Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 13, No. 3, pp. 385–402, July 2017

THE SHORTEST TIME AND OPTIMAL INVESTMENT CONTROL IN DUOPOLY*

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Abstract: In this paper, we consider an optimal duopoly competition problem in commencing period of horizontal expansion. First, we propose a type of optimal duopoly competition problem including the shortest time and the minimal investment cost to reach a balance for one product from two parties in terms of product occupancy rate. Such problem formulation includes a nonlinear differential-algebraic model with continuous inequality constraints and terminal state constraint. Due to the presence of the continuous state inequality constraints, maybe there are infinite constraints to be satisfied on the time horizon. Thus, it is challenging to obtain a feasible advertising strategy. In order to solve this problem effectively, we apply a constraint transcription technique to covert these constraints into some canonical forms. The converted problem is then solved via the control parameterization method. The essential idea of this method is to approximate the control function with a piecewise constant function in such a way that the optimal control problem is transformed into an optimal parameter selection problem, which can be solved as a nonlinear programming problem. Particularly, in order to optimize the switching time for reducing the investment cost, a time scaling transform is developed. Finally, a beer sales problem is studied to show the effectiveness of the proposed method, and we also compare the investment strategies with applying the time scaling transform and the coarsely equal partition over the time horizon.

Key words: *optimal control, nonlinear differential-algebraic model, control parameterization, time scaling transform, continuous state inequality constraints*

Mathematics Subject Classification: *90-08; 49M37*

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1 Introduction

Advertising is a kind of effective competition and an important strategy for product sale. By advertising, the enterprise can improve the market competition of its products. Hence, research of advertising strategy has attracted much more attention in history. In [2, 3, 5, 24, 27, 34, 35], the optimal dynamic competitive advertising strategy has been investigated with an aim to optimize the profit by using differential game theory. By utilizing this method, one can analytically obtain the open-loop and the closed-loop Nash equilibrium solutions. In [7], a numerical approach is developed to compute the stationary Markov perfect Nash equilibrium for advertising strategies with the Lanchester model. With dynamical analysis, the impact of advertising strategy on the bifurcating and chaos phenomenon in economic system is investigated in [25, 26, 28]. For different types of advertising strategies, researchers in [2, 4, 25, 28] focus on the generic and brand advertising strategies in a duopoly market,

*This work was supported by The National Natural Science Foundation of China under Grant 61273008.

while [11, 29] are on the optimal control problem by considering product price and advertising cost. We also notice that [23] gives correlational studies on the uncertain situation.

In fact, we observe that all above studies focus only on the enterprise's profit in dynamic market. However, in practice, more and more enterprises are required to develop new products for quick potential market occupancy while keeping their market occupancy for their current products. For example, Master Kong holds a dominant position in China's Instant-noodle market, and it also takes a certain percentage in soft drinks market. In this situation, it is important to effectively improve their impact of their new products in the market in a short time by advertising. With such motivation, we will investigate a new type problem in this paper by considering the invest time and the advertising cost as objectives. For simplicity, we assume that there are two enterprises selling one product in the market. One enterprise has occupied a certain amount of market for such product with certain competitiveness and it adopts the percentage of sales strategy [38] in advertise. The other enterprise has zero market occupancy at the beginning as this is a new product for this enterprise and the second enterprise aims to increase its market occupancy percentage in a short time with an aim to share the same market percentage. Previously, a nonlinear differential-algebraic model is developed but with different objectives in [33]. The main differences of this paper and [33] are in two folds. In [33], the time interval is given with an aim to maximize the occupancy of the product for the second enterprise and the time interval here is varying. On the other hand, in [33], its aim is to maximize the occupancy of the product for the second enterprise without constraint on investment cost and time length. In this paper, we put equal market occupancy for the product as a constraint and try to minimize the investment cost as well as invest time. The proposed problem in this paper is much harder than that in [33] mathematically.

The proposed problem in this paper is a nonlinear optimal control problem for a differential-algebraic system with free terminal time. This problem will be transformed into an optimal control problem with fixed terminal time [18]. Then it is reformulated to an equivalent nonlinear optimal control problem with continuous inequality constraints and terminal state constraint, which is difficult if not impossible to obtain an analytical solution. There are some approaches in the literature that we can use to solve this type of converted problem described below, such as the multiple shooting method [30] and the direct collocation method [6]. For the multiple shooting method, an accurate initial guess of the co-state variables is required and the optimal solution is very sensitive to this initial guess. Therefore, the failure of achieving convergence is common when using this method. For the direct collocation method, the computational burden can become enormous for effective implementation for large scale problems. Therefore, we shift our focus to find a numerical solution with control parameterization technique [12–14, 17, 21, 31, 32]. The essential idea of this method is to partition the time interval into a finite number of sub-intervals. Then, we can approximate the control function by a piecewise constant function. The optimal control problem is then transformed into a sequence of optimal parameter selection problems, which can be solved as nonlinear mathematical programing problems. It also needs to be emphasized that the continuous state inequality constraints cannot be handled easily by traditional methods. We should stress a fact that there is infinite number of points in the feasible time domain, leading to possible infinite number of constraints to be satisfied. To overcome this difficulty, a constraint transcription method is applied to convert this constraint into a canonical form [10, 21, 31, 36, 37]. Furthermore, in order to determine the switching time points, we consider two different types of partitions. The first one is to partition the sub-intervals equally. The second partition is more flexible with possible much more benefit to be obtained by applying the so-called 'time scaling transform' [15, 16, 19, 20], with an aim to optimize the switching time points. Finally, a practical beer sales example is used to demonstrate the effectiveness of the proposed method with the equal-partition method and time scaling transform for comparisons.

The rest of the paper is organized as follows. In Section 2 we present a nonlinear differential-algebraic model to describe the commencing period of the oligopoly market and then formulate the problem. In Section 3, some numerical computation methods are proposed to solve the proposed problem. A beer sales example is used to show the effectiveness of the proposed methods in Section 4. Finally, we conclude our paper in Section 5.

