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A POWER PENALTY METHOD FOR LINEAR COMPLEMENTARITY PROBLEMS OVER SYMMETIC CONES*

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Abstract: In this paper, we investigate a power penalty approach to linear complementarity problems over symmetric cones (SCLCP) based on a nonlinear equation approximation. Based on a Jordan-algebraic technique, we show that the solution to this equation converges to that of the SCLCP when the penalty parameter tends to infinity under some mild conditions.

Key words: *symmetric cone, linear complementary problem, power penalty approach, Z-transformation*

Mathematics Subject Classification: *26B05, 65K10, 90C33*

1 Introduction

In this paper, we focus on linear complementarity problems over symmetric cones (SCLCP for short, see Section 2), which include the well-known standard linear complementarity problem over nonnegative orthant (LCP), the semidefinite linear complementarity problem (SDLCP), and the second-order cone linear complementarity problem (SOCLCP) as special cases. Since this problem is closely related to the optimality conditions for symmetric conic linear programming (SCLP) and has wide applications in engineering, economics, management science, it has been the focus of recent studies, see, e.g., [4, 6, 7, 18, 19].

Existence of solutions for SCLCP and its algorithms for solving SCLCP, such as smooth/nonsmooth Newton-type methods and interior point methods, have been extensively investigated in the literature, see, e.g., [10, 12, 14, 15]. However, there is a limited study of penalty methods for SCLCP, even for special cases of SDLCP and SOCLCP, while penalty methods for LCP and continuous variational inequality have been discussed, see, e.g., [2, 5, 16, 17]. Recently, Wang and Yang [17] established convergence rates of the power penalty method for solving LCP for two instances of linear transformations: one is the positive definite matrix case where the diagonal entries are positive and off-diagonal entries are less than zero, the other is the indefinite matrix case where only the diagonal entries are nonzero. It is natural to ask whether their results can be generalized to the SCLCP. In this paper, we answer the question affirmatively.

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2 Preliminaries

We first recall some concepts and results on Euclidean Jordan algebras that will be used in this paper. Most of them can be found in the monograph by Faraut and Korányi [3].

2.1 Euclidean Jordan algebras

Let V be a n-dimensional vector space over $\mathcal{R}$ and $(x,s)\mapsto x\circ s : V\times V\to V$ be a bilinear mapping. We call $(V,\circ)$ a *Jordan algebra* iff the bilinear mapping satisfies the following conditions:

(i) $x\circ s = s\circ x$ for all $x,s\in V$,
(ii) $x\circ(x^2\circ s) = x^2\circ(x\circ s)$ for all $x,s\in V$,

where $x^2 := x\circ x$ and $x\circ s$ is the *Jordan product* of x and s. We call an element e the *identity* element if and only if $z\circ e = e\circ z = z$ for all $z\in V$. A Jordan algebra $(V,\circ)$ with an identity element e is called a *Euclidean Jordan algebra*, denoted by $\mathcal{V} := (V,\langle\cdot,\cdot\rangle,\circ)$, if and only if there is an inner product, $\langle\cdot,\cdot\rangle$, such that

$$\langle x\circ s, z\rangle = \langle x, s\circ z\rangle \quad \text{for all } x,s,z\in V.$$

Given a Euclidean Jordan algebra $\mathcal{V}$, we define *the set of squares* as $K := \{z^2 : z\in\mathcal{V}\}$. It is known by Theorem III 2.1 in [3] that K is the *symmetric cone*, i.e., K is a closed, convex, homogeneous and self-dual cone.

