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A GRADIENT ALGORITHM FOR FINDING MINIMUM-NORM SOLUTION OF THE SPLIT FEASIBILITY PROBLEM*

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Abstract: This paper discusses the problem of finding the minimum-norm solution of the split feasibility problem (SFP). Using the idea of Tikhonov's regularization, we first introduce a unconstrained optimization problem related to the SFP. Then, we design and analyze a gradient algorithm for finding the minimum-norm solution of the SFP. The global convergence of the algorithm is also established.

Key words: *gradient algorithm, split feasibility problem, minimum-norm solution*

Mathematics Subject Classification: *90C30, 90C90, 65K05*

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1 Introduction

Let C, Q be the nonempty closed convex sets of $\Re^N$ and $\Re^M$, respectively, and A be a matrix of $\Re^{M\times N}$. The split feasibility problem (SFP) is to find x^* with the property:

$$x^* \in C \text{ and } Ax^* \in Q. \tag{1.1}$$

It was first considered by Censor and Elfving [3], for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [4]. It has been founded that the SFP can also be used to model intensity-modulated radiation therapy (IMRT) [5–9]. In recent years, a wide variety of methods have been introduced for solving the split feasibility problem; see, for example, [3–14,19–22] and the references therein. Among these algorithms, a more popular algorithm that solves the SFP is the CQ algorithm proposed by Byren [1,4]. In fact, it is a gradient-projection method (GPM) in convex minimization. However, it remains a challenge how to implement the CQ algorithm in the case where the projections onto sets C,Q fail to have closed-form expressions. A special case of the SFP is the following convex constrained linear inverse problem:

$$\text{find } x \in C \text{ such that } Ax = b, \tag{1.2}$$

where $b \in \Re^M$. A classical method for solving this possibly ill-posed problem is the well-known Tikhonov regulation, which approximates a solution of problem (1.2) through the minimizer of the regularized problem

$$\min_{x\in C} \|Ax - b\|^2 + \alpha\|x\|^2,$$

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where $\alpha > 0$ is known the regularization parameter and, throughout this paper, $\|\cdot\|$ denotes the Euclidean norm.

Recently, Xu transfered the idea of Tikhonov's regulation method to solve the constrained linear inverse problem (1.2) to the case of the SFP (1.1). It easy to find that the SFP (1.1) is equivalent to the following optimization problem

$$\min_{x\in C} \|(I-P_Q)Ax\|.$$

Xu considered the minimization problem to solve the above optimization problem

$$\min_{x\in C} \|(I-P_Q)Ax\|^2 + \alpha\|x\|^2, \tag{1.3}$$

where $\alpha > 0$ is known the regularization parameter. And in [15, 18] he introduced some methods for solving the SFP based on the minimization problem (1.3) in Hilber spaces, which gave the minimum-norm solution of the split feasibility problem. In [2], Ceng proposed some relaxed extragradient methods for finding minimum-norm solution of the split feasibility problem.

In this paper, motivated by Xu's idea to find the minimum-norm solution for the SFP, we consider the following unconstrained optimization problem:

$$\min_{x\in\Re^N} f_\alpha(x) = \frac{1}{2}\|(I-P_C)x\|^2 + \frac{1}{2}\|(I-P_Q)Ax\|^2 + \frac{1}{2}\alpha\|x\|^2, \tag{1.4}$$

where $\alpha > 0$ is known the regularization parameter. Accordingly, we denote

$$f(x) = \frac{1}{2}\|(I-P_C)x\|^2 + \frac{1}{2}\|(I-P_Q)Ax\|^2. \tag{1.5}$$

Under some mild conditions, by solving this unconstrained optimization problem (1.4), we can obtain the minimum-solution of the SFP. In this paper, we design and analyze a gradient algorithm for finding the minimum-norm solution of the SFP through (1.4). The global convergence of the algorithm is also established.

The rest of the paper is organized as follows. In Section 2, we introduce some useful properties which will be used in the next sections. In Section 3, we propose an algorithm to obtain the minimum-norm solution of the SFP. Section 4 gives some conclusions.