2 Problem Formulation

In this section, we consider the duopoly enterprises, i.e. enterprise 1 and enterprise 2 selling the same product. In the initial stage, we assume that only enterprise 2's product is in current market, and it has gained considerable market occupancy. In this case, the enterprise 1 starts to put the same product on market and aims to achieve the same market occupancy in a short time. Also the enterprise 1 is assumed to use the strategy of advertising to obtain the market occupancy. A nonlinear differential-algebraic system based on the Vidale-Wolfe Model is proposed in [33] as follows.

$$\begin{cases} \dot{x}_1(t) = k_1 u(t)(1 - x_1(t)) - \beta_2 v(t) x_1(t) - \eta_1 x_1(t) \\ \dot{x}_2(t) = k_2 v(t)(1 - x_2(t)) - \beta_1 u(t) x_2(t) - \eta_2 x_2(t) \\ 0 = N(\delta_1 x_1(t) + \delta_2 x_2(t))[a - bN(x_1(t) + x_2(t))] - v(t) \\ x_1(0) = 0, x_2(0) = x_{20}, v(0) = v_0 \end{cases} \tag{2.1}$$

For each corresponding enterprise $i, (i = 1, 2)$, $x_i(t)$ represents the occupancy rate of the product, $\beta_i > 0$ represents the advertising competition coefficient, $k_i > 0$ is the influence coefficient of the advertising efficiency, and $\delta_i > 0$ represents the adjusting parameter of the investment intensity. $v(t)$ is the advertising cost of the enterprise 2, which is described by the percentage of sales method [38] and the inverse demand function [1]. a and b are positive constants, and $N > 0$ is the total number of customers in market. $x_{2,0}$ denotes the initial market share of the enterprise 2. v_0 is the initial advertising cost of the enterprise 2. The control variable $u(t) \in \mathbf{U}$ represents the advertising cost of the enterprise 1 and $\mathbf{U}$ denotes the admissible control set $\mathbf{U} = \{u(t) | 0 < u_{\min} \leq u(t) \leq u_{\max}\}$. We should notice that the market share variable $x_i(t)$ satisfies $0 \leq x_i(t) \leq 1$ according to the actual requirement, so $x_1(t) + x_2(t) \geq 0$ would be satisfied automatically.

For more details of this system, one can refer to [33]. With this model, we would study the problems of the optimal time and the lowest advertising cost as described below. The goal is to make the product market occupancy for the enterprise 1 to reach the same level as that for the enterprise 2 in the minimum time interval with the minimum advertising cost, i.e., $x_1(T) = x_2(T)$. We should notice that the customers are allowed to buy the product from both these two enterprises at the same time. In this case, $x_1(T) = x_2(T)$ does not imply that they are equal to 0.5. Also, the terminal time T in the following formulation is a variable and it will satisfy $0 < T < +\infty$. In summary, the objective function is defined as below.

$$\min_{u,T} J(u, T) = \int_0^T u^2(t) \mathrm{d}t + T \tag{2.2}$$

Remark 2.1. In fact, the objective function in (2.2) can be in a weighted form

$$J(u, T) = w_1 \int_0^T u^2(t) \mathrm{d}t + w_2 T$$

where w_1 and w_2 are the weighted coefficients, which can be used to adjust the impacts of corresponding parts on the whole performance index. In this paper, we consider the case of $w_1 = w_2 = 1$. Technically, the solving process is the same when the weighted coefficients are constants different from 1. In Section 4, we will investigate the influence of different weighted coefficients for the system.

As Problem (2.2) for system (2.1) is a nonlinear generalized optimal control problem, to best of our knowledge, there is no existing theoretical solution for such problem. Next we will try to develop a numerical approach to solve it in different situations.

3 Computational Solution

The aim of the optimal control problem mentioned above is to obtain the optimal investment strategy. It is difficult to solve an optimal control problem with nonlinear differential-algebraic system directly by using the optimal control theory in general. In this section, the above optimal control problem with free terminal time is first transformed into an optimal control problem with fixed terminal time [18], and then we apply a constraint transcription method [21, 31] together with the control parameterization technique [12–14, 17, 21, 31, 32] to solve the converted problem. In addition, the optimal control software package, MISER 3.2 [9], which is implemented based on the control parameterization technique, is suitable for this proposed problem. Particularly, for determining the switching points of the piece-wise constant function, we consider two possible partitions: the equal-partition and the partition with time scaling method to obtain two different investment strategies.

To begin with, we first convert system (2.1) into an equivalent nonlinear system as below

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), u(t)) \tag{3.1}$$

where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

$$\mathbf{f}(\mathbf{x}(t), u(t)) = \begin{bmatrix} k_1u(t)(1 - x_1(t)) - \beta_2Nx_1(t)(\delta_1x_1(t) + \delta_2x_2(t))[a - bN(x_1(t) + x_2(t))] - \eta_1x_1(t) \\ k_2N(1 - x_2(t))(\delta_1x_1(t) + \delta_2x_2(t))[a - bN(x_1(t) + x_2(t))] - \beta_1u(t)x_2(t) - \eta_2x_2(t) \end{bmatrix}$$

We should notice that although the differential-algebraic system is transformed into a normal system involving only differential equations, the algebraic equation still puts constraint on the value range of the system. The initial condition is now written as $\mathbf{x}(0) = [0, x_{20}]^T$. There are four continuous inequality constraints for this problem, i.e.,

$$x_1(t) \geq 0, 1 - x_1(t) \geq 0, x_2(t) \geq 0, 1 - x_2(t) \geq 0.$$

which are denoted as $g_j(\mathbf{x}(t)) \geq 0, j = 1, 2, 3, 4$. Then, we can define the Problem (P) as follows

$$\begin{array}{ll} \min\limits_{u,T} J(u,T) = \int_0^T u^2(t)\mathrm{d}t + T & \\ s.t. \quad (3.1) & \\ \quad x_1(T) - x_2(T) = 0 & \\ \quad g_j(\mathbf{x}(t)) \geq 0 \;\; , j = 1, 2, 3, 4 & \\ \quad u(t) \in \mathbf{U} & \end{array} \tag{3.2}$$

Next we will present how to solve Problem (P) in general.

