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# NEW NONLINEAR SEPARATION FUNCTIONS AND OPTIMALITY CONDITIONS FOR GENERALIZED KY FAN INEQUALITIES*

Y. Zhang, L. Wang and T. Chen

**Abstract:** In this paper, by using the Gertewitz nonlinear scalarization function, we first establish two weak separation functions and strong separation functions. Then, by virtue of the separation function, we obtain some optimality conditions for weakly efficient solution and globally efficient solution to generalized Ky Fan inequalities with constraints in the sense of image space analysis.

**Key words:** *image space analysis, optimality condition, nonlinear weak (strong) separation function, generalized Ky Fan inequality*

**Mathematics Subject Classification:** *90C46, 49K10, 49K27*

## 1 Introduction

Recently, there has been an increasing interest in the image space analysis (ISA) of optimization problems, e.g., see [4, 5, 11–13, 16–23]. The ISA is a powerful tool and a unifying scheme for studying both vector variational inequality problems and vector Ky Fan inequality problems. This approach can be applied to any kind of problem that can be expressed under the form of the impossibility of a parametric system. The impossibility of such a system is reduced to the disjunction of two suitable subsets of the image space. The disjunction between the two suitable subsets is proved by showing that they lie in two disjoint level sets of a separating functional. Dien et al. [4] established an overview of regularity conditions by using the image space approach. Li and Huang [11] investigated vector variational inequalities with matrixin inequality constraints by using the image space analysis. Li and Huang [12] investigated Lagrangian-type necessary and sufficient optimality conditions and gap functions and weak sharpness for variational inequalities with cone constraints by using the image space analysis approach. Besides, the results obtained have been applied to standard and time-dependent traffic equilibria. Recently, from the view of the image, Guu and Li [9] investigated vector quasi-equilibrium problems with a variable ordering relation by introducing the notion of the quasi relative interior which does not require the cone to have nonempty interior. Zhu and Li [22, 23] established a unified duality scheme and three

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special duality schemes (the Lagrange-type duality, Wolfe duality and Mond-Weir duality) for a constrained extremum problem by virtue of the image space analysis. Very recently, Li et al. [13,19–21] obtained some saddle-point sufficient optimality conditions and some necessary and sufficient optimality conditions associated with the constrained extremum problem by using the image space analysis.

On the other hand, as the unified model of vector optimization problems, vector variational inequality problems, variational inclusion problems and vector complementarity problems, generalized Ky Fan inequalities have been intensively studied. The existence results for various kinds of generalized Ky Fan inequalities have been established, e.g., see [1–3] and the references therein. There are many papers which deal with the properties of the solutions for the generalized Ky Fan inequalities. Gong [6] obtained the connectedness and path connectedness of weak efficient solution sets and various proper efficient solution sets of generalized Ky Fan inequalities. Gong [7,8] obtained some optimality conditions and the scalarization results for weakly efficient solution, Henig efficient solution, globally efficient solution and superefficient solution to the generalized Ky Fan inequalities with constraints by using the separation theorem of convex sets.

To the best of our knowledge, there are few papers to investigate optimality conditions for generalized Ky Fan inequalities with constraints by using the image space analysis. Motivated by the work of [7,13,19,21], two weak separation functions and strong separation functions are obtained. Then, some optimality conditions for weakly efficient solution and globally efficient solution to the generalized Ky Fan inequalities with constraints are also obtained.

The paper is organized as follows. In Section 2, we recall the main notions and definitions concerning the ISA. In Sections 3, we introduce two weak separation functions and strong separation functions. By virtue of the separation function, we obtain some optimality conditions for weakly efficient solution and globally efficient solution to the generalized Ky Fan inequalities with constraints.

## 2 Preliminaries

Let $X$, $Y$ and $Z$ be real Hausdorff topological vector spaces. Assume that $S$ and $C$ are two pointed closed convex cones in $Y$ and $Z$ with nonempty interior, i.e., $\text{int}S \neq \emptyset$ and $\text{int}C \neq \emptyset$, respectively.

**Definition 2.1** ([14])**.** Given $e \in \text{int}S$, the Gertewitz nonconvex separation function $\xi_e : Y \to R$ is defined by

$$\xi_e(z) := \sup\{t \in R : z \in te + S\}, \quad z \in Y.$$

Next, we give some useful properties of the above scalarization functions.

