2}{2(10\pi^2u^2 - 3\pi u + 3)}f_4(u), \end{aligned}$$

where

$$f_4(u) := 2(10\pi^2 u^2 - 3\pi u + 3) - (1+2u)^2 \tan(\pi u)\sec^2(\pi u).$$

We can easily verify that the minimum value of $f_4(u)$ in $(0, \frac{1}{3}]$ is

$$f_4\left(\frac{1}{3}\right) = \frac{20\pi^2 - 100\sqrt{3}}{9} + 6 - 2\pi \approx 2.4043 > 0,$$

which implies that $\omega_5(1; u, \lambda) > 0$, for all $u \in (0, \frac{1}{3}]$. Similarly, we can deduce that $\frac{\partial \omega_5(t;u,\lambda)}{\partial u}|_{t=1} > 0$ for all $u \in (0, \frac{1}{3}]$. From Lemma 2.3, we can easily verify that $\omega_5(t; u, \lambda) > 0$ for all $0 < u \leq \frac{1}{3}$ and $0 < \lambda \leq \lambda(u)$, which implies that $v'(t) < 0$ for $t > 1$. This finishes the proof of the lemma. □

Appendix C. Proof of Lemma 4.2

Proof. Let $\rho(s)$ be the inverse function of $-\frac{1}{2}\psi'(t)$. Then

$$-t + \frac{1}{t} - 2\lambda h'(t)\tan(h(t))\sec^2(h(t)) = 2s.$$

For all $t \in (0, 1]$, we have

$$\tan(h(t))\sec^2(h(t)) = -\frac{1}{2\lambda h'(t)}\left(2s + t - \frac{1}{t}\right) = \frac{(t+2u)^2}{2\lambda\pi u(1+2u)}\left(2s + t - \frac{1}{t}\right) \leq \frac{(1+2u)s}{\lambda\pi u}.$$

Hence, putting $t = \rho\left(\left(1 + \frac{1}{\sqrt{1+2\kappa}}\right)\delta\right)$, we have $-\psi'(t) = 2\left(1 + \frac{1}{\sqrt{1+2\kappa}}\right)\delta \leq 4\delta$. Then

$$\tan^3(h(t)) \leq \tan(h(t))\sec^2(h(t)) \leq \frac{4(1+2u)\delta}{\lambda\pi u} \quad \Rightarrow \tan(h(t)) \leq \left(\frac{4(1+2u)}{\lambda\pi u}\right)^{\frac{1}{3}} \delta^{\frac{1}{3}}.$$

We have, by Lemma 2.2,

$$1 + \tan(h(t)) > \frac{4u}{3\pi(1+2u)t}, \quad 0 < t \leq 1.$$

This means that

$$\frac{1}{t} < \frac{3\pi(1+2u)}{4u}\left(1 + \tan(h(t))\right), \quad 0 < t \leq 1.$$

Note that $h''(t) = \frac{2\pi u(1+2u)}{(t+2u)^3} < \frac{\pi(1+2u)}{4u^2}$ and $h'(t)^2 = \frac{\pi^2 u^2(1+2u)^2}{(t+2u)^4} < \frac{\pi^2(1+2u)^2}{16u^2}$ for all $0 < t \leq$