$\boxed{3.1}$ Constraint Transcriptions

In order to handle the continuous inequality constraints $g_j(\mathbf{x}(t)) \geq 0$ in Problem (P), we now introduce a constraint transcription method [21, 31] and transform these constraints, in which there are infinite constraints in nature, into the ones in a canonical form, in which the number of the constraints becomes finite. More specifically, we use

$$\gamma + G_{j,\varepsilon}(u, T) \geq 0 \tag{3.3}$$

to approximate $g_j(\mathbf{x}(t))$, for each $j = 1, 2, 3, 4$, where

$$G_{j,\varepsilon}(u, T) = \int_0^T L_{j,\varepsilon}(\mathbf{x}(t))dt \tag{3.4}$$

$$L_{j,\varepsilon}(\mathbf{x}(t)) = \begin{cases} g_j(\mathbf{x}(t)), & if \quad g_j(\mathbf{x}(t)) < -\varepsilon \\ -(g_j(\mathbf{x}(t)) - \varepsilon)^2/4\varepsilon \geq 0, & if \quad -\varepsilon \leq g_j(\mathbf{x}(t)) \leq \varepsilon \\ 0, & if \quad g_j(\mathbf{x}(t)) > \varepsilon \end{cases} \tag{3.5}$$

By replacing $g_j(\mathbf{x}(t)) \geq 0$ with (3.3) in Problem (P), we can formulate a related approximated problem based on Problem (P), referred to as Problem $(P^{\varepsilon,\gamma,T})$ in this paper. Since the terminal time in Problem $(P^{\varepsilon,\gamma,T})$ is also free, we now need to transform the problem into an equivalent problem with a fixed new time interval $[0, 1]$ by introducing a new variable z as stated in next section with technique reported in [8, 18], so that the problem can be transformed into an optimal control problem with fixed terminal time. In fact, z is the terminal time, which can be optimized in the following equivalent problem, which is denoted as Problem $(P^{\varepsilon,\gamma})$

$$\begin{array}{ll} \min\limits_{u,z} J(u, z) = z \int_0^1 \left(1 + u^2(s)\right) \mathrm{d}s & \\ s.t. \quad \dot{\mathbf{x}}(s) = z\mathbf{f}(\mathbf{x}(s), u(s)) & \\ \qquad x_1(1) - x_2(1) \ = 0 & \\ \qquad \gamma + G_{j,\varepsilon}(u(s)) \geq 0 \ \ , j = 1, 2, 3, 4 & \\ \qquad \mathbf{x}(0) = [0, x_{2,0}]^T & \\ \qquad u(s) \in \ \mathbf{U} & \end{array} \tag{3.6}$$

Problem $(P^{\varepsilon,\gamma})$ is in a canonical form with a fixed terminal time, which can be solved by applying the control parameterization method, and this will be investigated in next section.

Next, we will explore the relation between the optimal solution of Problem (P) and the optimal solution of Problem $(P^{\varepsilon,\gamma})$. The following lemma illustrates the relationship of feasible controls for Problem $(P^{\varepsilon,\gamma})$ and Problem (P), which was reported in [31].

Lemma 3.1. *There exists a $\gamma(\varepsilon) > 0$ such that for all $\gamma, 0 < \gamma < \gamma(\varepsilon)$, any feasible control $u_{\varepsilon,\gamma}(t)$ for Problem $(P^{\varepsilon,\gamma})$ is also a feasible control for Problem (P).*

Before we formally present the algorithm to solve Problem (P), we assume that Problem $(P^{\varepsilon,\gamma})$ can be solved numerically with the optimal solution $u^*_{\varepsilon,\gamma}(t)$, and in this optimal case, the continuous inequality constraints $g_j(\mathbf{x}(t)) \geq 0$ should become $g_j(\mathbf{x}(t|u^*_{\varepsilon,\gamma}(t))) \geq 0$. The algorithm for solving Problem $(P^{\varepsilon,\gamma})$ can be stated as follows.

Algorithm 1. Choose $\varepsilon^0 > 0$, and $\gamma^0 > 0$.
Step 1. Solve Problem $(P^{\varepsilon,\gamma})$ to obtain the optimal solution $u^*_{\varepsilon,\gamma}(t)$.
Step 2. Check the feasibility of $g_j(\mathbf{x}(t|u^*_{\varepsilon,\gamma}(t))) \geq 0$ for all $t \in [0, 1]$ and all $j = 1, 2, 3, 4$.

Step 3. If $u^*_{\varepsilon,\gamma}(t)$ is infeasible, go to Step 4. If $u^*_{\varepsilon,\gamma}(t)$ is a feasible control and ε meets the tolerance requirement, then stop. Otherwise go to Step 5.
Step 4. Set $\gamma=\frac{\gamma}{2}$ and go to Step 1.
Step 5. Set $\varepsilon=\frac{\varepsilon}{10}$, $\gamma=\frac{\gamma}{10}$ and go to Step 1.

The convergence of Algorithm 1 is guaranteed by the following results reported in [31].

Lemma 3.2. *Let $u^*_{\varepsilon,\gamma}(t)$ be the optimal control produced by Algorithm 1, then,*

$$J(u^*_{\varepsilon,\gamma}(t)) \to J(u^*(t)) \tag{3.7}$$

as $\varepsilon \to 0$, where $u^*(t)$ is an optimal control of Problem (P), which is obtained by using control parameterization method.

Remark 3.3. According to Lemma 3.1 and Lemma 3.2, we can see that, in Algorithm 1, γ is used to ensure the feasibility of $u^*_{\varepsilon,\gamma}(t)$, while ε is related to the accuracy of $u^*_{\varepsilon,\gamma}(t)$. In summary, we can achieve an optimal solution of Problem (P) by adjusting γ and ε in Algorithm 1 if Problem $(P^{\varepsilon,\gamma})$ has an optimal solution.