**Lemma 2.2** ([14,15])**.** *Let $e \in \text{int}S$. The following properties hold:*

(i) $\xi_e(z) > r \Leftrightarrow z \in re + \text{int}S$;

(ii) $\xi_e(z) \geq r \Leftrightarrow z \in re + S$;

(iii) $\xi_e(z) < r \Leftrightarrow z \notin re + S$;

(iv) $\xi_e(z) \leq r \Leftrightarrow z \notin re + \text{int}S$;

(v) $\xi_e(\cdot)$ is a continuous function.

Let $E$ be a nonempty subset of $X$, $f : E \times E \to Y$ be a vector-valued mapping and $g : E \to Z$ be a vector-valued mapping.

Consider the following Generalized Ky Fan Inequality with constraints (for short GKFIC): a vector $\bar{x} \in K$ satisfying

$$f(\bar{x}, y) \notin -\mathrm{int}S, \quad \forall y \in K, \tag{2.1}$$

is called a weakly efficient solution to the GKFIC, where the constraint set

$$K = \{x \in E : g(x) \in -C\}.$$

**Definition 2.3** ([6]). A vector $\bar{x} \in K$ is called a globally efficient solution to the GKFIC if there exists a pointed convex cone $H \subset Y$ with $S\backslash\{0\} \subset \mathrm{int}H$ such that

$$f(\bar{x}, y) \notin -H\backslash\{0\}, \qquad \forall y \in K.$$

If $f(x, y) = v(y) - v(x)$, $x, y \in K$, and if $\bar{x} \in K$ is a weakly efficient solution or a globally efficient solution to the GKFIC, then $\bar{x} \in K$ is a weakly efficient solution or a globally efficient solution to a vector optimization problem, where $v$ is a vector-valued mapping.

We recall the main features of the ISA for the problem (1). Let $\bar{x} \in E$; define the map

$$A_{\bar{x}} : E \to Y \times Z, \;\; A_{\bar{x}}(y) := (f(\bar{x}, y), g(y)),$$

and consider the sets

$$\mathcal{K}_{\bar{x}} := \{(u, v) \in Y \times Z : (u, v) = A_{\bar{x}}(y), \; y \in E\},$$

$$\mathcal{H} := \{(u, v) \in Y \times Z : u \in \mathrm{int}S, \; v \in C\},$$

and

$$\mathcal{H}_1 := \{(u, v) \in Y \times Z : u \in H\backslash\{0\}, \; v \in C\}.$$

$\mathcal{K}_{\bar{x}}$ is called the image of the problem (1), while $Y \times Z$ is the image space (IS). Obviously, $\bar{x} \in K$ is a weakly efficient solution to the GKFIC if and only if the generalized system

$$A_{\bar{x}}(y) \in -\mathcal{H}, \;\; y \in E, \tag{2.2}$$

has no solutions, or, equivalently,

$$\mathcal{K}_{\bar{x}} \cap (-\mathcal{H}) = \emptyset.$$

Similarly, $\bar{x} \in K$ is a globally efficient solution to the GKFIC if and only if the generalized system

$$A_{\bar{x}}(y) \in -\mathcal{H}_1, \;\; y \in E,$$

has no solutions, or, equivalently,

$$\mathcal{K}_{\bar{x}} \cap (-\mathcal{H}_1) = \emptyset.$$

In the sequel, $\Pi$ will denote a set of parameters to be specified case by case (see [13]).

**Definition 2.4** ([5]). The class of all the functions $w : Y \times Z \times \Pi \to R$, such that

(i) $lev_{\geq 0}\, w(\cdot; \pi) \supseteq \mathcal{H}(or, respectively, \mathcal{H}_1), \;\; \forall \pi \in \Pi,$

(ii) $\bigcap_{\pi \in \Pi} lev_{>0}\, w(\cdot; \pi) \subseteq \mathcal{H}(or, respectively, \mathcal{H}_1),$

is called the class of weak separation functions with respect to $\mathcal{H}$ (or, $\mathcal{H}_1$). The level sets $lev_{\geq 0} f$ and $lev_{>0} f$ are defined by

$$lev_{\geq 0} f := \{x \in X : f(x) \geq 0\}, \ \ lev_{>0} f := \{x \in X : f(x) > 0\}.$$

Besides the weak separation functions, another type of separation functions has been introduced in [5].