A Euclidean Jordan algebra is said to be *simple* if it is not a direct sum of two (non-trivial) Euclidean Jordan algebras. It is well known that any nonzero Euclidean Jordan algebra is a product of simple Euclidean Jordan algebras and every simple algebra is isomorphic to one of the algebras given below:

(i) The algebra $\mathcal{S}^n$ of $n\times n$ real symmetric matrices with trace inner product and the Jordan product $X\circ Y = \frac{1}{2}(XY+YX)$;

(ii) The Jordan spin algebra $\mathcal{L}^n$ $(n\geq 3)$ on R^n with standard inner product and the Jordan product

$$x\circ y := (x^T y, x_1y_2+y_1x_2,\cdots,x_1y_n+y_1x_n)^T;$$

(iii) The algebra $\mathcal{H}^n$ of all $n\times n$ complex Hermitian matrices with trace inner product and $X\circ Y = \frac{1}{2}(XY+YX)$;

(iv) The algebra $\mathcal{Q}^n$ of all $n\times n$ quaternion Hermitian matrices with (real) trace inner product and $X\circ Y = \frac{1}{2}(XY+YX)$;

(v) The algebra $\mathcal{O}^3$ of all 3×3 octonion Hermitian matrices with (real) trace inner product and $X\circ Y = \frac{1}{2}(XY+YX)$.

We state below the spectral decomposition theorem for elements in a Euclidean Jordan algebra.

Theorem 2.1 (Spectral Decomposition Type II (Theorem III.1.2, [3]))**.** *Let $\mathcal{V}$ be a Euclidean Jordan algebra with rank r. Then for $z\in\mathcal{V}$, there exist a Jordan frame $\{q_1,q_2,\cdots,q_r\}$ and real numbers $\lambda_1(z)\geq\lambda_2(z)\geq\cdots\geq\lambda_r(z)$, such that*

$$z = \lambda_1(z)q_1+\lambda_2(z)q_2+\cdots+\lambda_r(z)q_r. \tag{2.1}$$

The numbers $\lambda_i(z)$ $(i \in \{1, 2, \cdots, r\})$ are the eigenvalues of z. We call (2.1) the spectral decomposition (or the spectral expansion) of z.

$$\lambda(x) := (\lambda_1(x), \lambda_2(x), \ldots, \lambda_r(x)).$$

We define

$$z^+ := \Pi_K(z), \qquad z^- := z^+ - z, \qquad \text{and} \qquad |z| = z^+ + z^-,$$

where $\Pi_K(z)$ denotes the (orthogonal) projection of z onto K.

Given (2.1), we have

$$z = \sum_1^r {\lambda_i}^+ q_i - \sum_1^r {\lambda_i}^- q_i \qquad \text{and} \qquad \langle \sum_1^r {\lambda_i}^+ q_i, \sum_1^r {\lambda_i}^- q_i \rangle = 0,$$

where for a real number α, $\alpha^+ := \max\{0, \alpha\}$ and $\alpha^- := \alpha^+ - \alpha$.

From this we easily verify that

$$z^+ = \sum_1^r {\lambda_i}^+ q_i \qquad \text{and} \qquad x^- = \sum_1^r {\lambda_i}^- q_i,$$

and so

$$z = z^+ - z^- \qquad \text{with} \qquad \langle z^+, z^- \rangle = 0.$$

Given (2.1), $z^\alpha = \lambda_1^\alpha(z) q_1 + \lambda_2^\alpha(z) q_2 + \cdots + \lambda_r^\alpha(z) q_r$, when $z \in K$, $\alpha > 0$. The trace of z is defined by $trace(z) := \sum_{i=1}^r \lambda_i(z)$. We assume that V carries the canonical trace inner product $\langle x, y \rangle = trace(x \circ y)$. We let $\|z\|_p = [\sum_{i=1}^r |\lambda_i(z)|^p]^{1/p}$ for $p \geq 1$. When $p = 2$, it is the induced norm given by

$$\|z\|_{\mathcal{V}} := \|z\|_2 := \sqrt{\langle z, z \rangle} = \sqrt{\sum_{i=1}^r \lambda_i^2(z)}.$$

By Corollary 4.10 in [9], when V is simple,

$$\langle x, y \rangle \leq \langle \lambda(x), \lambda(y) \rangle \; \forall x, \, y \in V. \tag{2.2}$$