Remark 3.4. Lemma 3.1, Lemma 3.2 and Algorithm 1 are the results for optimal control problem with fixed terminal time. But so far, the optimal control problem in this paper is a free terminal time problem. We will transform such optimal control problem into one optimal control problem with a fix terminal time as shown in next section.

3.2 Control Parameterization

In this section, we will show how to solve Problem $(P^{\varepsilon,\gamma})$ numerically via the control parameterization method. The essential idea of this method is to parameterize the control function by partioning the time interval into a group of sub-intervals. Here, we consider two different types of partitions. One is the simple equal-partition, and the other is a time scaling transform method.

3.2.1 Equal-partition

We introduce a monotonically non-decreasing sequence $\{\tau_0, \tau_1, \ldots, \tau_p\}$, which carries out on an equal-partition of the time interval $[0,1]$ with $0 = \tau_0 < \tau_1 < \cdots < \tau_p = 1$. With such partition, the control function $u(s)$ is approximated by a piecewise constant function as follows

$$u_p(s) = \sum_{k=1}^{p} \sigma_k \chi_{[\tau_{k-1},\tau_k]}(s) \tag{3.8}$$

where $\sigma_k \in \mathbf{U}$, χ_I is the indicator function of I defined by

$$\chi_I = \begin{cases} 1, & s \in I \\ 0, & otherwise \end{cases}$$

Accordingly, we can denote the system (3.1) as

$$\dot{\mathbf{x}}(s) = z\bar{\mathbf{f}}(\mathbf{x}(s), \sigma^p) \tag{3.9}$$

where

$$\bar{\mathbf{f}}(\mathbf{x}(s), \sigma^p) = \mathbf{f}(\mathbf{x}(s), \sum_{k=1}^{p} \sigma_k \chi_{[\tau_{k-1},\tau_k]}(s))$$

$$\sigma^p = [\ \sigma_1, \quad \sigma_2, \quad \cdots \quad , \sigma_p\ ]^T$$

and the initial condition remains the same as $\mathbf{x}(0)$. Now let $\mathbf{x}_{\sigma^p}(s)$ be the solution of the system (3.9) with $u_p(s) = \sum_{k=1}^{p} \sigma_k \chi_{[\tau_{k-1},\tau_k]}(s)$ and define

$$\bar{G}_{j,\varepsilon}(\sigma^p, z) = z \int_0^1 L_{j,\varepsilon}(\mathbf{x}_{\sigma^p}(s)) ds$$

By using these notations and transformations, one can change (3.6) into

$$\begin{cases} \dot{\mathbf{x}}(s) = z\bar{\mathbf{f}}(\mathbf{x}(s), \sigma^p) \\ \mathbf{x}(0) = [0, x_{20}]^T \\ x_1(1) - x_2(1) = 0 \\ \gamma + \bar{G}_{j,\varepsilon}(\sigma^p, z) \geq 0 \\ \sigma^p \in \mathbf{U}, s \in [0,1] \end{cases} \tag{3.10}$$

Then the optimal control problem can be redefined as follows

$$(\bar{P}^{\varepsilon,\gamma}(p)): \qquad \min_{\sigma^p, z} \bar{J}(\sigma^p, z) = z \int_0^1 \left(1 + (u_p(s))^2\right) \mathrm{d}s$$
$$s.t. \qquad (3.10)$$

Problem $(\bar{P}^{\varepsilon,\gamma}(p))$ is in fact a sequence of optimal parameter selection problems in canonical form. To solve Problem $(\bar{P}^{\varepsilon,\gamma}(p))$ as a nonlinear optimization problem, the gradient formulas of the objective function and the constraint functions need to be derived. Next we will provide these gradient formulas in the following theorem and the proof is omitted here due to its similarity to that of Theorem 5.2.1 in [31].

Theorem 3.5. *The gradients of the objective function $\bar{J}(\sigma^p, z)$ and corresponding inequality continuous constraints with respect to σ^p are as follows.*

$$\frac{\partial \bar{J}(\sigma^p, z)}{\partial \sigma^p} = \int_0^1 \frac{\partial \bar{H}_0(\mathbf{x}(s), \sigma^p, \bar{\lambda}_0(s), z)}{\partial \sigma^p} \mathrm{d}s$$

$$\frac{\partial \bar{J}(\sigma^p, z)}{\partial z} = \int_0^1 \frac{\partial \bar{H}_0(\mathbf{x}(s), \sigma^p, \bar{\lambda}_0(s), z)}{\partial z} \mathrm{d}s$$

$$\frac{\partial \bar{G}_{j,\varepsilon}(\sigma^p, z)}{\partial \sigma^p} = \int_0^1 \frac{\partial \bar{H}_j(\mathbf{x}(s), \sigma^p, \bar{\lambda}_j(s), z)}{\partial \sigma^p} \mathrm{d}s$$

$$\frac{\partial \bar{G}_{j,\varepsilon}(\sigma^p, z)}{\partial z} = \int_0^1 \frac{\partial \bar{H}_j(\mathbf{x}(s), \sigma^p, \bar{\lambda}_j(s), z)}{\partial z} \mathrm{d}s$$

where $\bar{H}_0$ is the Hamiltonian for the objective function, $\bar{H}_j, j = 1, .., 4$ is the Hamiltonian for the canonical constraints,

$$\bar{H}_0(\mathbf{x}(s), \sigma^p, \bar{\lambda}_0(s), z) = z(1 + (\sigma^p)^2 + \bar{\lambda}_0^T(s)\bar{\mathbf{f}}(\mathbf{x}(s), \sigma^p))$$

$$\bar{H}_j(\mathbf{x}(s), \sigma^p, \bar{\lambda}_j(s), z) = L_{j,\varepsilon}(\mathbf{x}(s)) + z\bar{\lambda}_j^T(s)\bar{\mathbf{f}}(\mathbf{x}(s), \sigma^p)$$