**Definition 2.5** ([5])**.** The class of all the functions $s : Y \times Z \times \Pi \to R$, such that

(i) $lev_{>0}\, s(\cdot;\pi) \subseteq \mathcal{H}(or, respectively, \mathcal{H}_1), \ \ \forall \pi \in \Pi,$

(ii) $\bigcup_{\pi\in\Pi} lev_{>0}\, s(\cdot;\pi) = \mathrm{ri}\,\mathcal{H}(or, respectively, \mathrm{ri}\,\mathcal{H}_1),$

is called the class of strong separation functions with respect to $\mathcal{H}$ (or, $\mathcal{H}_1$).

**Remark 2.1** $\mathrm{ri}\,\mathcal{H} = \mathrm{int}\,\mathcal{H}$ when $\mathrm{int}\,\mathcal{H} \neq \emptyset$. Hence, $\bigcup_{\pi\in\Pi} lev_{>0} s(\cdot;\pi) = \mathrm{ri}\,\mathcal{H}$ is equivalent to $\bigcup_{\pi\in\Pi} lev_{>0} s(\cdot;\pi) = \mathrm{int}\,\mathcal{H}$ when $\mathrm{int}\,\mathcal{H} \neq \emptyset$.

**Remark 2.2** It is clear that $\mathrm{cl}\mathcal{H}$ and $\mathrm{cl}\mathcal{H}_1$ are closed convex cones, and $\mathrm{int}\ (\mathrm{cl}\mathcal{H}) = \mathrm{int}\mathcal{H} \neq \emptyset$ and $\mathrm{int}\ (\mathrm{cl}\mathcal{H}_1) = \mathrm{int}\mathcal{H}_1 \neq \emptyset$. However, $\mathrm{cl}\mathcal{H}$ and $\mathrm{cl}\mathcal{H}_1$ are not always pointed. In fact, if a pointed closed convex cone $S \subset Y$ has a base, then

$$S^\sharp = \{y^* \in Y^* : y^*(y) > 0, \forall y \in S\backslash\{0\}\} \neq \emptyset.$$

Take $y^* \in S^\sharp$, set $\mathcal{H} = \{y \in Y : y^*(y) > 0\} \bigcup \{0\}$. Then $\mathcal{H}$ is a pointed convex cone and $S\backslash\{0\} \subset \mathrm{int}\mathcal{H}$. We have

$$\mathrm{cl}\mathcal{H} = \{y \in Y : y^*(y) \geq 0\}.$$

There is $y \neq 0$ such that $y \in (\mathrm{cl}\mathcal{H}) \bigcap (-\mathrm{cl}\mathcal{H})$. It is clear that

$$(y,0) \in (\mathrm{cl}\mathcal{H}_1) \bigcap (-\mathrm{cl}\mathcal{H}_1) \quad \textit{and} \quad (y,0) \neq (0,0).$$

Thus, $\mathrm{cl}\mathcal{H}$ and $(\mathrm{cl}\mathcal{H}_1)$ are not pointed cones.

## 3 Separation functions and optimality conditions

In this section, we first introduce two nonlinear separation functions in the sense of image space analysis by using the Gertewitz separation functional.

**Lemma 3.1.** *The nonlinear function $h_1 : Y \times Z \times \Pi \to R$, if $\mathrm{cl}\mathcal{H} \neq Y \times Z$, then*

$$h_1(z,e) := \sup\{t \in R : z \in te + \mathrm{cl}\mathcal{H}\}, \ \ z \in Y \times Z \ \textit{and} \ e \in \Pi = \mathrm{int}\mathcal{H}$$

*is a weak separation function and strong separation function, and the nonlinear function $h_2 : Y \times Z \times \Pi \to R$, if $\mathrm{cl}\mathcal{H}_1 \neq Y \times Z$, then*

$$h_2(z,e) := \sup\{t \in R : z \in te + \mathrm{cl}\mathcal{H}_1\}, \ \ z \in Y \times Z \ \textit{and} \ e \in \Pi = \mathrm{int}\mathcal{H}_1$$

*is also a weak separation function and strong separation function.*

*Proof.* First, we show $h_1$ is a weak separation function.