Hence, by Hölder's inequality (see Theorem 4.2 in [13]),

$$|\langle x, y \rangle| \leq \|\lambda(x)\|_p \|\lambda(y)\|_q. \tag{2.3}$$

2.2 Nonlinear Penalized Equation

We consider SCLCP: find a vector $x \in \mathcal{V}$ such that

$$x \in K, \; y = \mathcal{M}(x) + q \in K, \text{and } \langle x, y \rangle = 0, \tag{2.4}$$

where $\mathcal{M} : \mathcal{V} \to \mathcal{V}$ is a linear transformation and $q \in \mathcal{V}$. In order to consider the power penalty method for SCLCP, we introduce the following *nonlinear equation (NE)*: Find $x \in \mathcal{V}$ such that

$$\mathcal{M}(x) + \mu(x^+)^{1/k} = q, \tag{2.5}$$

where $\mu > 1$ and $k > 0$ are parameters.

Replacing x by $-x$ in (2.5), we have

$$\mathcal{M}(x) - \mu(x^-)^{1/k} = -q. \tag{2.6}$$

This is a penalized equation with respect to SCLCP (2.4), where the penalty term penalizes the projection onto the symmetric cone of x. We expect that the solution of NE (2.6) converges to that of SCLCP (2.4) as $\mu \to +\infty$. Clearly, the rate of convergence depends on the parameters μ and k in the penalty term.

It is known that the SCLCP (2.4) is equivalent to the following variational inequality problem: Find $x \in K$ such that, for all $y \in K$,

$$\langle y - x, \mathcal{M}(x) + q\rangle \geq 0. \tag{2.7}$$

As mentioned in introduction, penalty methods for LCP and variational inequality problem have been discussed in the literature. Recently, Wang and Yang [17] established convergence rates of the power penalty method for solving LCP for a positive definite matrix where the diagonal entries are positive and off-diagonal entries are less than zero, and an indefinite matrix where only the diagonal entries are nonzero. However, there is a limited study of penalty methods for SCLCP, even for special cases of SDLCP and SOCLCP.

In order to study the power penalty methods for SCLCP, we introduce the following definitions of a linear transformation $\mathcal{M}$ on $\mathcal{V}$.

We say a linear transformation $\mathcal{M}$ is *strongly monotone* if there exists a constant $a_0 > 0$ such that

$$\langle x, \mathcal{M}(x)\rangle \geq a_0\|x\|_2^2 \ \ \forall\, x \in \mathcal{V}.$$

We say a linear transformation $\mathcal{M}$ is a *Z-transformation* (see [8]) if

$$x, y \in K, \quad \text{and } \langle x, y\rangle = 0 \Rightarrow \langle M(x), y\rangle \leq 0.$$

3 Main Results

In this section, we assume that $\mathcal{M}$ **is strongly monotone and Z-transformation**.

Example 3.1. $L_A(X) := AX + XA^T$ for a real $n \times n$ matrix A on the space $\mathcal{S}^n$. It is easy to verify that L_A is a Z-transformation. Also, when A is positive definite matrix, L_A is strongly monotone.

Now, we present two lemmas.

Lemma 3.2. *Let V be a simple Euclidean Jordan algebra of rank r. Let $x, y \in K$ and $0 < k \in \mathcal{R}$. Then $\langle x - y, x^{1/k} - y^{1/k}\rangle \geq 0$.*

Proof.

$$\begin{aligned}
\langle x - y, x^{1/k} - y^{1/k}\rangle &= \langle x, x^{1/k}\rangle - \langle y, x^{1/k}\rangle - \langle x, y^{1/k}\rangle + \langle y, y^{1/k}\rangle \\
&\geq \sum_1^r (\lambda_i(x))^{1+1/k} - \sum_1^r \lambda_i(y)(\lambda_i(x))^{1/k} \\
&\quad - \sum_1^r \lambda_i(y)(\lambda_i(y))^{1/k} + \sum_1^r (\lambda_i(y))^{1+1/k} \\
&= \sum_1^r (\lambda_i(x) - \lambda_i(y))((\lambda_i(x))^{1/k} - (\lambda_i(y))^{1/k}) \\
&\geq 0.
\end{aligned}$$