$\bar{\lambda}_0(s)$ and $\bar{\lambda}_j(s)$ are the solutions of the following co-state differential equations, respectively

$$(\dot{\bar{\lambda}}_0(s))^T = -\frac{\partial \bar{H}_0(\mathbf{x}(s), \sigma^p, \bar{\lambda}_0(s), z)}{\partial \mathbf{x}(s)}$$

$$(\dot{\bar{\lambda}}_j(s))^T = -\frac{\partial \bar{H}_j(\mathbf{x}(s), \sigma^p, \bar{\lambda}_j(s), z)}{\partial \mathbf{x}(s)}$$

with the boundary conditions

$$(\bar{\lambda}_0(1))^T = [0, 0]$$

$$(\bar{\lambda}_j(1))^T = [0, 0], j = 1, 2, 3, 4.$$

It should be reminded that Problem $(\bar{P}^{\varepsilon,\gamma}(p))$ can be solved easily with Theorem 3.5. Next we will discuss another flexible partition, which usually has more benefits as illustrated in experimental simulations.

$\boxed{3.2.2}$ Time Scaling Transform

In this section, we focus on the case in which the switching times can be varied. We investigate the problem of choosing optimal values for the switching times so that the switched system under consideration operates in the best possible manner. This is a switching time optimization problem [19]. In this section, we can obtain the optimal time intervals of advertising investment by switching time optimization. The time intervals are not equal-partition any more so that the advertising strategy can be better and more efficient, in the sense that the enterprise can predict when to invest on advertising can achieve better result.

In order to optimize the switching time points, we introduce a time scaling transform method as reported in [15, 19, 20]. For such purpose, we introduce a monotonically non-decreasing sequence $\{\tau_0, \tau_1, \ldots, \tau_p\}$ which satisfies $0 = \tau_0 < \tau_1 < \cdots < \tau_p = T$. Here, the switching time points $\tau_k, 1 \leq k \leq p-1$, are regarded as decision variables. Then, we map these switching time points into fixed time points $k/p, k = 1, 2, \ldots, p-1$, are regarded as decision variables. Then, we map these switching time points into fixed time points $k/p, k = 1, 2, \ldots, p-1$ in a new time horizon $[0, 1]$. This process is achieved by introducing the following differential equation

$$\begin{cases} \dot{t}(s) = v^p(s), s \in [0, 1] \\ t(0) = 0 \end{cases} \tag{3.11}$$

where

$$v^p(s) = \sum_{k=1}^{p} \theta_k \chi_{[\frac{k-1}{p}, \frac{k}{p})}(s) \tag{3.12}$$

with $\theta_k \geq 0, k = 1, 2, \ldots, p$. We denote $\theta^p = [\theta_1, \theta_2, \ldots, \theta_p]^T$ and let Θ be the set containing all possible θ^p. From (3.11) and (3.12), it follows that, for $k = 1, 2, \ldots, p-1$,

$$\tau_k = \sum_{i=1}^{k} \frac{\theta_i}{p} \tag{3.13}$$

and

$$t(1) = \sum_{i=1}^{p} \frac{\theta_i}{p} = T \tag{3.14}$$

In this case, system (3.1) can be rewritten in a new time horizon as follows

$$\frac{d\mathbf{x}(t)}{dt}\frac{dt(s)}{ds} = \dot{t}(s)\mathbf{f}(\mathbf{x}(t(s)), u(t(s))) = v^p(s)\mathbf{f}(\mathbf{x}(t(s)), u(t(s))) \tag{3.15}$$

In this case, the continuous state inequality constraints and the terminal constraint in the new time domain can be rewritten as

$$x_1(t(s)) \geq 0, 1 - x_1(t(s)) \geq 0, x_2(t(s)) \geq 0, 1 - x_2(t(s)) \geq 0 \tag{3.16}$$

$$x_1(1) - x_2(1) = 0 \tag{3.17}$$

Let $u(t(s)) = \sum_{k=1}^{p} \sigma_k \chi_{[\tau_{k-1},\tau_k]}(s)$, where σ_k is defined as in (3.8). Also define

$$\tilde{\mathbf{f}}(\mathbf{x}(s), \sigma^p, \theta^p) = \mathbf{f}(\mathbf{x}(s), \sum_{k=1}^{p} \sigma_k \chi_{[\tau_{k-1},\tau_k]}(s)) \tag{3.18}$$

If we denote $\mathbf{x}_{\sigma^p,\theta^p}(s)$ as the solution of system (3.18), and define

$$\tilde{G}_{j,\varepsilon}(\sigma^p, \theta^p) = \int_0^1 v^p(s) L_{j,\varepsilon}(\mathbf{x}_{\sigma^p,\theta^p}(t(s)))ds$$

then, we can formulate an optimal parameter selection problem in a canonical form in the same manner as we defined Problem $(\bar{P}^{\varepsilon,\gamma}(p))$, which is referred to as Problem $(\tilde{P}^{\varepsilon,\gamma}(p))$ here and is defined as follows

$$\begin{array}{ll}
\min\limits_{\sigma^p,\theta^p} & \tilde{J}(\sigma^p, \theta^p) = \int_0^1 v^p(s) \left(1 + (u_p(s))^2\right) \mathrm{d}s \\
s.t. & \dot{\mathbf{x}}(s) = v^p(s)\tilde{\mathbf{f}}(\mathbf{x}(s), \sigma^p, \theta^p) \\
& \dot{t}(s) = v^p(s) \\
& \mathbf{x}(0) = [0, x_{20}]^T \\
& x_1(1) - x_2(1) = 0 \\
& \gamma + \tilde{G}_{j,\varepsilon}(\sigma^p, \theta^p) \geq 0 \\
& t(1) = T \\
& \sigma_k \in \mathbf{U}, k = 1, 2, \ldots, p
\end{array} \tag{3.19}$$

The corresponding gradient formulas for this objective function and the constraint functions are given as below.