For any $z \in \mathcal{H} \subseteq \mathrm{cl}\mathcal{H}$, by Lemma 2.2 (ii), for each $e \in \Pi$,

$$h_1(z,e) \geq 0, \qquad \forall z \in \mathrm{cl}\mathcal{H}.$$

Hence, we have

$$lev_{\geq 0}h_1(\cdot, e) \supseteq \mathcal{H}, \quad \forall e \in \Pi.$$

Now, we show

$$\bigcap_{e\in\Pi} lev_{>0}h_1(\cdot, e) \subseteq \mathcal{H}. \tag{3.1}$$

Assume that (3.1) doesn't hold. Then, there exists $z \notin \mathcal{H}$ such that

$$h_1(z, e) > 0, \quad \forall e \in \Pi.$$

By Lemma 2.2 (i), we have

$$z \in \text{int}\mathcal{H} \subseteq \mathcal{H},$$

which contradicts $z \notin \mathcal{H}$. Hence, (3.1) holds and $h_1$ is a weak separation function.

Next, we show $h_1$ is also a strong separation function. First, we claim that

$$lev_{>0}h_1(\cdot, e) \subseteq \mathcal{H}, \quad \forall e \in \Pi.$$

In fact, for any $z \in lev_{>0}h_1(\cdot, e), \forall e \in \Pi$, by Lemma 2.2 (i),

$$z \in \text{int}\mathcal{H} \subseteq \mathcal{H}.$$

Now we show that

$$\bigcup_{e\in\Pi} lev_{>0}h_1(\cdot, e) = \text{int}\mathcal{H}.$$

By Lemma 2.2 (i), it is clear that

$$lev_{>0}h_1(\cdot, e) \subseteq \text{int}\mathcal{H}, \quad \forall e \in \Pi.$$

Namely,

$$\bigcup_{e\in\Pi} lev_{>0}h_1(\cdot, e) \subseteq \text{int}\mathcal{H}.$$

On the other hand, for each $z \in \text{int}\mathcal{H}$, by Lemma 2.2 (i), for each $e \in \Pi$, $h_1(z, e) > 0$. Thus, we have that for each $e \in \Pi$,

$$\text{int}\mathcal{H} \subseteq lev_{>0}h_1(\cdot, e) \subseteq \bigcup_{e\in\Pi} lev_{>0}h_1(\cdot, e).$$

Namely, $h_1$ is a strong separation function.

By assumptions and Lemma 2.16 in [10], for each $z \in Y \times Z, e \in \text{int}\mathcal{H}$, we have

$$h_1(z, e) < \infty.$$

Similarly, we can prove that $h_2$ is also a weak and strong separation function. This completes the proof. □

**Remark 3.2.** In [13, 19, 21], the authors also have been obtained some different separation functions by virtue of the oriented distance function and a set of parameters to be specified case by case. However, by using the Gertewitz separation functional, we obtained two new separation functions, which are weak and strong separation functions. Obviously, our results are different from ones in [13].

In all the sequel of this section, we discuss optimality conditions to the GKFIC by the above separation functions.

**Theorem 3.3.** *Let $E$ be a nonempty subset of $X$ and $f : E \times E \to Y$ be a vector-valued mapping. Let $g : E \to Z$ be a vector-valued mapping. If $\bar{x} \in K$ is a weakly efficient solution to the GKFIC, then for any $e \in \mathrm{int}\mathcal{H}$,*

$$h_1((-f(\bar{x}, y), -g(y)), e) \leq 0, \qquad \forall y \in E.$$

*Proof.* Let $\bar{x} \in K$ be a weakly efficient solution to the GKFIC and $e \in \mathrm{int}\mathcal{H}$. Thus, by image space analysis approach,

$$\mathcal{K}_{\bar{x}} \cap (-\mathcal{H}) = \emptyset.$$

Then,

$$\mathcal{K}_{\bar{x}} \cap (-\mathrm{int}\mathcal{H}) = \emptyset,$$

i.e.,

$$(-f(\bar{x}, y), -g(y)) \notin \mathrm{int}\mathcal{H}, \qquad \forall y \in E.$$

By Lemma 2.2 (iv), we have

$$h_1((-f(\bar{x}, y), -g(y)), e) \leq 0, \qquad \forall y \in E.$$

This completes the proof. □

When $f(x, y) = v(y) - v(x)$, $x, y \in K$, we can obtain the next result for the vector optimization problem with constraints.