Note that the first inequality is from (2.2). □

Lemma 3.3. *Let V be a simple Euclidean Jordan algebra of rank r. Let $f(x) = (x^+)^{1/k}$ for $0 < k \in \mathcal{R}$. Then $\langle x - y, f(x) - f(y)\rangle \geq 0$ for $x, y \in V$.*

Proof. Since

$$\begin{aligned}\langle x^- - y^-, (x^+)^{1/k} - (y^+)^{1/k}\rangle &= \langle x^-, (x^+)^{1/k}\rangle - \langle x^-, (y^+)^{1/k}\rangle - \langle y^-, (x^+)^{1/k}\rangle + \langle x^-, (x^+)^{1/k}\rangle \\ &= -\langle x^-, (y^+)^{1/k}\rangle - \langle y^-, (x^+)^{1/k}\rangle \\ &\leq 0,\end{aligned}$$

we have

$$\begin{aligned}\langle x - y, f(x) - f(y)\rangle &= \langle x^+ - x^- - (y^+ - y^-), (x^+)^{1/k} - (y^+)^{1/k}\rangle \\ &= \langle x^+ - y^+, x^{1/k} - y^{1/k}\rangle - \langle x^- - y^-, (x^+)^{1/k} - (y^+)^{1/k}\rangle \\ &\geq \langle x^+ - y^+, (x^+)^{1/k} - (y^+)^{1/k}\rangle \\ &\geq 0.\end{aligned}$$

Note that the last inequality is from Lemma 3.2. □

Remark 3.4. The above two lemmas also can be deduced from Theorem 3 in [**?**].

Theorem 3.5. *Let V be a simple Euclidean Jordan algebra of rank r. Then*

(i) *For any $q \in V$, (2.4) has a unique solution.*

(ii) *For any $q \in V$, (2.5) has a unique solution.*

Proof. (i) When $\mathcal{M}$ is strongly monotone, by Theorem 17 in [6], $q \in V$, (2.4) had a unique solution.

(ii) Let $F(x) := \mathcal{M}(x) + \mu(x^+)^{1/k}$. When $\mathcal{M}$ is strongly monotone, by Lemma 3.3, it is easy to show that F is strong monotone. Thus, apply VI (K, F), where $K = V$ (see Theorem 2.3.3 in [4]), we have the conclusion. □

Remark 3.6. It is easy to see that x_μ is a solution of (2.5) if and only if $-x_\mu$ is a solution of (2.6).

Lemma 3.7. *Let V be a simple Euclidean Jordan algebra of rank r. Let x_μ be the solution to NE (2.5). Then for given $q \in V$, k, and a_0, there exists a positive constant $C = max\{\|q\|_{k+1}^k, \sqrt{\|q\|_{k+1}^{k+1}/a_0}\}$ such that*

$$\|(x_\mu)^+\|_p \leq \frac{C}{\mu^k}, \tag{3.1}$$

$$\|(x_\mu)^+\|_2 \leq \frac{C}{\mu^{k/2}}, \tag{3.2}$$

where μ and k are the parameters used in (2.5), and $p = 1 + 1/k$.

