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LAGRANGE MULTIPLIER RULE FOR NONCONVEX SET-VALUED OPTIMIZATION PROBLEM*

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Abstract: In this paper, in terms of circatangent epiderivative, a new kind of weak subdifferential for set-valued mappings is introduced. An existence theorem of this kind of weak subgradient is established. In terms of Lagrange multiplier, necessary and sufficient optimality conditions are established for set-valued optimization problems with constraint sets being determined by a geometric set and a cone-convex set-valued mapping. In particular, the necessary optimality conditions do not require any convexity of the objective mapping and the geometric set.

Key words: *weak subgradient, set-valued mapping, existence, optimality condition*

Mathematics Subject Classification: *90C25, 49J53, 49K30*

1 Introduction

Rockafellar [15] first introduced the concept of subdifferential for convex functions in finite dimensional spaces. Many researchers studied subdifferential because it plays an important role in optimization and nonsmooth analysis. Today, the literature on subdifferential is vast. For more details about subdifferential, we refer the reader to [6, 11, 13, 14] and the references therein. Since set-valued mappings appear naturally in many branches of pure and applied mathematics, as pointed out by Jahn [8], set-valued optimization becomes a bridge between different areas, such as fuzzy programming, stochastic programming and optimal control in optimization. In recent years, many researchers [1-3, 5, 8, 16, 17, 20] generalized the concept of subdifferential to vector-valued or set-valued mappings and studied them. Yang [21] generalized the concept of weak subdifferential defined by Chen and Craven [4] from vector-valued mappings to set-valued mappings. He also obtained an existence theorem of the weak subgradient, and a weak Lagrange multiplier theorem for a set-valued optimization problem with the objective being a cone-convex set-valued mapping and the constraint being a convex set. Chen and Jahn [5] introduced another weak subdifferential for set-valued mappings, which is a subset of the Yang-weak subdifferential [21] at the same point. They also gave some existence results of the weak subgradient by the Eidelheit's separation theorem, and obtained a sufficient optimality condition for a set-valued optimization problem. Peng et al. [12] obtained some existence theorems of the Yang-weak subgradient and the Borwein-strong subgradient [3], by using the Hahn-Banach extension theorem of set-valued

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mappings. They also presented a Lagrange multiplier theorem for a set-valued optimization problem in terms of the Borwein-strong subgradient [3] with the multiple of the objective and the inverse of the constrained set-valued mapping being a cone-convex set-valued mapping. Song [16] introduced a weak subdifferential in terms of contingent epiderivative, and presented some property of this weak subdifferential. Li and Guo [9] presented some existence theorems of the Yang-weak subgradient and the Chen-Jahn-weak subgradient for set-valued mappings by the Hahn-Banach extension theorem given by Zălinescu [22]. In terms of the Chen-Jahn-weak subgradient, they also obtained a Sandwich theorem for cone-convex set-valued mappings, and obtain a formula for the subdifferentials of the sum of two cone-convex set-valued mappings. Finally, they discussed Lagrange rules for set-valued optimization problems in terms of the Yang-weak subdifferential and the Chen-Jahn-weak subdifferential, respectively, in which the multiple of the objective and the inverse of the constrained set-valued mapping is a cone-convex set-valued mapping. Hernández and Rodríguez-Marín [7] obtained some existence theorems of the Chen-Jahn subgradient and strong subgradient introduced by themselves, respectively. They also established several optimality conditions for set-valued optimization problems. Taa [19] established a formula for the Yang-weak subdifferential of the sum of two cone-convex set-valued mappings under the condition that the domain of one set-valued mapping and the interior of the domain of another have nonempty intersection. Using this formula, he obtained a Lagrange multiplier rule for a set-valued optimization with the objective and the constraint set-valued mappings being cone-subconvexlike. Taa [18] established a formula for the Yang-weak subdifferential of the sum of two cone-convex set-valued mappings under the Attouch-Brezis qualification condition which is weaker than [19]. Using this formula and the Attouch-Brezis qualification condition, he also established a Lagrange multiplier theorem for a set-valued optimization problem with the objective and the constraint set-valued mappings being cone-convex. Long, Peng and Li [10] obtained two existence theorems of the Chen-Jahn subgradient in terms of the contingent derivatives. They also discussed some properties of the Chen-Jahn subdifferential. All Lagrange multiplier theorems mentioned above require cone-convexity or nearly cone subconvexlike property of the considered set-valued mappings since these theorems are obtained by applying the separation theorem for convex sets in essence. However, in some nonconvex set-valued optimization problems, the Yang-weak subdifferential, Chen-Jahn subdifferential, Borwein-strong subdifferential and Song-weak subdifferential of the considered set-valued mappings are maybe empty. For example, all kinds of subdifferentials mentioned above of F at $(0,0)$ are empty, where $F(x) = [-|x|, +\infty)$ for all $x \in \mathbb{R}$. In order to avoid this drawback in dealing with the optimality conditions of nonconvex set-valued optimization problems, we introduce a new kind of weak subdifferential for set-valued mappings by the circatangent epiderivatives (Clarke epiderivative). In the special case when the set-valued mapping is cone-convex, this subdifferential is equivalent to the Yang-weak subdifferential and the Song-weak subdifferential. Then using this subdifferential, we established a Lagrange multiplier rule for set-valued optimization problems which covers more class than [9] and [18].

This paper is organized as follows. In Section 2, we recall some notions which will be needed in the sequel. Section 3 and 4 are the main contribution of this paper. In section 3, we introduce a new subdifferential, and present an existence theorem of this weak subgradient by applying the Eidelheit's separation theorem and the Closed Graph Theorem. In section 4, we derive a Lagrange multiplier rule as necessary and sufficient optimality condition for a constrained optimization problem. Unlike [9] and [18], Theorem 4.4 and Corollary 4.5 do not require any convexity of the objective set-valued mapping and the geometric set. Moreover, our Lagrange multiplier rule is given by the weak subdifferential of all set-valued mappings in the constrained optimization problem, which is consistent with the classic Lagrange multiplier rule for convex optimization problems. See Remark 4.8.

2 Preliminaries

Throughout this paper, we let X, Y and Z be Banach spaces, and let X^*, Y^* and Z^* be the topological duals of X, Y and Z, respectively. Let $\mathbf{B}(X,Y)$ denote the set of all linear continuous operators from X into Y. Let B_X and B_Y denote the unit open ball of X and Y, respectively. Let A be a subset of Y. We denote the interior and the closure of A by $\operatorname{int}(A)$ and $\operatorname{cl}(A)$, respectively. Let $C \subset Y$ be a closed pointed cone with $\operatorname{int}(C) \neq \emptyset$. We say that A is upper bounded if there exists $b \in Y$ such that $A \subset b - C$. Let C^* denote the dual cone of C, that is,

$$C^* = \{y^* \in Y^* : \langle y^*, c\rangle \geq 0, \ \ \forall\, c \in C\}.$$

We let $\operatorname{WMin}(A)$ denote the set of weak efficient points of A, that is,

$$\operatorname{WMin}(A) = \{a \in A : (A - a) \cap (-\operatorname{int}(C)) = \emptyset\}.$$

Let $a \in A$. Let $T(A,a)$ denote the contingent cone of A at a, that is, $v \in T(A,a)$ if and only if there exist a sequence $\{t_n\}$ in $(0,+\infty)$ decreasing to 0 and a sequence $\{v_n\}$ in Y converging to v such that $a + t_n v_n \in A$ for all n. Let $T_c(A,a)$ denote the circatangent (or Clarke tangent) cone of A at a, that is, $v \in T_c(A,a)$ if and only if, for each sequence $\{a_n\}$ in A converging to a and each sequence $\{t_n\}$ in $(0,+\infty)$ decreasing to 0, there exists a sequence $\{v_n\}$ in Y converging to v such that $a_n + t_n v_n \in A$ for all n. It is known that $T_c(A,a)$ is a closed convex cone and $T_c(A,a) \subset T(A,a)$. The Clarke normal cone of A at a is denoted by $N_c(A,a)$, that is,

$$N_c(A,a) = \{y^* \in Y^* : \langle y^*, v\rangle \leq 0, \ \ \forall\, v \in T_c(A,a)\}.$$

Clearly, if A is a convex set, then

$$N_c(A,a) = \{y^* \in Y^* : \langle y^*, y - a\rangle \leq 0, \ \ \forall\, y \in A\}.$$

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Let $\operatorname{Dom}(f)$ and $\operatorname{epi}(f)$ denote the domain and epigraph of f, respectively, that is,

$$\operatorname{Dom}(f) = \{x \in X : f(x) < +\infty\} \ \text{ and } \ \operatorname{epi}(f) = \{(x,t) \in X \times \mathbb{R} : f(x) \leq t\}.$$

For $x \in \operatorname{Dom}(f)$, the Clarke-Rockafellar subdifferential [6] of f at x is denoted by $\partial f(x)$, and is defined as

$$\partial f(x) = \{x^* \in X^* : (x^*, -1) \in N_c(\operatorname{epi}(f), (x, f(x)))\}.$$

It follows that $x^* \in \partial f(x)$ if and only if for all $(u,v) \in T_c(\operatorname{epi}(f), (x, f(x)))$,

$$v - \langle x^*, u\rangle \notin -\operatorname{int}(\mathbb{R}_+). \tag{2.1}$$

Let δ_A denote the indicator function of A. It is known [6] that $\partial \delta_A(a) = N_c(A,a)$.