**Corollary 3.4.** *Let $E$ be a nonempty subset of $X$ and $v : E \to Y$ be a vector-valued mapping. Let $g : E \to Z$ be a vector-valued mapping. If $\bar{x} \in K$ is a weakly efficient solution to the vector optimization problem with constraints, then for any $e \in \mathrm{int}\mathcal{H}$,*

$$h_1((v(\bar{x}) - v(y), -g(y)), e) \leq 0, \qquad \forall y \in E.$$

Next, we give an example to explain Corollary 3.1.

**Example 3.5.** Let $X = R$ and $Y = Z = R^2$. Let $E = [-1, 1]$ and

$$S = C = \{(u, v) \in R^2 | u \geq 0, v \geq 0\}.$$

We define two vector-valued mappings $v, g : [-1, 1] \to R^2$ by

$$v(x) = (x, x^2), \;\; x \in [-1, 1] \quad \textit{and} \quad g(x) = (x, x), \;\; x \in [-1, 1].$$

It is clear that $\mathrm{cl}\mathcal{H} = R^4_+$. We take $e = (1, 1, 1, 1) \in \mathrm{int}\mathcal{H}$. Then, by computing, we have

$$K = [-1, 0] \quad \textit{and} \quad V_{opw} = [-1, 0],$$

where $V_{opw}$ is the solution set of vector optimization. Thus, all conditions of Corollary 3.1 hold. Then, by computing, we have that for any $\bar{x} \in [-1, 0]$,

$$h_1((v(\bar{x}) - v(y), -g(y)), e) \leq 0, \qquad \forall y \in [-1, 1].$$

**Theorem 3.6.** *Let $E$ be a nonempty subset of $X$ and $f : E \times E \to Y$ be a vector-valued mapping. Let $g : E \to Z$ be a vector-valued mapping. For any $e \in \mathrm{int}\mathcal{H}$, if there exists $\bar{x} \in K$ such that*

$$h_1((-f(\bar{x}, y), -g(y)), e) < 0, \qquad \forall y \in E \setminus \{\bar{x}\},$$

*then $\bar{x}$ is a weakly efficient solution to the GKFIC.*

*Proof.* Assume that $\bar{x} \in K$ is not a weakly efficient solution to the GKFIC. Then there exists $\bar{y} \in K$ such that

$$f(\bar{x}, \bar{y}) \in -\mathrm{int}S.$$

Since $\bar{y} \in K$,

$$g(\bar{y}) \in -C.$$

Thus, by the definition of $\mathcal{H}$,

$$(-f(\bar{x}, \bar{y}), -g(\bar{y})) \in \mathcal{H} \subseteq \mathrm{cl}\mathcal{H}.$$

Then, by Lemma 2.2 (ii),

$$h_1((-f(\bar{x}, \bar{y}), -g(\bar{y}), e)) \geq 0.$$

This is a contradiction. Hence, $\bar{x} \in K$ is a weakly efficient solution to the GKFIC. This completes the proof. □

When $f(x, y) = v(y) - v(x)$, $x, y \in E$, we can also obtain the next result for the vector optimization problem with constraints.

**Corollary 3.7.** *Let $E$ be a nonempty subset of $X$ and $v : E \to Y$ be a vector-valued mapping. Let $g : E \to Z$ be a vector-valued mapping. For any $e \in \mathrm{int}\mathcal{H}$, if there exists $\bar{x} \in K$ such that*

$$h((v(\bar{x}) - v(y)), -g(y)), e) < 0, \qquad \forall y \in E \setminus \{\bar{x}\},$$

*then $\bar{x}$ is a weakly efficient solution of the vector optimization problem with constraints.*

Next, we give an example to explain Corollary 3.2.