Proof. Since x_μ is the solution to NE (2.5), $\mathcal{M}(x_\mu)+\mu[(x_\mu)^+]^{1/k}=q$. Taking the inner product both sides, we obtain

$$\langle (x_\mu)^+, \mathcal{M}(x_\mu)\rangle + \langle (x_\mu)^+, \mu[(x_\mu)^+]^{1/k}\rangle = \langle (x_\mu)^+, q\rangle.$$

Using $p=1+1/k$ and $x_\mu=(x_\mu)^+-(x_\mu)^-$, we get

$$\langle (x_\mu)^+, \mathcal{M}((x_\mu)^+)\rangle - \langle (x_\mu)^+, \mathcal{M}((x_\mu)^-)\rangle + \mu\|(x_\mu)^+\|_p^p = \langle (x_\mu)^+, q\rangle. \tag{3.3}$$

Since $(x_\mu)^+,(x_\mu)^-\in K$ and $\langle (x_\mu)^+,(x_\mu)^-\rangle=0$, and $\mathcal{M}$ is strongly monotone and Z-transformation, we have $\langle (x_\mu)^+, \mathcal{M}((x_\mu)^+)\rangle \geq 0$ and $\langle (x_\mu)^+, \mathcal{M}((x_\mu)^-)\rangle \leq 0$. These together with (3.3) yields

$$\mu\|(x_\mu)^+\|_p^p \leq \langle (x_\mu)^+, q\rangle.$$

From (2.3), we obtain

$$\mu\|(x_\mu)^+\|_p^p \leq \|(x_\mu)^+\|_p \cdot \|q\|_{k+1}.$$

Since $p-1=1/k$, direct calculations derives that

$$\|(x_\mu)^+\|_p \leq \frac{1}{\mu^k}\|q\|_{k+1}^k \leq \frac{C}{\mu^k}.$$

Now, from (3.3) and (3.1), we have

$$\langle (x_\mu)^+, \mathcal{M}((x_\mu)^+)\rangle \leq \langle (x_\mu)^+, q\rangle \leq \|(x_\mu)^+\|_p \cdot \|q\|_{k+1} \leq \frac{1}{\mu^k}\|q\|_{k+1}^{k+1}.$$

Since $a_0\|(x_\mu)^+\|^2 \leq \langle (x_\mu)^+, \mathcal{M}((x_\mu)^+)\rangle$, we have $\|(x_\mu)^+\|_2 \leq \frac{C}{\mu^{k/2}}$. □

Theorem 3.8. *Let V be a simple Euclidean Jordan algebra of rank r. Let x^* and x_μ^* be the solutions of the SCLCP (2.4) and NE (2.6), respectively. Then there exists a positive constant C' such that*

$$\|x^*-x_\mu^*\|_2 \leq \frac{C'}{\mu^{k/2}}, \tag{3.4}$$

where μ and k are the parameters used in (2.5).

Proof. Let x_μ be the solution to (2.5). From the decomposition $x_\mu=(x_\mu)^+-(x_\mu)^-$, letting $w=-x^*+(x_\mu)^-$, we can rewrite the vector $-x^*-x_\mu$ as

$$-x^*-x_\mu = -x^*-(x_\mu)^+ + (x_\mu)^- = w-(x_\mu)^+. \tag{3.5}$$

We will consider the estimation of w, as the estimates for $(x_\mu)^+$ have been established in Lemma 3.7. Since $(x_\mu)^-\in K$ and $\mathcal{M}(x^*)+q\in K$, it holds

$$\langle (x_\mu)^-, \mathcal{M}(x^*)+q\rangle \geq 0.$$

Noting that $\langle x^*, \mathcal{M}(x^*)+q\rangle=0$, we have

$$-\langle x^*-(x_\mu)^-, \mathcal{M}(x^*)+q\rangle \geq 0.$$

Thus,

$$-\langle w, \mathcal{M}(-x^*)-q\rangle \geq 0. \tag{3.6}$$

Taking the inner product on the both sides of the equation $\mathcal{M}(x_\mu) + \mu[(x_\mu)_+]^{1/k} = q$ with w, we obtain

$$\langle w, \mathcal{M}(x_\mu)\rangle + \langle w, \mu[(x_\mu)_+]^{1/k}\rangle - \langle w, q\rangle = 0. \tag{3.7}$$

Adding up (3.6) and (3.7), we obtain

$$\langle -w, \mathcal{M}(-x^* - x_\mu)\rangle + \langle w, \mu[(x_\mu)_+]^{1/k}\rangle \geq 0.$$