Let $F : X \rightrightarrows Y$ be a set-valued mapping. The domain, graph and epigraph of F are defined respectively by

$$\operatorname{Dom}(F) = \{x \in X : F(x) \neq \emptyset\},$$

$$\mathrm{Gr}(F) = \{x, y) \in X \times Y : y \in F(x)\},$$
$$\mathrm{epi}(F) = \{x, y) \in X \times Y : y \in F(x) + C\}.$$

Let $(\bar{x}, \bar{y}) \in \mathrm{Gr}(F)$. The contingent epiderivative of F at $(\bar{x}, \bar{y})$ is defined by $\mathrm{Gr}(DF(\bar{x}, \bar{y})) = T(\mathrm{epi}(F), (\bar{x}, \bar{y}))$. Clearly, $v \in DF(\bar{x}, \bar{y})(u)$ if and only if $(u, v) \in T(\mathrm{epi}(F), (\bar{x}, \bar{y}))$. The circatangent epiderivative of F at $(\bar{x}, \bar{y})$ is defined by $\mathrm{Gr}(D_c F(\bar{x}, \bar{y})) = T_c(\mathrm{epi}(F), (\bar{x}, \bar{y}))$. We say that F is closed if $\mathrm{Gr}(F)$ is a closed set. We say that F is a C-convex set-valued mapping, if for all $x_1, x_2 \in X$, $t \in [0, 1]$,

$$tF(x_1) + (1 - t)F(x_2) \subset F(tx_1 + (1 - t)x_2) + C.$$

We say that F is a convex process if $\mathrm{Gr}(F)$ is a convex cone. Clearly, $D_c F(\bar{x}, \bar{y})$ is a closed convex process. We say that F is locally Lipschitz at $\bar{x} \in \mathrm{Dom}(F)$, if there exist $\eta > 0$ and an open neighborhood U of $\bar{x}$ such that

$$F(x_1) \subset F(x_2) + \eta \|x_1 - x_2\| B_Y, \quad \forall\, x_1, x_2 \in U.$$

Definition 2.1 ([21])**.** Let $(\bar{x}, \bar{y}) \in \mathrm{Gr}(F)$. $T \in \mathrm{B}(X, Y)$ is called a Yang-weak subgradient of F at $(\bar{x}, \bar{y})$ if

$$(F(x) - \bar{y} - T(x - \bar{x})) \cap (-\mathrm{int}(C)) = \emptyset, \quad \forall\, x \in X.$$

The set of all Yang-weak subgradients of F at $(\bar{x}, \bar{y})$, denoted by $\partial^{\mathrm{y-w}} F(\bar{x}, \bar{y})$, is called the Yang-weak subdifferential of F at $(\bar{x}, \bar{y})$.

Definition 2.2 ([5])**.** Let $\bar{x} \in \mathrm{Dom}(F)$. $T \in \mathrm{B}(X, Y)$ is called a Chen-Jahn-weak subgradient of F at $\bar{x}$ if

$$(F(x) - F(\bar{x}) - T(x - \bar{x})) \cap (-\mathrm{int}(C)) = \emptyset, \quad \forall\, x \in X.$$

The set of all Chen-Jahn-weak subgradients of F at $\bar{x}$, denoted by $\partial^{\mathrm{c-w}} F(\bar{x})$, is called the Chen-Jahn-weak subdifferential of F at $\bar{x}$.

Definition 2.3 ([16])**.** Let $\bar{x} \in \mathrm{Dom}(F)$ and $\bar{y} \in F(\bar{x})$. $T \in \mathrm{B}(X, Y)$ is called a Song-weak subgradient of F at $\bar{x}$ if

$$(DF(\bar{x}, \bar{y})(u) - T(u)) \cap (-\mathrm{int}(C)) = \emptyset, \quad \forall\, u \in X.$$

The set of all Song-weak subgradients of F at $(\bar{x}, \bar{y})$, denoted by $\partial^{\mathrm{s-w}} F(\bar{x}, \bar{y})$, is called the Song-weak subdifferential of F at $(\bar{x}, \bar{y})$.

Definition 2.4 ([3])**.** Let $\bar{x} \in \mathrm{Dom}(F)$, $\bar{y} \in F(\bar{x})$. $T \in \mathrm{B}(X, Y)$ is called a Borwein-strong subgradient of F at $\bar{x}$ if for all $x \in \mathrm{Dom}(F)$ and $y \in F(x)$, we have

$$y - \bar{y} - T(x - \bar{x}) \subset C.$$

The set of all Borwein-strong subgradients of F at $(\bar{x}, \bar{y})$, denoted by $\partial^{\mathrm{b-s}} F(\bar{x}, \bar{y})$, is called the Borwein-strong subdifferential of F at $(\bar{x}, \bar{y})$.

Definition 2.5 ([2])**.** Let $\bar{x} \in \mathrm{Dom}(F)$, $\bar{y} \in F(\bar{x})$. $T \in \mathrm{B}(X, Y)$ is called a Baier-Jahn-strong subgradient of F at $\bar{x}$ if

$$DF(\bar{x}, \bar{y})(u) - T(u) \subset C.$$

The set of all Baier-Jahn-strong subgradients of F at $(\bar{x}, \bar{y})$, denoted by $\partial^{\mathrm{b-j}} F(\bar{x}, \bar{y})$, is called the Baier-Jahn-strong subdifferential of F at $(\bar{x}, \bar{y})$.

The following result gives a little improvement of [18, Theorem 3.1]. However, its proof is very similar to [19, Theorem 3.2], so we omit it.

Lemma 2.6. *Let $G_i : X \rightrightarrows Y$ $(i = 1, 2, ..., n)$ be n C-convex set-valued mappings with closed epigraphs, $\bar{x} \in \bigcap_{i=1}^{n} Dom(G_i)$, $y_i \in G_i(\bar{x})$, $T \in \mathbf{B}(X, Y)$ and $T \in \partial^{y\text{-}w}(G_1 + G_2 + \cdots + G_n)(\bar{x}, y_1 + y_2 + \cdots + y_n)$. Suppose that*

$$\mathbb{R}_+(Dom(G_i) - \bigcap_{j=i+1}^{n} Dom(G_j)) \quad (i = 1, 2, ..., n-1)$$

are closed vector subspaces of X. Then the following statements hold:

(a) *There exist $T_i \in \mathrm{B}(X, Y)$, $x_i^* \in X^*$ and $c_0 \in int(C)$ such that $T_i(x) = \langle x_i^*, x\rangle c_0$, $T_i \in \partial^{y\text{-}w} G_i(\bar{x}, y_i)$ $(i = 2, 3, ..., n)$ and $T - (T_2 + T_3 + \cdots + T_n) \in \partial^{y\text{-}w} G_1(\bar{x}, y_1)$.*

(b) *$\partial^{y\text{-}w}(G_1 + G_2 + \cdots + G_n)(\bar{x}, y_1 + y_2 + \cdots + y_n) \subset \partial^{y\text{-}w} G_1(\bar{x}, y_1) + \partial^{y\text{-}w} G_2(\bar{x}, y_2) + \cdots + \partial^{y\text{-}w} G_n(\bar{x}, y_n)$.*

Lemma 2.7 ([16])**.** *Let $F : X \rightrightarrows Y$ be a set-valued mapping, $\bar{x} \in X$ and $\bar{y} \in F(\bar{x})$. If $\bar{y} \in WMin(F(X))$, then $DF(\bar{x}, \bar{y})(u) \cap (-int(C)) = \emptyset$ for all $u \in X$.*

3 Existence of Weak Subgradient

In this section, we establish some existence results of weak subgradients for set-valued mappings.

Motivated by (2.1), we introduce the following definition.

Definition 3.1. Let $(\bar{x}, \bar{y}) \in \mathrm{Gr}(F)$. $T \in \mathrm{B}(X, Y)$ is called a weak subgradient of F at $(\bar{x}, \bar{y})$ if

$$(D_c F(\bar{x}, \bar{y})(u) - T(u)) \cap (-\mathrm{int}(C)) = \emptyset, \quad \forall\, u \in X.$$

The set of all weak subgradients of F at $(\bar{x}, \bar{y})$, denoted by $\partial F(\bar{x}, \bar{y})$, is called the weak subdifferential of F at $(\bar{x}, \bar{y})$.

Now, we compare the difference among new subdifferential, Yang-weak subdifferential, Jahn-Chen subdifferential, Song-weak sudifferential, Borwein-strong subdifferential and Baier-Jahn-strong subdifferential. It is known [5] that $\partial^{\mathrm{c\text{-}w}} F(\bar{x}) \subset \partial^{\mathrm{y\text{-}w}} F(\bar{x}, \bar{y})$, but the converse is not true. It is also known [2] that $\partial^{\mathrm{b\text{-}s}} F(\bar{x}, \bar{y}) \subset \partial^{\mathrm{b\text{-}j}} F(\bar{x}, \bar{y})$, and $\partial^{\mathrm{b\text{-}s}} F(\bar{x}, \bar{y}) = \partial^{\mathrm{b\text{-}j}} F(\bar{x}, \bar{y})$ if F is C-convex. By Definition 2.3 and 2.5, we can easily obtain that $\partial^{\mathrm{b\text{-}j}} F(\bar{x}, \bar{y}) \subset \partial^{\mathrm{s\text{-}w}} F(\bar{x}, \bar{y})$.