Theorem 3.6. *The gradients of the objective function $\tilde{J}(\sigma^p, \theta^p)$ and corresponding inequality continuous constraints with respect to σ^p and θ^p are*

$$\frac{\partial \tilde{J}(\sigma^p, \theta^p)}{\partial \sigma^p} = \int_0^1 \frac{\partial \tilde{H}_0(\mathbf{x}(t(s)), \sigma^p, \theta^p, \tilde{\lambda}_0(t(s)))}{\partial \sigma^p} \mathrm{d}s$$

$$\frac{\partial \tilde{J}(\sigma^p, \theta^p)}{\partial \theta^p} = \int_0^1 \frac{\partial \tilde{H}_0(\mathbf{x}(t(s)), \sigma^p, \theta^p, \tilde{\lambda}_0(t(s)))}{\partial \theta^p} \mathrm{d}s$$

$$\frac{\partial \tilde{G}_{j,\varepsilon}(\sigma^p, \theta^p)}{\partial \sigma^p} = \int_0^1 \frac{\partial \tilde{H}_j(\mathbf{x}(t(s)), \sigma^p, \theta^p, \tilde{\lambda}_j(t(s)))}{\partial \sigma^p} \mathrm{d}s$$

$$\frac{\partial \tilde{G}_{j,\varepsilon}(\sigma^p, \theta^p)}{\partial \theta^p} = \int_0^1 \frac{\partial \tilde{H}_j(\mathbf{x}(t(s)), \sigma^p, \theta^p, \tilde{\lambda}_j(t(s)))}{\partial \theta^p} \mathrm{d}s$$

where $\tilde{H}_0$ is the Hamiltonian for the objective function, $\tilde{H}_j, j = 1, .., 4$ is the Hamiltonian for the canonical constraints,

$$\tilde{H}_0(\mathbf{x}(t(s)), \sigma^p, \theta^p, \tilde{\lambda}_0(t(s))) = \sum_{k=1}^{p} \frac{\theta_k}{p}(1 + (\sigma_k)^2) + \tilde{\lambda}_0^T(t(s)) v^p(s) \tilde{f}(\mathbf{x}(t(s)), \sigma^p)$$

$$\tilde{H}_j(\mathbf{x}(t(s)),\sigma^p,\theta^p,\tilde{\lambda}_j(t(s))) = L_{j,\varepsilon}(\mathbf{x}(t(s))) + \tilde{\lambda}_j^T(t(s))v^p(s)\tilde{f}(\mathbf{x}(t(s)),\sigma^p)$$

and $\tilde{\lambda}_0(t(s))$ *and* $\tilde{\lambda}_j(t(s))$ *are the solution of the following co-state differential equations, respectively*

$$(\dot{\tilde{\lambda}}_0(t(s)))^T = -\frac{\partial \tilde{H}_0(\mathbf{x}(t(s)),\sigma^p,\theta^p,\tilde{\lambda}_0(t(s)))}{\partial \mathbf{x}(t(s))}$$

$$(\dot{\tilde{\lambda}}_j(t(s)))^T = -\frac{\partial \tilde{H}_j(\mathbf{x}(t(s)),\sigma^p,\theta^p,\tilde{\lambda}_j(t(s)))}{\partial \mathbf{x}(t(s))}$$

with the boundary conditions

$$(\tilde{\lambda}_0(1))^T = [0,0]$$

$$(\tilde{\lambda}_j(1))^T = [0,0], j = 1,2,3,4.$$

With Theorem 3.5 and Theorem 3.6, Problem $(\bar{P}^{\varepsilon,\gamma}(p))$ and Problem $(\tilde{P}^{\varepsilon,\gamma}(p))$ can be solved easily by the optimal control software package MISER 3.2 [9].

4 Illustrative Example

In this section, we will use a beer sales dynamic system to verify the validity and practicability of the obtained results in this paper. As we know that all kinds of beer advertisements are a primary means to increase beer sales, and would have impact on consumers' consumption preferences. Moreover, high advertisement investment usually can achieve higher sales. First, we use the data from [22] for beer advertisement and market share in America market during 1993 – 2003. In detail, all the parameters related to the second enterprise in the proposed model are from [22] and the following parameters are used: $k_1 = 0.5$, $k_2 = 0.5$, $\beta_1 = 0.5$, $\beta_2 = 0.7$, $\eta_1 = 0.1$, $\eta_2 = 0.2$, $\delta_1 = 15$, $\delta_2 = 10$, $a = 1.2\times 10^{-5}$, $b = 10^{-10}$, $N = 10^5$. The proposed system model is shown as below.

$$\begin{cases} \dot{x}_1(t) = 0.5u(t)(1-x_1(t)) - 0.7v(t)x_1(t) - 0.1x_1(t) \\ \dot{x}_2(t) = 0.5v(t)(1-x_2(t)) - 0.5u(t)x_2(t) - 0.2x_2(t) \\ 0 = (15x_1(t)+10x_2(t))[1.2-(x_1(t)+x_2(t))] - v(t) \end{cases} \tag{4.1}$$

where the initial conditions are $x_1(0) = 0$, $x_2(0) = 0.6$ and $v(0) = 3.6$. We now solve Problem (P) subject to (4.1). And the parameters ε^0 and γ^0 in Algorithm 1 are chosen to be 1.0000e − 02 and 2.5000e − 03 respectively. With Algorithm 1, their final values are obtained as $\varepsilon = 1.0000\text{e}-04$ and $\gamma = 2.5000\text{e}-05$. The implementation is carried out by using the optimal control software MISER 3.2 [9]. Next we consider Problem (P) in following two cases.

4.1 The equal-partition case

We adopt the method mentioned in Section 3.1 to solve the problem (4.1) and set $u \in [0,10]$ and $p = 10$. After transforming into an equivalent form shown in (3.10), the optimal states and optimal control are shown in Fig.1, and the corresponding function value is 9.3827 and the optimal terminal time (i.e. the value of the parameter z) is 0.5431.