2 Preliminaries

In this section, we introduce some useful preliminaries which will be used in the next sections. Throughout the paper, $\langle .,.\rangle$ denotes the inner product; P_Ω denotes the projection from $\Re^N$ onto a nonempty closed convex subset Ω of $\Re^N$, and it is defined as $P_\Omega(x) = \arg\min_{y\in\Omega}\|x-y\|$. Let $T : \Re^N \to \Re^N$ be a nonlinear operator, T is called β-strongly monotone with $\beta > 0$, if

$$\langle x-y, Tx-Ty\rangle \geq \beta\|x-y\|^2,\ x,y\in\Re^N.$$

Lemma 2.1 ([16])**.** *Let Ω be a nonempty closed convex subset of $\Re^N$,then*

(1) $\langle P_\Omega(x)-x, y-P_\Omega(x)\rangle \geq 0, x\in\Re^N, y\in\Omega$;
(2) $\langle P_\Omega(x)-P_\Omega(y), x-y\rangle \geq \|P_\Omega(x)-P_\Omega(y)\|^2, x,y\in\Re^N$;
(3) $\|P_\Omega(x)-P_\Omega(y)\|^2 \leq \|x-y\|^2 - \|P_\Omega(x)-x+y-P_\Omega(y)\|^2, x,y\in\Re^N$.

Remark 2.2. From Lemma 2.1, it is not hard to find that

(1) $\|P_\Omega(x)-P_\Omega(y)\| \leq \|x-y\|, \quad x,y\in\Re^N$;
(2) $\|P_\Omega(x)-x+y-P_\Omega(y)\| \leq \|x-y\|, \quad x,y\in\Re^N$.

Lemma 2.3 ([17]). *A proper convex function $g : \Re^N \to (-\infty, +\infty]$ is strongly convex with the coefficient $\sigma > 0$ if and only if $\tilde{g}(x) = g(x) - \frac{1}{2}\sigma\|x\|^2$ is also a proper convex function.*

Lemma 2.4 ([17]). *Let g be a convex and differentiable function. Then $x^* \in \Re^N$ is a solution of the problem:*

$$\min_{x \in \Re^N} g(x)$$

if and only if $x^* \in \Re^N$ satisfies the following optimality condition:

$$\nabla g(x^*) = 0.$$

Moreover, if g is strictly convex, then the problem $\min_{x \in \Re^N} g(x)$ has a unique solution.

Theorem 2.5. *The function $f(x)$ defined by (1.5) is convex and ∇f is Lipschitz continuous with Lipschitz constant $1 + \|A\|^2$.*

Proof. Since $\nabla f = (I - P_C) + A^T(I - P_Q)A$, we have

$$\begin{aligned}
&\langle \nabla f(x) - \nabla f(y), x - y\rangle \\
&= \langle (I - P_C)x - (I - P_C)y + A^T(I - P_Q)Ax - A^T(I - P_Q)Ay, x - y\rangle \\
&= \|x - y\|^2 - \langle P_C(x) - P_C(y), x - y\rangle + \|Ax - Ay\|^2 - \langle P_Q(Ax) - P_Q(Ay), Ax - Ay\rangle \\
&\geq \|x - y\|^2 - \|P_C(x) - P_C(y)\|\|x - y\| + \|Ax - Ay\|^2 - \|P_Q(Ax) - P_Q(Ay)\|\|Ax - Ay\| \\
&\geq \|x - y\|^2 - \|x - y\|^2 + \|Ax - Ay\|^2 - \|Ax - Ay\|^2 \\
&= 0.
\end{aligned}$$

Combing with the fact that $f(x)$ is continuous and differentiable, we can conclude that it is a convex function.