Proposition 3.2. *Let $\bar{x} \in Dom(F)$ and $\bar{y} \in F(\bar{x})$. Then*

$$\partial^{y\text{-}w} F(\bar{x}, \bar{y}) \subset \partial^{s\text{-}w} F(\bar{x}, \bar{y}) \subset \partial F(\bar{x}, \bar{y}). \tag{3.1}$$

Furthermore, if F is a C-convex set-valued mapping, then they are equal.

Proof. By [16, Proposition 2.2], we have $\partial^{\mathrm{y\text{-}w}} F(\bar{x}, \bar{y}) \subset \partial^{\mathrm{s\text{-}w}} F(\bar{x}, \bar{y})$. Suppose that $T \in \partial^{\mathrm{s\text{-}w}} F(\bar{x}, \bar{y})$. Then

$$(DF(\bar{x}, \bar{y})(u) - T(u)) \cap (-\mathrm{int}(C)) = \emptyset, \quad \forall\, u \in X.$$

Since $D_cF(\bar{x},\bar{y})(u) \subset DF(\bar{x},\bar{y})(u)$, we have

$$(D_cF(\bar{x},\bar{y})(u) - T(u)) \cap (-\mathrm{int}(C)) = \emptyset, \quad \forall\, u \in X.$$

which implies that $T \in \partial F(\bar{x},\bar{y})$.

Now, suppose that F is a C-convex set-valued mapping. Let $T \in \partial F(\bar{x},\bar{y})$. Then

$$(D_cF(\bar{x},\bar{y})(u) - T(u)) \cap (-\mathrm{int}(C)) = \emptyset, \quad \forall\, u \in X.$$

Since F is a C-convex set-valued mapping, we have $F(x) - \bar{y} \subset D_cF(\bar{x},\bar{y})(x - \bar{x})$. It follows that

$$(F(x) - \bar{y} - T(x - \bar{x})) \cap (-\mathrm{int}(C)) = \emptyset, \quad \forall\, x \in X,$$

which implies that $T \in \partial^{\text{y-w}} F(\bar{x},\bar{y})$. □

Remark 3.3. The inclusion in equation (3.1) may be strict. See the following example.

Example 3.4. Let $F : \mathbb{R} \rightrightarrows \mathbb{R}$ be defined as

$$F(x) = \begin{cases} [x^2 \sin \frac{1}{x}, x^2 \sin \frac{1}{x} + 1], & x \neq 0, \\ \{0\}, & x = 0. \end{cases}$$

It is easy to calculate that $DF(0,0)(u) = [0,+\infty)$, $D_cF(0,0)(u) = [|u|,+\infty)$ for all $u \in \mathbb{R}$, $\partial^{\text{y-w}} F(0,0) = \emptyset$, $\partial^{\text{s-w}} F(0,0) = \{0\}$ and $\partial F(0,0) = [-1,1]$.

Remark 3.5. $T \in \partial F(\bar{x},\bar{y})$ if and only if $T \in \partial^{\text{y-w}}(D_cF(\bar{x},\bar{y}))(0,0)$.

Remark 3.6. According to the above analysis, for a given set-valued mapping, the Yang-weak subdifferential, Jahn-Chen subdifferential, Song-weak sudifferential, Borwein-strong subdifferential and Baier-Jahn-strong subdifferential are subsets of our new subdifferential at the same point.

Theorem 3.7. *Let $F : X \rightrightarrows Y$ be a set-valued mapping and $(\bar{x},\bar{y}) \in Gr(F)$. Suppose that the following conditions are satisfied:*

(i) $0 \in int(Dom(D_cF(\bar{x},\bar{y})))$;

(ii) $T_c(epi(F),(\bar{x},\bar{y}))$ *is a proper subset of* $X \times Y$.

Then $\partial F(\bar{x},\bar{y}) \neq \emptyset$.

Proof. Firstly, we prove that $\mathrm{int}(T_c(\mathrm{epi}(F),(\bar{x},\bar{y}))) \neq \emptyset$. By the definition of circatangent epiderivative, we have $C \subset D_cF(\bar{x},\bar{y})(0)$. Take $\bar{c} \in \mathrm{int}(C)$. Then there exists $\rho > 0$ such that $\bar{c} + \rho B_Y \subset D_cF(\bar{x},\bar{y})(0) \cap C$. Clearly,

$$2\bar{c} - (\bar{c} + \rho B_Y) \subset C. \tag{3.2}$$

Since $D_cF(\bar{x},\bar{y})$ is a closed convex process, by (i), we have $\mathrm{Dom}(D_cF(\bar{x},\bar{y})) = X$. By the Closed Graph Theorem [1, Theorem 2.2.6], there exists $\eta > 0$ such that

$$\bar{c} \in D_cF(\bar{x},\bar{y})(0) \subset D_cF(\bar{x},\bar{y})(x) + \eta\|x\|B_Y, \quad \forall\, x \in X. \tag{3.3}$$

Let $x \in \frac{\rho}{\eta}B_X$. By (3.3), there exists $b_x \in B_Y$ such that $\bar{c} + \eta\|x\|b_x \in D_cF(\bar{x},\bar{y})(x)$. Since $\|\eta\|x\|b_x\| < \rho$, we have $\bar{c}+\eta\|x\|b_x \in \bar{c}+\rho B_Y$. It follows from (3.2) that $2\bar{c}-(\bar{c}+\eta\|x\|b_x) \in C$, and so,

$$2\bar{c} \in \bar{c} + \eta\|x\|b_x + C \subset D_cF(\bar{x},\bar{y})(x) + D_cF(\bar{x},\bar{y})(0) \subset D_cF(\bar{x},\bar{y})(x),$$

which leads to

$$2\bar{c} + \mathrm{int}(C) \in D_cF(\bar{x}, \bar{y})(x) + D_cF(\bar{x}, \bar{y})(0) \subset D_cF(\bar{x}, \bar{y})(x).$$

This implies that $(x, 2\bar{c} + \mathrm{int}(C)) \in T_c(\mathrm{epi}(F), (\bar{x}, \bar{y}))$. Therefore, $(\frac{\rho}{\eta}B_X, 2\bar{c} + \mathrm{int}(C)) \subset T_c(\mathrm{epi}(F), (\bar{x}, \bar{y}))$, and so $\mathrm{int}(T_c(\mathrm{epi}(F), (\bar{x}, \bar{y}))) \neq \emptyset$.

Secondly, we show that

$$(0, 0) \notin \mathrm{int}(T_c(\mathrm{epi}(F), (\bar{x}, \bar{y}))). \tag{3.4}$$

Suppose to the contrary that there exists $\tau > 0$ such that

$$\tau B_X \times \tau B_Y \subset T_c(\mathrm{epi}(F), (\bar{x}, \bar{y})).$$

Since $T_c(\mathrm{epi}(F), (\bar{x}, \bar{y}))$ is a cone, one has

$$X \times Y = \bigcup_{\lambda > 0} \lambda(\tau B_X \times \tau B_Y) = T_c(\mathrm{epi}(F), (\bar{x}, \bar{y})),$$

which contradicts the assumption that $T_c(\mathrm{epi}(F), (\bar{x}, \bar{y}))$ is a proper subset of $X \times Y$. Therefore, (3.4) is justified. Applying the Eidelheit's separation theorem to (3.4), there exists $(x^*, y^*) \in X^* \times Y^*$ with $\|x^*\| + \|y^*\| = 1$ such that

$$0 \leq \langle x^*, x\rangle + \langle y^*, y\rangle, \qquad \forall\ (x, y) \in T_c(\mathrm{epi}(F), (\bar{x}, \bar{y})). \tag{3.5}$$

For each $c \in C$, taking $(x, y) = (0, c) \in T_c(\mathrm{epi}(F), (\bar{x}, \bar{y}))$ in (3.5), we have $\langle y^*, c\rangle \geq 0$, and so $y^* \in C^*$. We claim that $y^* \neq 0$. Suppose to the contrary that $y^* = 0$. Then by (3.5), we have

$$0 \leq \langle x^*, x\rangle, \qquad \forall\ x \in \mathrm{Dom}(D_cF(\bar{x}, \bar{y})) = X.$$

This implies that $x^* = 0$, which contradicts $(x^*, y^*) \neq (0, 0)$. Hence $y^* \neq 0$. Then there exists $c_0 \in \mathrm{int}(C)$ such that $\langle y^*, c_0\rangle = 1$. Now, we define a mapping $T : X \to Y$ by

$$T(x) = -\langle x^*, x\rangle c_0, \quad \forall\ x \in X.$$

Clearly, T is linear and continuous. We claim that T satisfies

$$(D_cF(\bar{x}, \bar{y})(u) - T(u)) \cap (-\mathrm{int}(C)) = \emptyset, \quad \forall\ u \in X. \tag{3.6}$$