We observe that the objective function value is decreasing as the upper bound of the control $u_{\max}$ is decreasing, and the results with different $u_{\max}$ are shown in Table 1. The cost functions are the same when $u_{\max}$ is larger than 7.4462. More importantly, we can imagine that there should have a critical value $u_{\max}$ such that these two market occupancies never reach a balance with any admissible control (if investment is not sufficient). In order

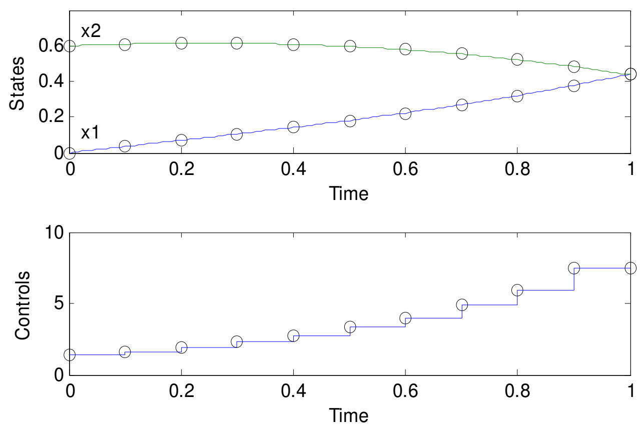

Figure 1: The solutions of problem (4.1) with $u \in [0, 10]$

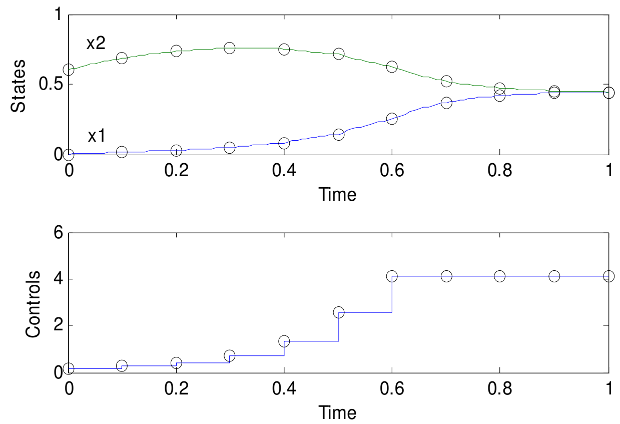

Figure 2: The results of problem (4.1) with $u \in [0, 4.1]$

to find such critical value, we let the upper bound value decrease to a certain degree, such as $u_{\max} = 4.1$. In this case, the optimal $\bar{J} = 16.3109$ and state changes in this are shown in Fig.2. If we reduce $u_{\max}$ slightly further to 4, we can see the result in Fig.3 showing that the market occupancies never reach a balance. In fact, people are more interested on the smallest upper bound of the advertising investment to make both market occupancies equal. Next we will show that one can improve such upper bound via the time scaling approach.

$\boxed{4.2}$ The time scaling partition

Now we use the time scaling transform to solve problem (3.19). By setting $u \in [0, 10]$ and $p = 10$, we can obtain the optimal function value is 9.3704 and the corresponding optimal states and optimal control are shown in Fig.4. In comparison with the results in Fig.1, it is easy to observe that the results with time scaling are better than equal-partition method for

Table 1: The results with different $u_{\max}$

$u_{\max}$	10	9	8	7	6	5
J	9.3827	9.3827	9.3827	9.3978	9.5715	10.3602

the same problem in terms of reducing the objective function value. Notice that each switch time is derived rather than predetermined, and this shows that the time scaling method can bring much more benefit for enterprise.

In order to compare the performance of the two partition methods in more detail, we calculate the function values of equal-partition and that obtained from time scaling transform with different number of partitions in Table 2. From Table 2, we can see the improvement by applying the time scaling transform. We also obverse that, as expected, the more the number of partition is, the more effective the time scaling transform becomes. Clearly, for the cost function, we consider two factors, i.e., the investment time and the investment cost, so the comparison results in Table 2 show that the time scaling method is better at the expense of investment time, in terms of less advertising cost. And the notation ΔJ is equal to $\bar{J} - \tilde{J}$ in Table 2.

Furthermore, the function value would approach to the optimal value when we increase value p according to [31], and they will approach to constant values. The results are shown in Table 3.

Last, we investigate the effect of weighted coefficients in the original control problem, i.e., $J(u(t)) = w_1 \int_0^T u^2(t)\mathrm{d}t + w_2 T$, where w_1 and w_2 represent the corresponding weights respectively. Now we consider two situations, i.e., (1) $w_1 = 0.3, w_2 = 0.7$ and (2) $w_1 = 0.7, w_2 = 0.3$, and all other conditions are unchanged. The results are shown in Fig.5 and Fig.6 respectively. We should notice that in order to show these results clearly, the numeric value in time coordinates is magnified 10 times. And now we can compare the results of these two situations in Table 4. First, there is an obvious difference between these cost functions in these two cases. If $w_1 < w_2$, this implies that the enterprise 2 pay more attention to the investment time. Oppositely, if $w_1 > w_2$, the enterprise 2 focuses on saving advertising cost. So by adjusting the weight parameters w_1 and w_2, different advertising policies can be provided to the enterprise.