$$\begin{aligned}
\|\nabla f(x) - \nabla f(y)\| &= \|(I - P_C)x - (I - P_C)y + A^T(I - P_Q)Ax - A^T(I - P_Q)Ay\| \\
&\leq \|x - P_C(x) + P_C(y) - y\| + \|A^T(I - P_Q)Ax - A^T(I - P_Q)Ay)\| \\
&\leq \|x - y\| + \|A\|\|(I - P_Q)Ax - (I - P_Q)Ay)\| \\
&\leq \|x - y\| + \|A\|\|Ax - Ay\| \\
&\leq \|x - y\| + \|A\|^2\|x - y\| \\
&= (1 + \|A\|^2)\|x - y\|.
\end{aligned}$$

So, ∇f is Lipschitz continuous and the Lipschitz constant is $1 + \|A\|^2$. □

Remark 2.6. Following the same line as in the proof for Theorem 2.5, we can obtain that ∇f_α is Lipschitz continuous and the Lipschitz constant is $1 + \|A\|^2 + \alpha$.

Proposition 2.7. *∇f_α is α-strongly monotone.*

Proof. Since $\nabla f_\alpha(x) = \nabla f(x) + \alpha x$, we have

$$\begin{aligned}
f_\alpha(x) - \nabla f_\alpha(y), x - y\rangle &= \langle \nabla f(x) - \nabla f(y) + \alpha(x - y), x - y\rangle \\
&= \alpha\|x - y\|^2 + \langle \nabla f(x) - \nabla f(y), x - y\rangle.
\end{aligned} \tag{2.1}$$

From Theorem 2.5, we know that f is a convex function. So

$$\langle \nabla f(x) - \nabla f(y), x - y\rangle \geq 0,$$

which, together with (2.1), implies that

$$\langle \nabla f_\alpha(x) - \nabla f_\alpha(y), x - y\rangle \geq \alpha\|x-y\|^2.$$

This tells us that ∇f_α is α-strongly monotone. $\square$

Proposition 2.8. *For any γ which satisfies $0 < \gamma < \frac{\alpha}{(1+\alpha+\|A\|^2)^2}$, $I - \gamma\nabla f_\alpha$ is contraction with coefficient $1 - \frac{1}{2}\alpha\gamma$, that is*

$$\|(I-\gamma\nabla f_\alpha)x - (I-\gamma\nabla f_\alpha)y\| \leq (1 - \frac{1}{2}\alpha\gamma)\|x-y\|.$$

Proof. From the condition that $0 < \gamma < \frac{\alpha}{(1+\alpha+\|A\|^2)^2}$, we can get that $\gamma^2(1+\alpha+\|A\|^2)^2 < \gamma\alpha < 1$. So

$$\begin{aligned}
\|(I-\gamma\nabla f_\alpha)x - (I-\gamma\nabla f_\alpha)y\|^2 &= \|x-y+\gamma\nabla f_\alpha y - \gamma\nabla f_\alpha x\|^2 \\
&= \|x-y\|^2 - 2\gamma\langle\nabla f_\alpha x - \nabla f_\alpha y, x-y\rangle + \gamma^2\|\nabla f_\alpha x - \nabla f_\alpha y\|^2 \\
&\leq \|x-y\|^2 - 2\gamma\alpha\|x-y\|^2 + \gamma^2(1+\alpha+\|A\|^2)^2\|x-y\|^2 \\
&\leq \|x-y\|^2 - 2\gamma\alpha\|x-y\|^2 + \gamma\alpha\|x-y\|^2 \\
&= (1-\gamma\alpha)\|x-y\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|(I-\gamma\nabla f_\alpha)x - (I-\gamma\nabla f_\alpha)y\| &\leq \sqrt{1-\gamma\alpha}\|x-y\| \\
&\leq (1 - \tfrac{1}{2}\gamma\alpha)\|x-y\|.
\end{aligned}$$

This completes the proof. $\square$

Now we consider the optimization problem (1.4). Combing the fact that $f(x)$ and $f_\alpha(x)$ are proper convex functions and Lemma 2.3, it easy to observe that $f_\alpha(x)$ is a strongly convex function, which guarantees that the solution of (1.4) is unique. Let x_α be the unique solution to the optimization problem (1.4).