Suppose to the contrary that there exist $\bar{u} \in X$ and $\bar{v} \in D_cF(\bar{x}, \bar{y})(\bar{u})$ such that $\bar{v} - T(\bar{u}) \in -\mathrm{int}(C)$. Since $y^* \in C^* \setminus \{0\}$, we have

$$\begin{aligned} \langle y^*, \bar{v} - T(\bar{u})\rangle &= \langle y^*, \bar{v}\rangle + \langle x^*, \bar{u}\rangle\langle y^*, c_0\rangle \\ &= \langle y^*, \bar{v}\rangle + \langle x^*, \bar{u}\rangle < 0. \end{aligned} \tag{3.7}$$

On the other hand, taking $(x, y) = (\bar{u}, \bar{v})$ in (3.5), we have $\langle x^*, \bar{u}\rangle + \langle y^*, \bar{v}\rangle \geq 0$, which contradicts (3.7). Therefore, (3.6) is justified, and so $T \in \partial F(\bar{x}, \bar{y})$. □

Corollary 3.8. *Let $F : X \rightrightarrows Y$ be a set-valued mapping, $\bar{x} \in Dom(F)$ and $\bar{y} \in F(\bar{x})$. Suppose that the following conditions are satisfied:*

(i) *$F(\bar{x})$ is upper bounded and F is upper semicontinuous at $\bar{x}$;*

(ii) *there exists $u_0 \in X$ such that $D_cF(\bar{x}, \bar{y})(u_0) \neq Y$.*

Then $Dom(D_cF(\bar{x},\bar{y}))=X$ *and* $\partial F(\bar{x},\bar{y})\neq\emptyset$.

Proof. Since $F(\bar{x})$ is upper bounded and F is upper semicontinuous at $\bar{x}$, by [7, Lemma 4.1], there exists an open neighborhood U of $\bar{x}$ such that

$$\bar{y}+\mathrm{int}(C)\subset F(x)+\mathrm{int}(C),\quad \forall\ x\in U.$$

Take $\tau>0$ sufficiently small such that $\bar{x}+\tau B_X\subset U$. Then $(\bar{x}+\tau B_X,\bar{y}+\mathrm{int}(C))\subset \mathrm{epi}(F)$. By the definition of circatangent cone, we obtain $(\tau B_X,\mathrm{int}(C))\subset T_c(\mathrm{epi}(F),(\bar{x},\bar{y}))$. Therefore $\tau B_X\subset \mathrm{Dom}(D_cF(\bar{x},\bar{y}))$. Since $D_cF(\bar{x},\bar{y})$ is a process, we obtain $\mathrm{Dom}(D_cF(\bar{x},\bar{y}))=X$. By condition (ii), there exists an $v_0\in Y$ such that $v_0\notin D_cF(\bar{x},\bar{y})(u_0)$, and so $(u_0,v_0)\notin T_c(\mathrm{epi}(F),(\bar{x},\bar{y}))$. Therefore, $T_c(\mathrm{epi}(F),(\bar{x},\bar{y}))$ is a proper subset of $X\times Y$. By Theorem 3.7, $\partial F(\bar{x},\bar{y})\neq\emptyset$. □

Now, we give an example to illustrate Theorem 3.7.

Example 3.9. Let $X=\mathbb{R}$, $Y=\mathbb{R}^2$, $C=\mathbb{R}^2_+$ and the set-valued mapping $F:\mathbb{R}\rightrightarrows\mathbb{R}^2$ be defined as

$$F(x)=\{(u,v):x^3\leq u,v\leq x^3+1\},\quad \forall\ x\in\mathbb{R}.$$

Let $\bar{x}=0$ and $\bar{y}=(0,0)$. Then

$$T_c(\mathrm{epi}(F),(\bar{x},\bar{y}))=\{(u,v,w):u\in\mathbb{R},v\geq 0,w\geq 0\}.$$

It is easy to verify that all conditions of Theorem 3.7 are satisfied. Therefore, $\partial F(\bar{x},\bar{y})\neq\emptyset$. In fact,

$$\partial F(\bar{x},\bar{y})=\{(T_1,T_2)\in\mathbb{R}^2:(T_1\leq 0,T_2\geq 0)\text{ or }(T_1\geq 0,T_2\leq 0)\}.$$

4 Generalized Lagrange Multiplier Rule

In this section, in terms of Lagrange multiplier, we give necessary and sufficient conditions for the solution of set-valued optimization problems.

Let $K\subset Z$ be a closed convex cone, $F:X\rightrightarrows Y$ be a set-valued mapping, $H:X\rightrightarrows Z$ be a K-convex set-valued mapping, and $\Omega\subset X$ be a nonempty closed set. Consider the following two set-valued optimization problems:

$$(\mathrm{SOP1})\quad\begin{cases}\min & F(x),\\ \text{s.t.} & H(x)\cap(-K)\neq\emptyset,\ x\in\Omega.\end{cases}$$

$$(\mathrm{SOP2})\quad\begin{cases}\min & F(x),\\ \text{s.t.} & x\in\Omega.\end{cases}$$

Definition 4.1. We say that $(\bar{x},\bar{y})\in X\times Y$ is a weak minimizer of optimization problem (SOP1) if $\bar{x}\in D:=\{x\in X\ :\ H(x)\cap(-K)\neq\emptyset,\ x\in\Omega\}$ and $\bar{y}\in F(\bar{x})\cap\mathrm{WMin}(F(D))$. We say that $(\bar{x},\bar{y})\in X\times Y$ is a weak minimizer of optimization problem (SOP2) if $\bar{x}\in\Omega$ and $\bar{y}\in F(\bar{x})\cap\mathrm{WMin}(F(\Omega))$.

For convenience in the sequel, we define the set

$$\mathcal{L}=\{T\in\mathbf{B}(X,Y)\ :\ \text{there exist } x^*\in X^*\ \text{ and }\ c\in\mathrm{int}(C)\text{ such that }T=\langle x^*,\cdot\rangle c\}.$$

We need the following blanket assumption.

Assumption (A) Let $\bar{z} \in H(\bar{x}) \cap (-K)$. Suppose that for each $v \in T_c(-K, \bar{z})$, each sequence $\{t_n\} \subset (0, +\infty)$ decreasing to 0, each sequence $\{(u_n, v_n)\} \subset X \times Z$ satisfying $(u_n, v_n) \to (u, v)$, $\bar{z}+t_n v_n \in H(\bar{x}+t_n u_n)+K$ and $\bar{x}+t_n u_n \in \Omega$, there exist two subsequences $\{t_{n_l}\} \subset \{t_n\}$, $\{u_{n_l}\} \subset \{u_n\}$ and a sequence $\{\tilde{v}_l\} \subset Z$ with $\tilde{v}_l \to v$ such that $\bar{z} + t_{n_l}\tilde{v}_l \in H(\bar{x} + t_{n_l}u_{n_l}) + K$ and $\bar{z} + t_{n_l}\tilde{v}_l \in -K$ when l sufficiently large.

Remark 4.2. Clearly, if $\bar{z} \in H(\bar{x}) \cap (-\text{int}(K))$, then Assumption (A) is satisfied by taking $t_{n_l} = t_n$, $u_{n_l} = u_n$ and $\tilde{v}_l = v_n$.

Example 4.3. Let $X = Z = \mathbb{R}$, $H(x) = [x, x + 1]$ for all $x \in \mathbb{R}$, $K = [0, +\infty)$, $\Omega = [-\pi, 0]$, $\bar{x} = 0$ and $\bar{z} = 0$. Let $v \in T_c(-K, \bar{z}) = (-\infty, 0]$, $\{t_n\} \subset (0, +\infty)$ decreasing to 0, $\{(u_n, v_n)\} \subset X \times Z$ satisfying $(u_n, v_n) \to (u, v)$, $\bar{z}+t_n v_n \in H(\bar{x}+t_n u_n)+K$ and $\bar{x}+t_n u_n \in \Omega$. In the case when $v = 0$, we take $\tilde{v}_n = 0$. In the case when $v < 0$, we take $\tilde{v}_n = v_n$. Then $\bar{z} + t_n\tilde{v}_n \in H(\bar{x} + t_n u_n) + K$ and $\bar{z} + t_n\tilde{v}_n \in -K$ when n sufficiently large. The assumption (A) is satisfied.

Theorem 4.4. *Suppose that $\bar{x} \in D$, $\bar{y} \in F(\bar{x})$, $\bar{z} \in H(\bar{x}) \cap (-K)$, F and H are locally Lipschitz at $\bar{x}$, $F(\bar{x})$ and $H(\bar{x})$ have upper bound, Assumption (A) is satisfied, and*

$$T_c(-K, \bar{z}) - D_c H(\bar{x}, \bar{z})(T_c(\Omega, \bar{x})) = Z. \tag{4.1}$$

If $(\bar{x}, \bar{y})$ is a weak minimizer of (SOP1), then there exist $\Lambda \in \mathbf{B}(Z, Y)$, $z^* \in Z^*$ and $c_0 \in$ int(C) such that

(i) $\Lambda(z) = \langle z^*, z\rangle c_0, \quad \forall\ z \in Z;$

(ii) $\Lambda(K) \subset C;$

(iii) $\Lambda(\bar{z}) = 0;$

(iv) $0 \in \partial F(\bar{x}, \bar{y}) + \partial(\Lambda \circ H)(\bar{x}, \Lambda(\bar{z})) \cap \mathcal{L} + N_c(\Omega, \bar{x})\text{int}(C),$

where $N_c(\Omega, \bar{x})\text{int}(C) := \{\langle x^*, \cdot\rangle c : x^* \in N_c(\Omega, \bar{x}), c \in \text{int}(C)\}$.