5 Conclusions

In this paper, a duopoly competition system between two enterprises is investigated and it can be formulated as a nonlinear differential-algebraic system with inequality continuous constraints and free time terminal. In order to solve this problem effectively, we transfer the free terminal time problem into a fixed terminal time problem. In addition, the continuous inequality constraints are transformed into the constraints in a canonical form by applying the constraint transcription technique. Hence, we can solve the original optimal control problem by solving an approximated problem in a canonical form, and the corresponding convergence results are also provided. Finally, the approximated problem can be solved by control parameterization technique via two partitioning techniques: the equal-partition and time scaling method so that two different investment strategies can be obtained. Particularly, the required gradients for the objective function and the constraint functions are derived. In the simulation, we illustrated the effectiveness of the proposed methods. By comparing these two partitioning methods, we find that the time scaling method is superior to the

Table 2: Comparing two methods

		Equal-partition	
p	$\bar{J}$	Terminal time	Advertising investment
2	9.8980	0.3907	9.5073
4	9.5299	0.4809	9.0490
6	9.4381	0.5140	8.9241
8	9.4002	0.5327	8.8675
10	9.3827	0.5431	8.8396
12	9.3695	0.5479	8.8216
		Time scaling	
p	$\tilde{J}$	Terminal time	Advertising investment
2	9.8075	0.4280	9.3795
4	9.4727	0.5159	8.9568
6	9.4073	0.5396	8.8677
8	9.3815	0.5490	8.8325
10	9.3704	0.5538	8.8166
12	9.3608	0.5549	8.8059
		Comparison results	
p	ΔJ	Terminal time	Advertising investment
2	0.0905	−0.0373	0.1278
4	0.0573	−0.0350	0.0922
6	0.0308	−0.0256	0.0564
8	0.0187	−0.0163	0.0350
10	0.0123	−0.0107	0.0230
12	0.0087	−0.0070	0.0157

Table 3: The results of different segments by time scaling

p	$\tilde{J}$	Terminal time	Advertising investment
2	9.8075	0.4280	9.3795
4	9.4727	0.5159	8.9568
6	9.4073	0.5396	8.8677
8	9.3815	0.5490	8.8325
10	9.3704	0.5538	8.8166
12	9.3608	0.5549	8.8059
14	9.3571	0.5565	8.8006
16	9.3584	0.5573	8.8011
18	9.3641	0.5600	8.8041
20	9.3556	0.5588	8.7968

Table 4: The results of different weights by time scaling

	$w_1 = 0.3, w_2 = 0.7$	$w_1 = 0.7, w_2 = 0.3$
$\tilde{J}$	3.0005	6.3083
Terminal time	9.0222	8.6928
Advertising investment	0.4198	0.7447

equal-partition method in terms of reducing the investment cost. But the equal-partition investment strategy takes less time.

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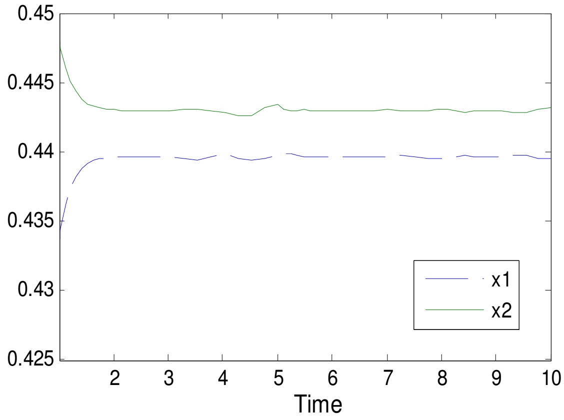

Figure 3: The results of problem (4.1) with $u_{\max} = 4$

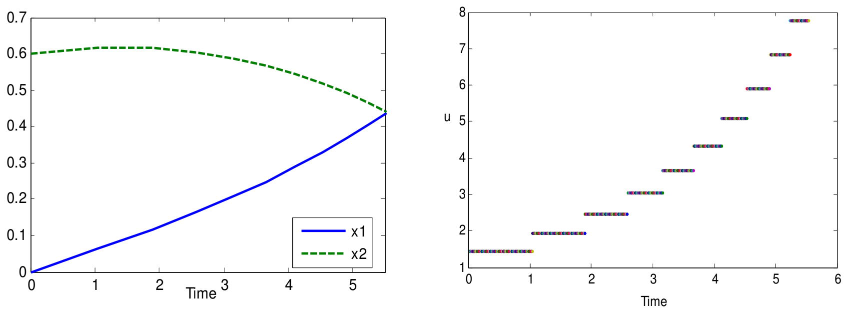

Figure 4: The results of problem (4.1) by using time scaling method

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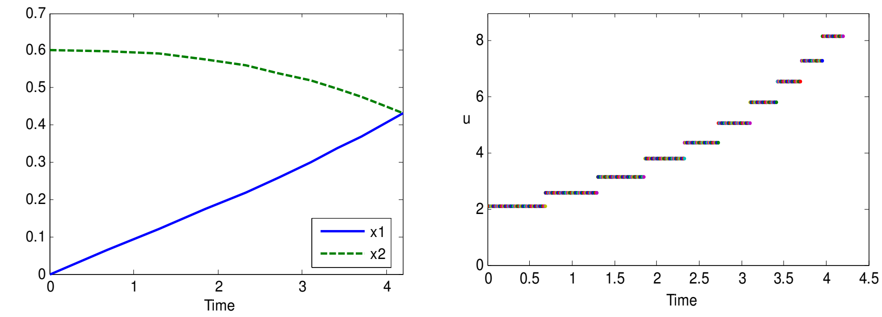

Figure 5: The situation of $w_1 = 0.3, w_2 = 0.7$

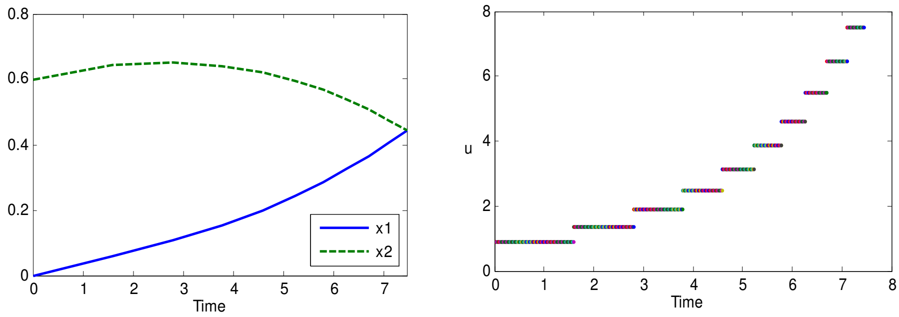

Figure 6: The situation of $w_1 = 0.7, w_2 = 0.3$

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Manuscript received 2 December 2014
revised 6 March 2015
accepted for publication 1 October 2015

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