That is,

$$\langle w, \mathcal{M}(-x^* - x_\mu)\rangle \leq \mu\langle w, [(x_\mu)_+]^{1/k}\rangle. \tag{3.8}$$

Note that $\langle w, [(x_\mu)^+]^{1/k}\rangle = \langle -x^* + (x_\mu)^-, [(x_\mu)^+]^{1/k}\rangle = \langle -x^*, [(x_\mu)^+]^{1/k}\rangle \leq 0$ because of $x^* \in K$ and $\langle (x_\mu)^-, ((x_\mu)^+)^{1/k}\rangle = 0$. Thus, we have

$$\langle w, \mathcal{M}(w - (x_\mu)^+)\rangle = \langle w, \mathcal{M}(-x^* - x_\mu)\rangle \leq 0.$$

Hence, $\langle w, \mathcal{M}(w)\rangle \leq \langle w, \mathcal{M}((x_\mu)^+)\rangle \leq ||\mathcal{M}||\, \|w\|_2\, \|(x_\mu)^+\|_2$. Therefore, it is straightforward to derive from strong monotonicity of $\mathcal{M}$ and (3.2) of Lemma 3.7 that

$$\|w\|_2 \leq ||\mathcal{M}||\, \|(x_\mu)^+\|_2 \leq \frac{||\mathcal{M}||C/a_0}{\mu^{k/2}}. \tag{3.9}$$

This together with (3.2), (3.5) and the triangular inequality yields

$$\|x^* + x_\mu\|_2 \leq \|w\|_2 + \|(x_\mu)^+\|_2 \leq \frac{||\mathcal{M}||C/a_0 + C}{\mu^{k/2}}. \tag{3.10}$$

Now, letting $C' = ||\mathcal{M}||C/a_0 + C$, we have

$$\|x^* + x_\mu\|_2 \leq \frac{C'}{\mu^{k/2}}.$$

Now, from Remark 3.6, we have $x_\mu{}^* = -x_\mu$. Thus,

$$\|x^* - x_\mu{}^*\|_2 \leq \|w\|_2 + \|(x_\mu)^+\|_2 \leq \frac{C'}{\mu^{k/2}}.$$

□

Since all norms in a finite dimensional space are equivalent, we can improve the error bound in (3.4) in the following theorem.

Theorem 3.9. *Let V be a simple Euclidean Jordan algebra of rank r. Let x^* and $x_\mu{}^*$ be the solutions to the SCLCP (2.4) and NE (2.6), respectively. Then there exists a positive constant C'' such that*

$$\|x^* - x_\mu{}^*\|_2 \leq \frac{C''}{\mu^k}, \tag{3.11}$$

where μ and k are the parameters used in (2.5).

Proof. Since all norms on $\mathcal{V}$ are equivalent, there exists $d > 0$ such that

$$\|(x_\mu)^+\|_2 \leq d\|(x_\mu)^+\|_p.$$

Thus, from Lemma 3.7, we have

$$\|(x_\mu)^+\|_2 \leq \frac{dC}{\mu^k},$$

Now, letting $C'' = dC'$, where C' is in Theorem 3.8, we have from (3.9) and (3.10),

$$\|x^* - {x_\mu}^*\|_2 \leq \frac{C''}{\mu^k}.$$

□

4 Conclusion

In this paper, we have generalized the power penalty method for LCP in [17] to SCLCP. As noted in [17], one advantage of these estimates is that, to obtain the same level of accuracy of the approximate solution to that of SCLCP, the penalty parameter required for $k > 1$ is smaller than that required for $k = 1$, while the penalized problem with $k > 1$ is non-smooth and non-Lipschitz. In the standard LCP, Wang, Yang and Teo [16] proposed a smoothing technique in real computation. Clearly, one interesting research topic is to extend the smoothing or nonsmooth techniques in the power penalty approach of SCLCP.

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