Proof. Suppose that $(\bar{x}, \bar{y})$ is a weak minimizer of (SOP1). Let $F_i : X \times Z \rightrightarrows Y$ $(i = 1, 2, 3, 4)$ be defined by

$$F_1(x, z) = F(x), \quad \forall\ (x, z) \in X \times Z,$$

$$F_2(x, z) = \delta_{-K}(z), \quad \forall\ (x, z) \in X \times Z,$$

$$F_3(x, z) = \delta_{\Omega}(x), \quad \forall\ (x, z) \in X \times Z,$$

$$F_4(x, z) = \delta_{\text{epi}(H)}(x, z), \quad \forall\ (x, z) \in X \times Z,$$

where $\delta_{-K}(z)$ equals to $\{0\}$ if $z \in -K$, $\emptyset$ otherwise. Then $(\bar{x}, \bar{z}, \bar{y})$ is a weak minimizer of the following optimization problem

$$\begin{cases} \min & F_1(x, z) + F_2(x, z) + F_3(x, z) + F_4(x, z), \\ \text{s.t.} & (x, z) \in X \times Z. \end{cases}$$

By Lemma 2.7, we have

$$D(F_1 + F_2 + F_3 + F_4)(\bar{x}, \bar{z}, \bar{y})(u, v) \bigcap (-\text{int}(C)) = \emptyset, \quad \forall\ (u, v) \in X \times Z. \tag{4.2}$$

Now, we show that for all $(u,v) \in X \times Z$,

$$\begin{aligned} &D_cF_1(\bar{x},\bar{z},\bar{y})(u,v) + D_cF_2(\bar{x},\bar{z},0)(u,v) + D_cF_3(\bar{x},\bar{z},0)(u,v) \\ &+ D_cF_4(\bar{x},\bar{z},0)(u,v) \subset D(F_1+F_2+F_3+F_4)(\bar{x},\bar{z},\bar{y})(u,v). \end{aligned} \tag{4.3}$$

Let $w_1 \in D_cF_1(\bar{x},\bar{z},\bar{y})(u,v)$ and $w_i \in D_cF_i(\bar{x},\bar{z},0)(u,v)$ $(i=2,3,4)$. Let $\{t_n\}$ be a sequence in $(0,+\infty)$ decreasing to 0. By the definition of circatangent epiderivative, there exist $\{(u_n^{(i)}, v_n^{(i)}, w_n^{(i)})\} \subset X \times Z \times Y$ $(i=1,2,3,4)$ with $(u_n^{(i)}, v_n^{(i)}, w_n^{(i)}) \to (u,v,w)$ such that

$$\bar{y} + t_nw_n^{(1)} \in F_1(\bar{x}+t_nu_n^{(1)}, \bar{z}+t_nv_n^{(1)}) + C = F(\bar{x}+t_nu_n^{(1)}) + C,$$

$$0 + t_nw_n^{(2)} \in F_2(\bar{x}+t_nu_n^{(2)}, \bar{z}+t_nv_n^{(2)}) + C = \delta_{-K}(\bar{z}+t_nv_n^{(2)}) + C, \tag{4.4}$$

$$0 + t_nw_n^{(3)} \in F_3(\bar{x}+t_nu_n^{(3)}, \bar{z}+t_nv_n^{(3)}) + C = \delta_{\Omega}(\bar{x}+t_nu_n^{(3)}) + C, \tag{4.5}$$

$$0 + t_nw_n^{(4)} \in F_4(\bar{x}+t_nu_n^{(4)}, \bar{z}+t_nv_n^{(4)}) + C = \delta_{\mathrm{epi}(H)}(\bar{x}+t_nu_n^{(4)}, \bar{z}+t_nv_n^{(4)}) + C. \tag{4.6}$$

Since F is locally Lipschitz at $\bar{x}$, there exist a positive integer number n_1 and $\eta_1 > 0$ such that

$$\bar{y} + t_nw_n^{(1)} \in F(\bar{x}+t_nu_n^{(1)}) + C \subset F(\bar{x}+t_nu_n^{(3)}) + \eta_1 t_n\|u_n^{(1)} - u_n^{(3)}\|B_Y + C$$

when $n > n_1$. Then there exists $b_n^{(1)} \in B_Y$ such that

$$\bar{y} + t_n\left(w_n^{(1)} + \eta_1\|u_n^{(1)} - u_n^{(3)}\|b_n^{(1)}\right) \in F(\bar{x}+t_nu_n^{(3)}) + C \tag{4.7}$$

when $n > n_1$. By the definition of the indicator function and (4.6), we have $t_nw_n^{(4)} \in C$ and $\bar{z}+t_nv_n^{(4)} \in H(\bar{x}+t_nu_n^{(4)}) + K$. Since H is locally Lipschitz at $\bar{x}$, there exist a sequence $\{b_n^{(4)}\} \subset B_Z$, a positive integer number n_4 and $\eta_4 > 0$ such that

$$\bar{z} + t_n\left(v_n^{(4)} + \eta_4\|u_n^{(4)} - u_n^{(3)}\|b_n^{(4)}\right) \in H(\bar{x}+t_nu_n^{(3)}) + K$$

when $n > n_4$. By (4.4), we have $\bar{z}+t_nv_n^{(2)} \in -K$, and so $v \in T_c(-K,\bar{z})$. By (4.5), we have $\bar{x}+t_nu_n^{(3)} \in \Omega$. Noting

$$\left(u_n^{(3)}, v_n^{(4)} + \eta_4\|u_n^{(4)} - u_n^{(3)}\|b_n^{(4)}\right) \to (u,v),$$

by Assumption (A), we may assume that there exist a positive integer number n_3 and a sequence $\{\tilde{v}_n\} \subset Z$ with $\tilde{v}_n \to v$ such that

$$\bar{z} + t_n\tilde{v}_n \in H(\bar{x}+t_nu_n^{(3)}) + K \tag{4.8}$$

and

$$\bar{z} + t_n\tilde{v}_n \in -K \tag{4.9}$$

when $n > n_3$. It follows from (4.6) and (4.8) that

$$0 + t_nw_n^{(4)} \in \delta_{\mathrm{epi}(H)}\left(\bar{x}+t_nu_n^{(3)}, \bar{z}+t_n\tilde{v}_n\right) + C \tag{4.10}$$

when $n > \{n_3, n_4\}$. It follows from (4.4) and (4.9) that

$$0 + t_nw_n^{(2)} \in \delta_{-K}\left(\bar{z}+t_n\tilde{v}_n\right) + C \tag{4.11}$$

when $n > \{n_3, n_4\}$. Adding (4.5), (4.7), (4.10) and (4.11), we have

$$\begin{aligned}
&\bar{y} + t_n \left(w_n^{(1)} + \eta_1 \|u_n^{(1)} - u_n^{(3)}\| b_n^{(1)} + w_n^{(2)} + w_n^{(3)} + w_n^{(4)}\right) \\
&\in F(\bar{x} + t_n u_n^{(3)}) + \delta_{-K}(\bar{z} + t_n \tilde{v}_n) + \delta_\Omega(\bar{x} + t_n u_n^{(3)}) \\
&+ \delta_{\mathrm{epi}(H)}\left(\bar{x} + t_n u_n^{(3)}, \bar{z} + t_n \tilde{v}_n\right) + C \\
&= (F_1 + F_2 + F_3 + F_4)\left(\bar{x} + t_n u_n^{(3)}, \bar{z} + t_n \tilde{v}_n\right) + C
\end{aligned} \tag{4.12}$$

when $n > \max\{n_1, n_3, n_4\}$. Since

$$w_n^{(1)} + \eta_1 \|u_n^{(1)} - u_n^{(3)}\| b_n^{(1)} + w_n^{(2)} + w_n^{(3)} + w_n^{(4)} \to w_1 + w_2 + w_3 + w_4,$$

$$\left(u_n^{(3)}, \tilde{v}_n\right) \to (u, v),$$

it follows from (4.12) that $w_1 + w_2 + w_3 + w_4 \in D(F_1 + F_2 + F_3 + F_4)(\bar{x}, \bar{z}, \bar{y})(u, v)$. Therefore, (4.3) holds. Combining (4.2) with (4.3), we have

$$\left(D_c F_1(\bar{x}, \bar{z}, \bar{y})(u, v) + \sum_{i=2}^{4} D_c F_i(\bar{x}, \bar{z}, 0)(u, v)\right) \bigcap (-\mathrm{int}(C)) = \emptyset.$$

This implies that

$$(0, 0) \in \partial^{\text{y-w}}(D_c F_1(\bar{x}, \bar{z}, \bar{y}) + \sum_{i=2}^{4} D_c F_i(\bar{x}, \bar{z}, 0))(0, 0, 0).$$