1. We have

$$
\begin{aligned}
\bar{\alpha} &\geq \frac{1}{(1+2\kappa)\psi''(t)}\\
&= \frac{1}{(1+2\kappa)} \frac{1}{1+\frac{1}{t^2}+2\lambda\sec^2(h(t))(h''(t)\tan(h(t))+h'(t)^2(3\tan^2(h(t))+1))}\\
&= \frac{1}{(1+2\kappa)} \frac{1}{1+\frac{1}{t^2}+2\lambda h''(t)\tan(h(t))\sec^2(h(t))+2\lambda h'(t)^2\left(3\tan^2(h(t))\sec^2(h(t))+\sec^2(h(t))\right)}\\
&\geq \frac{1}{(1+2\kappa)}\left(1+\frac{9\pi^2(1+2u)^2}{16u^2}\left(1+\left(\frac{4(1+2u)}{\lambda\pi u}\right)^{\frac{1}{3}}\delta^{\frac{1}{3}}\right)^2\right.\\
&\quad\left.+2\lambda h''(t)\frac{4(1+2u)\delta}{\lambda\pi u}+2\lambda h'(t)^2\left(3\left(\frac{4(1+2u)}{\lambda\pi u}\right)^{\frac{4}{3}}\delta^{\frac{4}{3}}+\left(\frac{4(1+2u)}{\lambda\pi u}\right)^{\frac{2}{3}}\delta^{\frac{2}{3}}+1\right)\right)^{-1}\\
&\geq \frac{1}{(1+2\kappa)}\left(1+\frac{9\pi^2(1+2u)^2}{16u^2}\left(1+\left(\frac{4(1+2u)}{\lambda\pi u}\right)^{\frac{1}{3}}\delta^{\frac{1}{3}}\right)^2\right.\\
&\quad\left.+\frac{2(1+2u)^2\delta}{u^3}+\frac{\lambda\pi^2(1+2u)^2}{8u^2}\left(3\left(\frac{4(1+2u)}{\lambda\pi u}\right)^{\frac{4}{3}}\delta^{\frac{4}{3}}+\left(\frac{4(1+2u)}{\lambda\pi u}\right)^{\frac{2}{3}}\delta^{\frac{2}{3}}+1\right)\right)^{-1}
\end{aligned}
$$

Furthermore, we have, by Corollary 2.7 (i.e., $2\delta \geq \sqrt{\Psi(v)} \geq 1$),

$$
\begin{aligned}
\bar{\alpha} \geq \frac{1}{(1+2\kappa)}&\left((2\delta)^{\frac{4}{3}}+\frac{9\pi^2(1+2u)^2}{16u^2}\left((2\delta)^{\frac{2}{3}}+\left(\frac{4(1+2u)}{\lambda\pi u}\right)^{\frac{1}{3}}\delta^{\frac{1}{3}}(2\delta)^{\frac{1}{3}}\right)^2+\frac{2(1+2u)^2\delta}{u^3}(2\delta)^{\frac{1}{3}}\right.\\
&\left.+\frac{\lambda\pi^2(1+2u)^2}{8u^2}\left(3\left(\frac{4(1+2u)}{\lambda\pi u}\right)^{\frac{4}{3}}\delta^{\frac{4}{3}}+\left(\frac{4(1+2u)}{\lambda\pi u}\right)^{\frac{2}{3}}\delta^{\frac{2}{3}}(2\delta)^{\frac{2}{3}}+(2\delta)^{\frac{4}{3}}\right)\right)^{-1}
\end{aligned}
$$

Let

$$
\begin{aligned}
C(\lambda,u) := 2^{\frac{4}{3}}&\left(1+\frac{9\pi^2(1+2u)^2}{16u^2}\left(1+\left(\frac{2(1+2u)}{\lambda\pi u}\right)^{\frac{1}{3}}\right)^2+\frac{(1+2u)^2}{u^3}\right.\\
&\left.+\frac{\lambda\pi^2(1+2u)^2}{8u^2}\left(3\left(\frac{2(1+2u)}{\lambda\pi u}\right)^{\frac{4}{3}}+\left(\frac{2(1+2u)}{\lambda\pi u}\right)^{\frac{2}{3}}+1\right)\right). \qquad (5.6)
\end{aligned}
$$

Then

$$
\bar{\alpha} \geq \frac{1}{(1+2\kappa)C(\lambda,u)\delta^{\frac{4}{3}}}.
$$

This finishes the proof of the lemma. □

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L. Li
College of Fundamental Studies, Shanghai University of
Engineering Science, Shanghai 201620, China
E–mail address: lilu@sues.edu.cn

J.Y. Tao
Department of Mathematics and Statistics
Loyola University Maryland, Baltimore, Maryland 21210, U.S.A.
E–mail address: jtao@loyola.edu

M. El Ghami
Faculty of Education and Arts, Mathematics Section
Nord University-Nesna 8700, Nesna, Norway
E–mail address: mohamed.el-ghami@nord.no

X.Z. Cai
College of Advanced Vocational Technology, Shanghai University of
Engineering Science, Shanghai 201620, China
E–mail address: xzhcai@163.com

G.Q. Wang
College of Fundamental Studies, Shanghai University of
Engineering Science, Shanghai 201620, China
E–mail address: guoq_wang@hotmail.com