Next we establish the relationship between x_α and the minimum-norm solution of the SFP. Note that finding the minimum-norm solution of the SFP is equal to solve the following optimization problem:

$$\min_{s.t. x\in C, Ax\in Q} \|x\|^2$$

It is easy to find that this is a convex programming and the objective function is strictly convex. So, it has a unique solution. Namely the minimum-norm solution of the SFP is unique.

Lemma 2.9. *$\{\|x_\alpha\|\}$ is bounded for $\alpha \in (0, \infty)$.*

Proof. Let $\alpha > \beta > 0$ and x_α and x_β be the solutions of optimization problems $\min_{x\in\Re^N} f_\alpha(x)$ and $\min_{x\in\Re^N} f_\beta(x)$, respectively. Then we have

$$\begin{cases} f_\alpha(x_\alpha) \leq f_\alpha(x_\beta); \\ f_\beta(x_\beta) \leq f_\beta(x_\alpha). \end{cases}$$

i.e.,

$$f(x_\alpha) + \frac{1}{2}\alpha\|x_\alpha\|^2 \leq f(x_\beta) + \frac{1}{2}\alpha\|x_\beta\|^2;$$

$$f(x_\beta) + \frac{1}{2}\beta\|x_\beta\|^2 \leq f(x_\alpha) + \frac{1}{2}\beta\|x_\alpha\|^2.$$

Adding these two inequalities, we obtain that

$$\alpha\|x_\alpha\|^2 + \beta\|x_\beta\|^2 \leq \alpha\|x_\beta\|^2 + \beta\|x_\alpha\|^2.$$

It implies that

$$(\alpha - \beta)(\|x_\alpha\|^2 - \|x_\beta\|^2) \leq 0.$$

That is to say $\|x_\alpha\| \leq \|x_\beta\|$. So $\|x_\alpha\|$ decreases for $\alpha \in (0, \infty)$. Moreover $0 \leq \|x_\alpha\| < \|x_0\|$, which guarantees that $\|x_\alpha\|$ is bounded. □

Theorem 2.10. *As $\alpha \to 0, \{x_\alpha\}$ converges to the minimum-norm solution of the SFP.*

Proof. Let $\tilde{x}$ be the unique minimum-norm solution of the SFP. We first show that

$$\|x_\alpha\| \leq \|\tilde{x}\|. \tag{2.2}$$

Since x_α be the unique solution to the optimization problem (1.4), we have

$$f_\alpha(x_\alpha) \leq f_\alpha(\tilde{x}),$$

i.e.,

$$\frac{1}{2}\|(I - P_C)x_\alpha\|^2 + \frac{1}{2}\|(I - P_Q)Ax_\alpha\|^2 + \frac{1}{2}\|x_\alpha\|^2 \leq \frac{1}{2}\|\tilde{x}\|^2.$$

Then we get

$$\|x_\alpha\|^2 \leq \|\tilde{x}\|^2 - \frac{1}{\alpha}(\|(I - P_C)x_\alpha\|^2 + \|(I - P_Q)Ax_\alpha\|^2) \leq \|\tilde{x}\|^2.$$

So (2.2) holds. Now we suppose that $\{\alpha_n\}$ is a sequence such that $\alpha_n \to 0$ as $n \to \infty$. The optimal solution x_{α_n} of $\min_{x\in\Re^N} f_{\alpha_n}(x)$ is abbreviated as x_n for convenience. From Lemma 2.9, we know that $\{x_n\}$ is bounded. Let $\bar{x}$ be any accumulation point of $\{x_n\}$. Then there exists its subsequence $\{x_{n_i}\}$ converging to $\bar{x}$. Next we show that $\bar{x}$ is a solution to the SFP and $\bar{x} = \tilde{x}$. Because f_{α_n} is continuous and differentiable, from Lemma 2.4, we can easily get

$$\nabla f_{\alpha_n}(x_n) = 0.$$

Then

$$\langle (I - P_C)x_n + (I - P_Q)Ax_n + \alpha_n x_n, \tilde{x} - x_n\rangle = 0. \tag{2.3}$$