Now, we show that the condition of Lemma 2.6 is satisfied (taking $G_1 = D_c F_1(\bar{x}, \bar{z}, \bar{y})$, $G_i = D_c F_i(\bar{x}, \bar{z}, 0)$ $(i = 2, 3, 4)$). Since F and H are locally Lipschitz at $\bar{x}$, and $F(\bar{x})$ and $H(\bar{x})$ have upper bound, by Corollary 3.8, we have $\mathrm{Dom}(D_c F(\bar{x}, \bar{y})) = \mathrm{Dom}(D_c H(\bar{x}, \bar{z})) = X$. It is easy to verify that

$$\mathrm{Dom}(D_c F_1(\bar{x}, \bar{z}, \bar{y})) = \mathrm{Dom}(D_c F(\bar{x}, \bar{y})) \times Z = X \times Z,$$

$$\mathrm{Dom}(D_c F_2(\bar{x}, \bar{z}, 0)) = X \times T_c(-K, \bar{z}),$$

$$\mathrm{Dom}(D_c F_3(\bar{x}, \bar{z}, 0)) = T_c(\Omega, \bar{x}) \times Z,$$

$$\mathrm{Dom}(D_c F_4(\bar{x}, \bar{z}, 0)) = T_c(\mathrm{epi}(H), (\bar{x}, \bar{z})).$$

It is easy to verify that

$$\mathrm{Dom}(D_c F_1(\bar{x}, \bar{z}, \bar{y})) - \bigcap_{i=2}^{4} \mathrm{Dom}(D_c F_i(\bar{x}, \bar{z}, 0)) = X \times Z.$$

Now, we show that

$$\mathrm{Dom}(D_c F_2(\bar{x}, \bar{z}, 0)) - \bigcap_{i=3}^{4} \mathrm{Dom}(D_c F_i(\bar{x}, \bar{z}, 0)) = X \times Z,$$

that is,

$$X \times T_c(-K, \bar{z}) - (T_c(\Omega, \bar{x}) \times Z) \bigcap T_c(\mathrm{epi}(H), (\bar{x}, \bar{z})) = X \times Z. \tag{4.13}$$

Let $(x,z)\in X\times Z$. By (4.1), there exist $\bar v\in T_c(-K,\bar z)$, $\tilde u\in T_c(\Omega,\bar x)$ and $\tilde v\in D_cH(\bar x,\bar z)(\tilde u)$ such that $z=\bar v-\tilde v$. Clearly, $(x+\tilde u,\bar v)\in X\times T_c(-K,\bar z)$, $(\tilde u,\tilde v)\in (T_c(\Omega,\bar x)\times Z)\cap T_c(\mathrm{epi}(H),(\bar x,\bar z))$ and $(x,z)=(x+\tilde u,\bar v)-(\tilde u,\tilde v)$. Therefore, (4.13) holds. Using the fact $\mathrm{Dom}(D_cH(\bar x,\bar z))=X$, we can easily show that

$$T_c(\Omega,\bar x)\times Z-T_c(\mathrm{epi}(H),(\bar x,\bar z))=X\times Z,$$

that is,

$$\mathrm{Dom}(D_cF_3(\bar x,\bar z,0))-\mathrm{Dom}(D_cF_4(\bar x,\bar z,0))=X\times Z.$$

By Lemma 2.6, there exist $(x_i^*,z_i^*)\in X^*\times Z^*$, $c_0\in\mathrm{int}(C)$, $(T_i(x),\Lambda_i(z))=(\langle x_i^*,x\rangle,\langle z_i^*,z\rangle)c_0$ such that $(T_1,\Lambda_1)\in\partial^{\text{y-w}}(D_cF_1(\bar x,\bar z,\bar y))(0,0,0)$, $(T_i,\Lambda_i)\in\partial^{\text{y-w}}(D_cF_i(\bar x,\bar z,0))(0,0,0)$ $(i=2,3,4)$ and

$$(0,0)=(T_1+T_2+T_3+T_4,\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4). \tag{4.14}$$

Since $D_cF_1(\bar x,\bar z,\bar y)(u,v)=D_cF(\bar x,\bar y)(u)$ and $(T_1,\Lambda_1)\in\partial^{\text{y-w}}(D_cF_1(\bar x,\bar z,\bar y))(0,0,0)$, one has

$$(D_cF(\bar x,\bar y)(u)-\langle x_1^*,u\rangle c_0-\langle z_1^*,v\rangle c_0)\cap(-\mathrm{int}(C))=\emptyset,\quad \forall\,(u,v)\in X\times Z. \tag{4.15}$$

Taking $u=0$ in (4.15) and noting $0\in D_cF(\bar x,\bar y)(0)$, one obtain

$$(-\langle z_1^*,v\rangle c_0)\notin-\mathrm{int}(C),\quad\forall\,v\in Z.$$

As $c_0\in\mathrm{int}(C)$, the above relation implies that

$$\langle z_1^*,v\rangle\le 0,\quad\forall\,v\in Z,$$

and so $z_1^*=0$, that is, $\Lambda_1=0$. Replacing z_1^* by 0 in (4.15), one has

$$(D_cF(\bar x,\bar y)(u)-\langle x_1^*,u\rangle c_0)\cap(-\mathrm{int}(C))=\emptyset,\quad\forall\,u\in X,$$

which implies that $T_1\in\partial F(\bar x,\bar y)$.

It is not hard to verify that $w\in D_cF_2(\bar x,\bar z,0)(u,v)$ if and only if $w\in C$, $u\in X$ and $v\in T_c(-K,\bar z)$. As $(T_2,\Lambda_2)\in\partial^{\text{y-w}}(D_cF_2(\bar x,\bar z,0))(0,0,0)$, we have

$$(C-\langle x_2^*,u\rangle c_0-\langle z_2^*,v\rangle c_0)\cap(-\mathrm{int}(C))=\emptyset,\quad\forall\,u\in X\text{ and }v\in T_c(-K,\bar z).$$

Since $0\in C$, we get

$$\langle x_2^*,u\rangle c_0+\langle z_2^*,v\rangle c_0\notin\mathrm{int}(C),\quad\forall\,u\in X\text{ and }v\in T_c(-K,\bar z). \tag{4.16}$$

Taking $v=0$ in (4.16), we get

$$\langle x_2^*,u\rangle\le 0,\quad\forall\,u\in X.$$

This implies that $x_2^*=0$, and so $T_2=0$. Replacing x_2^* by 0 in (4.16), we get

$$\langle z_2^*,v\rangle\le 0,\quad\forall\,v\in T_c(-K,\bar z). \tag{4.17}$$

As $-K-\bar z\subset T_c(-K,\bar z)$ and $\bar z\in -K$, taking $v=0-\bar z$ and $v=2\bar z-\bar z$ in (4.17), respectively, we obtain $\langle z_2^*,\bar z\rangle\ge 0$ and $\langle z_2^*,\bar z\rangle\le 0$, and so $\langle z_2^*,\bar z\rangle=0$. Therefore,

$$\Lambda_2(\bar z)=0. \tag{4.18}$$

Taking $v = -k - \bar{z}$ in (4.17) for all $k \in K$, we have

$$\langle z_2^*, -k - \bar{z}\rangle = \langle z_2^*, -k\rangle \leq 0,$$

and so

$$\Lambda_2(K) = \langle z_2^*, K\rangle c_0 \subset C. \tag{4.19}$$

It is not hard to verify that $w \in D_cF_3(\bar{x}, \bar{z}, 0)(u, v)$ if and only if $w \in C$, $u \in T_c(\Omega, \bar{x})$ and $v \in Z$. As $(T_3, \Lambda_3) \in \partial^{\text{y-w}}(D_cF_3(\bar{x}, \bar{z}, 0))(0, 0, 0)$, we have

$$(C - \langle x_3^*, u\rangle c_0 - \langle z_3^*, v\rangle c_0) \cap (-\text{int}(C)) = \emptyset, \quad \forall\ u \in T_c(\Omega, \bar{x}) \text{ and } v \in Z. \tag{4.20}$$

Taking $u = 0$ in (4.20) and noting $0 \in C$, we get

$$\langle z_3^*, v\rangle \leq 0, \quad \forall\ v \in Z.$$

This implies that $z_3^* = 0$, and so $\Lambda_3 = 0$. Replacing z_3^* by 0 in (4.20), we get

$$\langle x_3^*, u\rangle \leq 0, \quad \forall\ u \in T_c(\Omega, \bar{x}).$$

This implies that $x_3^* \in N_c(\Omega, \bar{x})$, and so $T_3 = \langle x_3^*, \cdot\rangle c_0 \in N_c(\Omega, \bar{x})\text{int}(C)$.