**Example 3.8.** Considering the Example 3.1. By Example 3.1, we have

$$K = [-1, 0] \quad \textit{and} \quad V_{opw} = [-1, 0].$$

Take $e = (1, 1, 1, 1) \in \mathrm{int}\mathcal{H}$ and $\bar{x} = -1 \in K$. By computing, we have

$$\max \bigcup_{y \in E} h((v(-1) - v(y)), -g(y)), e) = -2 < 0.$$

Then, we have $-1 \in V_{opw}$.

**Theorem 3.9.** *Let $E$ be a nonempty subset of $X$ and $f : E \times E \to Y$ be a vector-valued mapping. Let $g : E \to Z$ be a vector-valued mapping. If $\bar{x} \in K$ is a globally efficient solution to the GKFIC, then for any $e \in \mathrm{int}\mathcal{H}_1$ such that*

$$h_2((-f(\bar{x}, y), -g(y)), e) \leq 0, \qquad \forall y \in E.$$

*Proof.* Let $\bar{x} \in K$ be a globally efficient solution to the GKFIC and $e \in \text{int}\mathcal{H}_1$. Thus, by image space analysis approach,

$$\mathcal{K}_{\bar{x}} \cap (-\mathcal{H}_1) = \emptyset.$$

Then,

$$\mathcal{K}_{\bar{x}} \cap (-\text{int}\mathcal{H}_1) = \emptyset,$$

i.e.,

$$(-f(\bar{x}, y), -g(y)) \notin \text{int}\mathcal{H}_1, \qquad \forall y \in E.$$

By Lemma 2.2 (iv), we have

$$h_2((-f(\bar{x}, y), -g(y)), e) \leq 0, \qquad \forall y \in E.$$

This completes the proof. □

**Theorem 3.10.** *Let $E$ be a nonempty subset of $X$ and $f : E \times E \to Y$ be a vector-valued mapping. Let $g : E \to Z$ be a vector-valued mapping. For any $e \in \text{int}\mathcal{H}_1$, if there exists $\bar{x} \in K$ such that*

$$h_2((-f(\bar{x}, y), -g(y)), e) < 0, \qquad \forall y \in E \setminus \{\bar{x}\},$$

*then $\bar{x}$ is globally efficient solution to the GKFIC.*

*Proof.* Assume that $\bar{x} \in K$ is not a globally efficient solution to the GKFIC. Then there exists $\bar{y} \in K$ such that

$$f(\bar{x}, \bar{y}) \in -H \backslash \{0\}.$$

Since $\bar{y} \in K$,

$$g(\bar{y}) \in -C.$$

Thus, by the definition of $\mathcal{H}_1$,

$$(-f(\bar{x}, \bar{y}), -g(\bar{y})) \in \mathcal{H}_1 \subseteq \text{cl}\mathcal{H}_1.$$

Then, by Lemma 2.2 (ii),

$$h_2((-f(\bar{x}, \bar{y}), e)) \geq 0.$$

This is a contradiction. Hence, $\bar{x} \in K$ is a weakly efficient solution to the GKFIC. This completes the proof. □

**Remark 3.11.** In [7], under some convexity assumptions, the author has investigated the optimality conditions for weakly efficient solution and globally efficient solution to GKFIC by applying the separation theorem of convex sets. However, we also obtain some optimality conditions for weakly efficient solution and globally efficient solution to GKFIC without any convexity assumptions in the sense of image space analysis.

## 4 Concluding Remarks

In this paper, we first established two weak and strong separation functions by virtu of the Gertewitz nonlinear scalarization function. Then, by the separation function, we obtained some optimality conditions for weakly efficient solution and globally efficient solution to the GKFIC in the sense of image space analysis without any convexity assumptions, which are different from existing results in the literatures.

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Y. Zhang
College of Statistics and Mathematics
Yunnan University of Finance and Economics
Kunming 650221, China
E-mail address: zhangyu198606@sina.com

L. Wang
College of Statistics and Mathematics
Yunnan University of Finance and Economics
Kunming 650221, China
E-mail address: WL64mail@aliyun.com

T. Chen
College of Public Foundation
Yunnan Open University
Kunming 650223, China
E-mail address: chentao1cq@126.com