Note that

$$\begin{aligned}\langle x_n - P_C(x_n), x_n - \tilde{x}\rangle &= \langle x_n - P_C(x_n), x_n - P_C(x_n) + P_C(x_n) - \tilde{x}\rangle \\ &= \|x_n - P_C(x_n)\|^2 + \langle x_n - P_C(x_n), P_C(x_n) - \tilde{x}\rangle \\ &\geq \|x_n - P_C(x_n)\|^2.\end{aligned} \tag{2.4}$$

By similar proof to (2.4), we can also obtain that

$$\langle (I - P_Q)Ax_n, Ax_n - A\tilde{x}\rangle \geq \|(I - P_Q)Ax_n)\|^2. \tag{2.5}$$

Combining (2.2)-(2.5), we get

$$\begin{aligned}\|x_n - P_C(x_n)\|^2 + \|(I-P_Q)Ax_n)\|^2 &\le \langle (I-P_C)x_n + A^T(I-P_Q)Ax_n, x_n - \tilde{x}\rangle \\ &= \alpha_n\langle x_n, \tilde{x} - x_n\rangle \\ &\le \alpha_n\|x_n\|\|\tilde{x} - x_n\| \\ &\le \alpha_n\|x_n\|(\|\tilde{x}\| + \|x_n\|) \\ &\le 2\alpha_n\|\tilde{x}\|^2.\end{aligned}$$

So

$$\|x_{n_i} - P_C(x_{n_i})\|^2 + \|(I-P_Q)Ax_{n_i})\|^2 \le 2\alpha_{n_i}\|\tilde{x}\|^2. \tag{2.6}$$

Taking limits on both side of (2.6), we conclude that

$$\|\bar{x} - P_C(\bar{x})\|^2 + \|A\bar{x} - P_Q(A\bar{x})\|^2 = 0.$$

We can immediately get

$$\bar{x} = P_C(\bar{x}) \text{ and } A\bar{x} = P_Q(A\bar{x}).$$

That is to say that $\bar{x}$ is a solution to the SFP. We have proved that

$$\|x_n\| \le \tilde{x}.$$

Then

$$\|\bar{x}\| \le \lim_{n\to\infty} \sup\|x_n\| \le \tilde{x}.$$

i.e.,

$$\|\bar{x}\| \le \tilde{x}.$$

Due to the uniqueness of minimum-norm solution, we must have $\bar{x} = \tilde{x}$. So $\{x_n\}$ is convergent. And its subsequence $\{x_{n_i}\}$ converges to $\tilde{x}$, so is $\{x_n\}$, which is the required result. □

3 Gradient algorithm and its convergence

In this section, we will first introduce a gradient algorithm for solving the minimum-norm solution of the SFP, and then establish the convergence of it.

Algorithm 3.1. Given any $x^0 \in \Re^N$. For $k = 0, 1, 2, ...,$
calculate

$$x^{k+1} = x^k - \gamma_k \nabla f_{\alpha_k}(x^k),$$

where $0 < \gamma_k < \frac{\alpha_k}{(1+\|A\|^2+\alpha_k)^2}$.

Lemma 3.2 ([15, 18]). *Assume that $\{\varphi_k\}$ be a sequence of nonnegative real numbers such that*

$$\varphi_{k+1} \le (1-\sigma_k)\varphi_k + \sigma_k\delta_k, \quad k \ge 0.$$

where $\{\sigma_k\}, \{\delta_k\}$ are sequences of real number satisfying
(a) $\{\sigma_k\} \subset [0,1]$ *and* $\sum_{k=0}^{\infty}\sigma_k = \infty$;
(b) $\lim_{k\to\infty} sup\delta_k \le 0$, *or* $\sum_{k=0}^{\infty}\sigma_k\delta_k$ *is convergent.*
Then, $\lim_{k\to\infty}\varphi_k = 0$.

Now we establish the convergence of Algorithm 3.1.