It is not hard to verify that $w \in D_cF_4(\bar{x}, \bar{z}, 0)(u, v)$ if and only if $w \in C$ and $(u, v) \in T_c(\text{epi}(H), (\bar{x}, \bar{z}))$. Since $(T_4, \Lambda_4) \in \partial^{\text{y-w}}(D_cF_4(\bar{x}, \bar{z}, 0))(0, 0, 0)$, then

$$(C - T_4(u) - \Lambda_4(v)) \cap (-\text{int}(C)) = \emptyset, \quad \forall\ (u, v) \in T_c(\text{epi}(H), (\bar{x}, \bar{z})).$$

Noting $0 \in C$, we get

$$(0 - T_4(u) - \Lambda_4(D_cH(\bar{x}, \bar{z})(u))) \cap (-\text{int}(C)) = \emptyset, \qquad \forall\ u \in X.$$

Since $\Lambda_1 = \Lambda_3 = 0$, by (4.14), we get $\Lambda_2 = -\Lambda_4$. The above relation can be rewritten as

$$(\Lambda_2(D_cH(\bar{x}, \bar{z})(u)) - T_4(u)) \cap (-\text{int}(C)) = \emptyset, \qquad \forall\ u \in X. \tag{4.21}$$

Now, let $u \in X$ and $v \in D_c(\Lambda_2 \circ H)(\bar{x}, \Lambda_2(\bar{z}))(u)$. Since H is a C-convex set-valued mapping, by [1, Proposition 5.3.4], we have $\text{Gr}(D_c(\Lambda_2 \circ H)(\bar{x}, \Lambda_2(\bar{z}))) = \text{cl}(\text{Gr}(\Lambda_2(D_cH(\bar{x}, \bar{z}))))$. Then there exists $\{(u_n, v_n)\} \subset X \times Y$ with $v_n \in \Lambda_2(D_cH(\bar{x}, \bar{z})(u_n))$ such that $(u_n, v_n) \to (u, v)$. By (4.21), we have $v_n - T_4(u_n) \in Y \setminus (-\text{int}(C))$. Letting $n \to \infty$, we have $v - T_4(u) \in Y \setminus (-\text{int}(C))$. This implies that

$$(D_c(\Lambda_2 \circ H)(\bar{x}, \Lambda_2(\bar{z}))(u) - T_4(u)) \cap (-\text{int}(C)) = \emptyset, \qquad \forall\ u \in X.$$

Therefore, $T_4 \in \partial(\Lambda_2 \circ H)(\bar{x}, \Lambda_2(\bar{z}))$.

Since $T_2 = 0$, $T_3 = \langle x_3^*, \cdot\rangle c_0$, by (4.14), we have

$$\begin{aligned} 0 &= T_1 + T_4 + \langle x_3^*, \cdot\rangle c_0 \\ &\in \partial F(\bar{x}, \bar{y}) + \partial(\Lambda_2 \circ H)(\bar{x}, \Lambda_2(\bar{z})) \cap \mathcal{L} + N_c(\Omega, \bar{x})\text{int}(C). \end{aligned} \tag{4.22}$$

Letting $z^* := z_2^*$, $\Lambda := \Lambda_2$, then (i) holds. (ii)-(iv) follow from (4.18), (4.19) and (4.22). □

Corollary 4.5. *Suppose that $\bar{x} \in Dom(F) \cap \Omega$, $\bar{y} \in F(\bar{x})$, F is locally Lipschitz at $\bar{x}$ and $F(\bar{x})$ has upper bound. If $(\bar{x}, \bar{y})$ is a weak minimizer of optimization problem (SOP2), then*

$$0 \in \partial F(\bar{x}, \bar{y}) + N_c(\Omega, \bar{x})\,int(C).$$

Proof. In Theorem 4.4, we define $Z = Y$, $K = Y$, $H(x) = 0$, $\bar{z} = 0$. It is easy to see that all conditions of Theorem 4.4 are satisfied. By (4.22), it suffices to show that $T_4 = 0$. Since

$$T_4 = \langle x_4^*, \cdot\rangle c_0 \in \partial(\Lambda_2 \circ H)(\bar{x}, \Lambda_2(\bar{z})) = \partial(\mathbf{0})(\bar{x}, 0),$$

we have

$$-\langle x_4^*, u\rangle c_0 \notin -\mathrm{int}(C), \quad \forall\, u \in X.$$

The above inequality implies that $x_4^* = 0$, and so $T_4 = 0$. □

Theorem 4.6. *Let F be a C-convex set-valued mapping, Ω be a closed convex set, $\bar{x} \in D$, $\bar{y} \in F(\bar{x})$ and $\bar{z} \in H(\bar{x}) \cap (-K)$. If there exists $\Lambda \in \mathbf{B}(Z, Y)$ such that*

(i) $\Lambda(K) \subset C$;

(ii) $\Lambda(\bar{z}) = 0$;

(iii) $0 \in \partial F(\bar{x}, \bar{y}) + \partial(\Lambda \circ H)(\bar{x}, \Lambda(\bar{z})) \cap \mathcal{L} + N_c(\Omega, \bar{x})\mathit{int}(C)$,

then $(\bar{x}, \bar{y})$ is a weak minimizer of problem (SOP1).

Proof. Suppose that there exists $\Lambda \in \mathbf{B}(Z, Y)$ such that (i), (ii) and (iii) hold. It suffices to show that $(\bar{x}, \bar{y})$ is a weak minimizer of (SOP1). Suppose to the contrary that there exist $\tilde{x} \in \Omega$ with $H(\tilde{x}) \cap (-K) \neq \emptyset$ and $\tilde{y} \in F(\tilde{x})$ such that $\tilde{y} - \bar{y} \in -\mathrm{int}(C)$. By (iii), there exist $T_1 \in \partial F(\bar{x}, \bar{y})$, $T_2 \in \partial(\Lambda \circ H)(\bar{x}, \Lambda(\bar{z})) \cap \mathcal{L}$, $x_1^* \in N_c(\Omega, \bar{x})$ and $c_1 \in \mathrm{int}(C)$ such that $0 = T_1 + T_2 + \langle x_1^*, \cdot\rangle c_1$. Since $x_1^* \in N_c(\Omega, \bar{x})$, we have $\langle x_1^*, \tilde{x} - \bar{x}\rangle \leq 0$, and so

$$\langle x_1^*, \tilde{x} - \bar{x}\rangle c_1 \in -C.$$

Since H is a K-convex set-valued mapping, by (i), $\Lambda \circ H$ is a C-convex set-valued mapping. By Proposition 3.2 and condition (ii), we have $T_2 \in \partial(\Lambda \circ H)(\bar{x}, \Lambda(\bar{z})) = \partial^{\text{y-w}}(\Lambda \circ H)(\bar{x}, 0)$, and so

$$(\Lambda(H(\tilde{x})) - T_2(\tilde{x} - \bar{x})) \cap (-\mathrm{int}(C)) = \emptyset. \tag{4.23}$$

Take $\tilde{z} \in H(\tilde{x}) \cap (-K)$. By (i), there exists $\tilde{c} \in C$ such that $\Lambda(\tilde{z}) = -\tilde{c}$. Since $T_2 \in \mathcal{L}$, there exist $x_2^* \in X^*$ and $c_2 \in \mathrm{int}(C)$ such that $T_2 = \langle x_2^*, \cdot\rangle c_2$. Replacing $H(\tilde{x})$ by $\tilde{z}$ in (4.23), we obtain

$$-\tilde{c} - \langle x_2^*, \tilde{x} - \bar{x}\rangle c_2 \notin -\mathrm{int}(C).$$

Since $\tilde{c} \in C$ and $c_2 \in \mathrm{int}(C)$, the above relation implies that $\langle x_2^*, \tilde{x} - \bar{x}\rangle \leq 0$. Therefore,

$$\begin{aligned} \tilde{y} - \bar{y} - T_1(\tilde{x} - \bar{x}) &\in -\mathrm{int}(C) + T_2(\tilde{x} - \bar{x}) + \langle x_1^*, \tilde{x} - \bar{x}\rangle c_1 \\ &\subset -\mathrm{int}(C) + \langle x_2^*, \tilde{x} - \bar{x}\rangle c_2 + \langle x_1^*, \tilde{x} - \bar{x}\rangle c_1 \\ &\subset -\mathrm{int}(C) - C - C \subset -\mathrm{int}(C). \end{aligned} \tag{4.24}$$

On the other hand, since $T_1 \in \partial F(\bar{x}, \bar{y}) = \partial^{\text{y-w}} F(\bar{x}, \bar{y})$, we have

$$(F(\tilde{x}) - \bar{y} - T_1(\tilde{x} - \bar{x})) \cap (-\mathrm{int}(C)) = \emptyset.$$

Since $\tilde{y} - \bar{y} \in F(\tilde{x}) - \bar{y}$, we have

$$\tilde{y} - \bar{y} - T_1(\tilde{x} - \bar{x}) \notin -\mathrm{int}(C),$$

which contradicts (4.24). Therefore, $(\bar{x}, \bar{y})$ is a weak minimizer of (SOP1). □

Corollary 4.7. *Let F be a C-convex set-valued mapping, Ω be a closed convex set, $\bar{x} \in Dom(F) \cap \Omega$ and $\bar{y} \in F(\bar{x})$. If*

$$0 \in \partial F(\bar{x}, \bar{y}) + N_c(\Omega, \bar{x}) int(C), \tag{4.25}$$

then $(\bar{x}, \bar{y})$ is a weak minimizer of optimization problem (SOP2).

Proof. In Theorem 4.6, we define $Z = Y$, $K = \{0\}$, $\bar{z} = 0$, $H(x) = 0$ and $\Lambda = 0$. Then (4.25) holds by Theorem 4.6. □

Remark 4.8. In [9, Corollary 5.1], Li and Guo proved the following result. Let $\Omega = X$ and $\bar{x} \in D$. If $0 \in \mathrm{int}(H(X))$, $F \circ H^{-1}$ is C-convex, then for every $\bar{y} \in F(\bar{x})$, $(\bar{x}, \bar{y})$ is a weak minimizer of optimization problem (SOP1) if and only if there exists $\Lambda \in \mathbf{B}(Z, Y)$ such that $\Lambda(K) \subset C$, $\Lambda \circ (H(\bar{x}) \cap (-K)) = 0$, and $0 \in \partial^{\text{C-W}}(F + \Lambda \circ (H(\cdot) \cap (-K)))(\bar{x})$.

In [18, Theorem 4.1], Taa proved the following result. If F is C-convex, H is K-convex,

$$\inf\{\varphi_e(y) : y \in F(x) + C\} > -\infty, \quad \forall\, x \in \mathrm{Dom}(F),$$

and $\mathbb{R}_+(H(\mathrm{Dom}(F) \cap \Omega \cap \mathrm{Dom}(H)) + K)$ is a closed vector subspace of Z, then $(\bar{x}, \bar{y})$ is a weak minimizer of optimization problem (SOP1) if and only if for any $\bar{z} \in H(\bar{x}) \cap (-K)$, there exists $\Lambda \in \mathbf{B}(Z, Y)$ such that $\Lambda(K) \subset C$, $\Lambda(\bar{z}) = 0$, and $0 \in \partial^{\text{Y-W}}(F + \delta_\Omega + \Lambda \circ H)(\bar{x}, \bar{y})$, where $\varphi_e(y) = \inf\{t \in \mathbb{R} : y \in te - Y\}$ for a given element $e \in \mathrm{int}C$.

Now, we summarizes the differences between our results and the results mentioned above.

(i) Let us recall the classic Lagrange multiplier rule for convex optimization problems. Let $f, h : X \to \mathbb{R} \cup \{+\infty\}$ be two proper convex functions, Ω be a closed convex subset of X. Consider the following convex optimization problem:

$$\text{(SOP3)} \quad \begin{cases} \min & f(x), \\ \text{s.t.} & h(x) \le 0, \quad x \in \Omega. \end{cases}$$

Suppose that $D = \{x \in X : x \in \Omega,\ x \in \mathrm{dom}(f),\ h(x) \le 0\} \neq \emptyset$ and $\bar{x} \in D$. Consider the following statements:

(a) $\bar{x}$ is a solution of (SOP3);

(b) there exists $\mu \ge 0$ such that $0 \in \partial f(\bar{x}) + \mu \partial h(\bar{x}) + N_c(\Omega, \bar{x})$.

It follows from [14] that (b)⇒(a). Moreover, if $h(\bar{x}) < 0$, then (a) and (b) are equivalent. Our Lagrange multiplier rule is expressed by the subdifferential of all set-valued mappings in (SOP1), that is, $0 \in \partial F(\bar{x}, \bar{y}) + \partial(\Lambda \circ H)(\bar{x}, \Lambda(\bar{z})) \cap \mathcal{L} + N_c(\Omega, \bar{x})\mathrm{int}(C)$, which is consistent with the form of the classical Lagrange multiplier rule for (SOP3) and different from [9, Corollary 5.1] and [18, Theorem 4.1].

(ii) Theorem 4.4 and Corollary 4.5 do not require cone-convexity of F. Hence they can be applied to some nonconvex optimization problems. See the following example. In Theorem 4.4, let $X = Y = \mathbb{R}$, $C = K = [0, +\infty)$, $\Omega = [-\pi, 0]$, and $F, H : \mathbb{R} \rightrightarrows \mathbb{R}$ be defined as

$$F(x) = \left[x^2 |\sin x|,\ x^2 |\sin x| + 1\right], \quad \forall\, x \in \mathbb{R},$$

$$H(x) = [x,\ x + 1], \quad \forall\, x \in \mathbb{R}.$$

Clearly, F is not a C-convex set-valued mapping since

$$\frac{1}{2}F(0) + \frac{1}{2}F(-\pi) = [0, 1] \not\subseteq F\Big(\frac{1}{2}0 + \frac{1}{2}(-\pi)\Big) + C = \Big[\frac{\pi^2}{4}, +\infty\Big).$$

Therefore, [18, Theorem 4.1] can not be applied to optimization problem (SOP1). Similarly, $F \circ H^{-1}$ is not a C-convex set-valued mapping since

$$\frac{1}{2}(F \circ H^{-1})(0) + \frac{1}{2}(F \circ H^{-1})(-\pi) = \left[0, \frac{1}{2}\sin 1 + \frac{1}{2}(\pi+1)^2 \sin 1 + 1\right]$$
$$\not\subseteq (F \circ H^{-1})\Big(\frac{1}{2}0 + \frac{1}{2}(-\pi)\Big) + C = \Big[\frac{\pi^2}{4}, +\infty\Big).$$

Therefore, [9, Corollary 5.1] can not be applied to optimization problem (SOP1). Take $\bar{x} = \bar{y} = \bar{z} = 0$. Now, we verify that all conditions of Theorem 4.4 are satisfied. Clearly, $(\bar{x}, \bar{y})$ is a weak minimizer of (SOP1). Since

$$F(x_1) \subset F(x_2) + 3|x_1 - x_2|B_X, \quad \forall\, x_1, x_2 \in [-1, 1],$$

$$H(x_1) \subset H(x_2) + |x_1 - x_2|B_X, \quad \forall\, x_1, x_2 \in \mathbb{R},$$

F and H are locally Lipschitz at $\bar{x}$. $F(\bar{x})$, $H(\bar{x})$ have upper bound 1. By Example 4.3, the assumption (A) is satisfied. It is easy to calculate that

$$D_c H(\bar{x}, \bar{z})(u) = D_c H(0, 0)(u) = [u, +\infty), \quad \forall\, u \in \mathbb{R},$$

$$T_c(-K, \bar{z}) - D_c H(\bar{x}, \bar{z})(T_c(\Omega, \bar{x})) = (-\infty, +\infty).$$

All conditions of Theorem 4.4 are fulfilled.

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References

[1] J.P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, 1990.

[2] J. Baier and J. Jahn, On subdifferentials of set-valued maps, *J. Optim. Theory App.* 100 (1999) 233–240.

[3] J.M. Borwein, A Lagrange multiplier theorem and a sandwich theorem for convex relations, *Math. Scand.* 48 (1981) 189–204.

[4] G.Y. Chen and B.D. Craven, A vector variational inequality and optimization over an efficient set, *Math. Method Oper. Res.* 34 (1990) 1–12.

[5] G.Y. Chen and J. Jahn, Optimality conditions for set-valued optimization problems, *Math. Method Oper. Res.* 48 (1998) 187–200.

[6] F.H. Clarke, *Optimization and Nonsmooth Analysis*, John Wiley & Sons, New York, 1983.

[7] E. Hernández and L. Rodríguez-Marín, Weak and strong subgradients of set-valued maps, *J. Optim. Theory App.* 149 (2011) 352–365.

[8] J. Jahn, *Vector Optimization: Theory, Applications, and Extensions*, Springer, Berlin, 2004.

[9] S.J. Li and X.L. Guo, Weak subdifferential for set-valued mappings and it applications, *Nonlinear Anal-Theor.* 71 (2009) 5781–5789.

[10] X.J. Long, J.W. Peng and X.B. Li, Weak subdifferentials for set-valued mappings, *J. Optim. Theory App.* 162 (2014) 1–12.

[11] B.S. Mordukhovich, *Variational Analysis and Generalized Differentiation. I: Basic Theory*, Springer, Berlin, 2006.

[12] W.J. Peng, H.W.J. Lee, W.D. Rong and X.M. Yang, Hahn-Banach theorems and subgradients of set-valued maps, *Math. Method Oper. Res.* 61 (2005) 281–297.

[13] R.T. Rockafellar, *Conjugate duality and Optimization*, SIAM Coll Regional Conference Series in Applied Mathematics, Philadelphia, 1976.

[14] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1970.

[15] R.T. Rockafellar, *Convex Functions and Dual Extremum Problems*, Thesis Harvard, MA, 1963.

[16] W. Song, Weak subdifferential of set-valued mappings, *Optimization*, 52 (2003) 263–276.

[17] C. Swartz, Pschenichnyis' theorem for vector optimization, *J. Optim. Theory App.* 53 (1987) 309–317.

[18] A. Taa, On subdifferential calculus for set-valued mappings and optimality conditions, *Nonlinear Anal-Theor.* 74 (2011) 7312-7324.

[19] A. Taa, Subdifferentials of multifunctions and Lagrange multipliers for multiobjective optimization problems, *J. Optim. Theory App.* 283 (2003) 398-415.

[20] T. Tanino, Conjugate duality in vector optimization, *J. Math. Anal. Appl.* 167 (1992) 84–97.

[21] X.Q. Yang, A Hahn-Banach theorem in ordered linear spaces and its applications, *Optimization* 25 (1992) 1–9.

[22] C. Zălinescu, Hahn-Banach extension theorems for multifunctions revisited, *Math. Method Oper. Res.* 68 (2008) 493–508.

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