Theorem 3.3. *Assuming that the solution set of the SFP is nonempty. The sequences* $\{\gamma_k\}, \{\alpha_k\}$ *satisfies the following conditions:*

(i) $0 < \gamma_k < \frac{\alpha_k}{(1+\|A\|^2+\alpha_k)^2}$;

(ii) $\alpha_k \to 0$ *and* $\gamma_k \to 0$;

(iii) $\sum_{k=0}^{\infty} \alpha_k \gamma_k = \infty$;

(iv) $(|\gamma_{k+1} - \gamma_k| + \gamma_k |\alpha_{k+1} - \alpha_k|)/(\alpha_{k+1}\gamma_{k+1})^2 \to 0$.

Then the sequence $\{x^k\}$ *generated by Algorithm 3.1 globally converges to the minimum-norm solution of the SFP.*

Proof. Because f_{α_k} is continuous and differentiable. Through Lemma 2.4, we can easily get

$$\nabla f_{\alpha_k}(x_{\alpha_k}) = 0 \text{ and } \lim_{k\to\infty} x_{\alpha_k} = \tilde{x}. \tag{3.1}$$

Now we only need to prove that $\|x^{k+1} - x_{\alpha_k}\| \to 0$. Due to (3.1), x_{α_k} can be express as: $x_{\alpha_k} = (I - \gamma_k \nabla f_{\alpha_k})x_{\alpha_k}$. Note that

$$\begin{aligned}
\|x^{k+1} - x_{\alpha_k}\| &= \|(I - \gamma_k \nabla f_{\alpha_k})x^k - (I - \gamma_k \nabla f_{\alpha_k})x_{\alpha_k}\| \\
&\le (1 - \frac{1}{2}\alpha_k\gamma_k)\|x^k - x_{\alpha_k}\| \\
&= (1 - \frac{1}{2}\alpha_k\gamma_k)\|x^k - x_{\alpha_{k-1}} + x_{\alpha_{k-1}} - x_{\alpha_k}\| \\
&\le (1 - \frac{1}{2}\alpha_k\gamma_k)\|x^k - x_{\alpha_{k-1}}\| + \|x_{\alpha_k} - x_{\alpha_{k-1}}\|.
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
\|x_{\alpha_k} - x_{\alpha_{k-1}}\| &= \|x_{\alpha_k} - (I - \gamma_k \nabla f_{\alpha_k})x_{\alpha_{k-1}} + (I - \gamma_k \nabla f_{\alpha_k})x_{\alpha_{k-1}} - x_{\alpha_{k-1}}\| \\
&\le \|x_{\alpha_k} - (I - \gamma_k \nabla f_{\alpha_k})x_{\alpha_{k-1}}\| + \|\gamma_k \nabla f_{\alpha_k}(x_{\alpha_{k-1}})\| \\
&= \|(I - \gamma_k \nabla f_{\alpha_k})x_{\alpha_k} - (I - \gamma_k \nabla f_{\alpha_k})x_{\alpha_{k-1}}\| + \|\gamma_k \nabla f_{\alpha_k}(x_{\alpha_{k-1}})\| \\
&\le (1 - \frac{1}{2}\alpha_k\gamma_k)\|x_{\alpha_k} - x_{\alpha_{k-1}}\| + \|\gamma_k \nabla f_{\alpha_k}(x_{\alpha_{k-1}})\| \\
&= (1 - \frac{1}{2}\alpha_k\gamma_k)\|x_{\alpha_k} - x_{\alpha_{k-1}}\| + \|\gamma_k \nabla f_{\alpha_k}(x_{\alpha_{k-1}}) - \gamma_{k-1} \nabla f_{\alpha_{k-1}}(x_{\alpha_{k-1}})\|.
\end{aligned}$$

So we get

$$\begin{aligned}
&\|x_{\alpha_k} - x_{\alpha_{k-1}}\| \\
&\le \frac{2}{\alpha_k\gamma_k}\|\gamma_k \nabla f_{\alpha_k}(x_{\alpha_{k-1}}) - \gamma_{k-1} \nabla f_{\alpha_{k-1}}(x_{\alpha_{k-1}})\| \\
&= \frac{2}{\alpha_k\gamma_k}(\|(\gamma_k - \gamma_{k-1})\nabla f_{\alpha_k}(x_{\alpha_{k-1}}) + \gamma_{k-1}(\nabla f_{\alpha_k}(x_{\alpha_{k-1}}) - \nabla f_{\alpha_{k-1}}(x_{\alpha_{k-1}}))\|) \\
&\le \frac{2}{\alpha_k\gamma_k}(|\gamma_k - \gamma_{k-1}|\|\nabla f_{\alpha_k}(x_{\alpha_{k-1}})\| + \gamma_{k-1}\|\nabla f_{\alpha_k}(x_{\alpha_{k-1}}) - \nabla f_{\alpha_{k-1}}(x_{\alpha_{k-1}})\|) \\
&= \frac{2}{\alpha_k\gamma_k}(|\gamma_k - \gamma_{k-1}|\|\nabla f_{\alpha_k}(x_{\alpha_{k-1}})\| + \frac{1}{2}\gamma_{k-1}|\alpha_k - \alpha_{k-1}|\|x_{\alpha_{k-1}}\|).
\end{aligned} \tag{3.3}$$

From Lemma 2.9, we know that $\{x_{\alpha_k}\}$ is bounded. So $\{\nabla f_{\alpha_k}(x_{\alpha_k})\}$ is also bounded as $\alpha_k \to 0$.

Let
$$M = \sup\{\|\nabla f_{\alpha_k}(x_{\alpha_{k-1}})\|, \|x_{\alpha_{k-1}}\|\} < \infty.$$
Then we get the following inequality from (3.3)
$$\|x_{\alpha_k} - x_{\alpha_{k-1}}\| \leq \frac{2}{\alpha_k \gamma_k}(|\gamma_k - \gamma_{k-1}| + \frac{1}{2}\gamma_{k-1}|\alpha_k - \alpha_{k-1}|)M.$$
Let
$$\rho_k = \frac{4}{(\alpha_k \gamma_k)^2}(|\gamma_k - \gamma_{k-1}| + \frac{1}{2}\gamma_{k-1}|\alpha - k - \alpha_{k-1}|)M.$$
Then the above inequality can be rewritten as follows
$$\|x_{\alpha_k} - x_{\alpha_{k-1}}\| \leq \frac{\alpha_k \gamma_k}{2}\rho_k. \tag{3.4}$$
Combine (3.2) and (3.4), we get
$$\|x^{k+1} - x_{\alpha_k}\| \leq (1 - \frac{1}{2}\alpha_k \gamma_k)\|x^k - x_{\alpha_{k-1}}\| + \frac{\alpha_k \gamma_k}{2}\rho_k. \tag{3.5}$$
Under the conditions $(i) - (iv)$ in theorem, we obtain that
$$\frac{1}{2}\alpha_k \gamma_k \subset [0,1]; \quad \sum_{k=0}^{\infty} \frac{1}{2}\alpha_k \gamma_k = \infty; \quad \rho_k \to 0, \text{nemaly}, \lim_{k\to\infty} \sup \rho_k \leq 0.$$
Now from Lemma 3.2, we get $\lim_{k\to\infty} \|x^{k+1} - x_{\alpha_k}\| = 0$. So, $\lim_{k\to\infty} x^{k+1} = \lim_{k\to\infty} x_{\alpha_k}$. Then we obtain the required result, i.e., $\lim_{k\to\infty} x^{k+1} = \tilde{x}$. □

Remark 3.4. Note that, if $\alpha_k = k^{-\delta}, \gamma_k = k^{-\sigma}$ and $0 < \delta < \sigma < 1, \sigma + 2\delta < 1$, then $\{\alpha_k\}, \{\gamma_k\}$ satisfy the conditions $(i) - (iv)$ in Theorem 3.3.

4 Conclusions

In this paper we mainly consider to find the minimum-norm solution of the split feasibility problem. We first construct an unconstrained optimization problem related to the split feasibility problem and analyze the relationship between them. Then, we present a gradient algorithm to find the minimum-norm solution of the split feasibility problem. Under some mild conditions, we also establish the global convergence of the algorithm.

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