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OPTIMALITY CONDITIONS FOR BILEVEL OPTIMIZATION PROBLEM WITH BOTH LEVELS PROBLEMS BEING MULTIOBJECTIVE*

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Abstract: This paper mainly studies necessary optimality conditions for a class of bilevel optimization problem, of which both levels are multiobjective optimization problem. We first of all transform this problem into an auxiliary semivectorial bilevel programming problem, by applying scalarization approach for the lower level multiobjective optimization problem. Then by replacing the lower level scalar optimization problem by its KKT conditions, we get a necessary optimality condition for the bilevel multiobjective optimization problem. Moreover, using the optimal value function of the lower level scalar optimization problem, we transform the auxiliary semivectorial bilevel programming problem into a multiobjective optimization problem. Further, we obtain another necessary optimality condition for this bilevel multiobjective optimization problem. Optimality conditions for this bilevel problem are first obtained in this paper. The two optimality conditions could aid the design of algorithm and convergence analysis.

Key words: *bilevel programming, multiobjective programming, optimality conditions*

Mathematics Subject Classification: *90C29, 90C30, 90C33*

1 Introduction

Bilevel programming is an active research area in mathematical programming at present [1, 6, 13, 17, 18, 22, 31, 33]. It has a framework to deal with decision processes involving two decision makers with a hierarchical nesting structure. The upper level decision maker (leader) has the first choice, and the lower level decision maker (follower) reacts optimal solution to the learder's selection. In [9], Bonnel and Morgan concerned a semivectorial bilevel programming problem whose upper level is a scalar optimization problem and lower level is a multiobjective optimization problem. They proposed a penalty approach to solve it. Bonnel [8] derived a necessary optimality condition for the semivectorial bilevel optimization problem, while considering efficient and weakly efficient solutions for the lower level problem. Dempe and Gadni [16] derived another optimality conditions for the problem by transforming the lower level multiobjective optimization problem into a scalar optimization problem. Ye [34] discussed another semivectorial bilevel programming problem whose upper level is a multiobjective optimization problem and lower level is a scalar optimization

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problem. By transforming the semivectorial bilevel optimization problem into a mathematical programming with equilibrium constraint, she gave a necessary optimality condition. Some algorithms have been proposed to solve this semivectoral bilevel optimization problem [3, 26, 36].

As we know that, it is needed to consider a bilevel optimization problem whose both levels are multiobjective optimization problem, in transportation planning, management problems, supply chain problem, cloud market and so on. This can be see in [21, 30, 35]. This problem is named bilevel multiobjective optimization problem (BMOP) and different from the bilevel problem with upper-level multiobjective problem proposed in [34] by Ye.

In this paper we consider an optimistic BMOP described as follows. Let $\preceq_{R^l_+}$ be a partial order for vectors in R^l, and $\preceq_{\text{int}R^r_+}$ be a partial order for vectors in R^r, which are induced by R^l_+ and $\text{int}R^r_+$ respectively (See Section 2.2). We consider the following bilevel multiobjective optimization problem.

$$(BMOP) \quad \min_{x,y} F(x,y) \tag{1.1}$$

$$\begin{array}{rrcl} s.t. & G(x) & \leq & 0, \\ & H(x) & = & 0, \\ & y & \in & \psi_{we}(x), \end{array} \tag{1.2}$$

where $x \in R^n$ and $y \in R^m$ denote the upper level and the lower level decision variables respectively. $G : R^n \to R^{p_1}$ and $H : R^n \to R^{p_2}$ denote the upper level constraint function. $F : R^n \times R^m \to R^l$ is the upper level objective function. The term "min" here is used to symbolize that the upper level decision maker (leader) intends to minimize his objective function F with respective to the variable x while taking into account the reaction y of the lower level decision maker (follower). $\psi_{we}(x)$ is the weakly efficient optimal solutions set of the following parametric multiobjective optimization problem

$$\min_{y} \ f(x,y) \tag{1.3}$$

$$s.t. \quad g(x,y) \quad \leq \quad 0, \tag{1.4}$$

where $g : R^n \times R^m \to R^q$ is the lower level constraint function, $f : R^n \times R^m \to R^r$ is the lower level objective function. For simplicity, let

$$X := \{x \in R^n : G(x) \leq 0, H(x) = 0\},$$

denotes the upper level constraint set. The lower level constraint set w.r.t. x is given by

$$K(x) := \{y \in R^m : g(x,y) \leq 0\},$$

here $K : R^n \rightrightarrows R^m$ is a set-valued mapping. For $\bar{x} \in X$, a point $\bar{y} \in \psi_{we}(\bar{x})$ if (See Section 2.2)

$$f(\bar{x}, \bar{y}) - f(\bar{x}, y) \notin intR^r_+, \ \forall y \in K(x).$$

This model can be suitably applied in cloud market pricing. The cloud computing market pricing has evolved into a highly complex economic system made up of a variety of services, which are typically classified into three categories [4]: Infrastructure-as-a-Service (IaaS); Platform-as-a-Service (PaaS); Software-as-a-Service (SaaS). But the end user may deal directly with SaaS. Both end-users and SaaS wish their multi-criteria attain to optimum. Most of the time end-users and SaaS are noncooperative. If SaaS are able to influence the end-users' choice, then the price-equilibrium of end-user and SaaS can be modeled by BMOP. The same in principal-agent problem, of which bath principal and agent may wish their multi-criteria attain to the optimum. It also can be modeled by BMOP, if principal and agent are noncooperative, and principal is able to influence the agent's choice.

In the decade years, many algorithms ware proposed to solve this problem. Dempe and Franke [15] proposed a k-th best algorithm for a class of BMOP whose both levels are liner multiobjective optimization problem. Zheng, Wan and Wang [37] proposed a fuzzy interactive method to solve it in this linear case also. Alves and Costa [2] proposed a particle swarm algorithm to solve BMOP, same other evolutionary algorithms can been see in [11, 12, 39].

To the best of our knowledge, there are few results on the optimality conditions for the bilevel multiobjective optimization problem. Since optimality conditions are essential to the design of algorithm and the convergence analysis, thus in this paper, we will discuss the optimality conditions for BMOP. In order to get the optimality conditions for BMOP, we firstly need to construct an auxiliary semivectorial bilevel programming problem by transforming the lower level multiobjective programming problem into a scalar optimization problem. Then we transform the auxiliary semivectorial bilevel programming problem into two kinds of different multiobjective optimization problems by KKT approach and optimal value function approach respectively. Finally, we obtain two necessary optimality conditions via the two kinds of different multiobjective optimization problems.

The rest of this paper is organized as follows. In section 2, we recall some important results about variational analysis and multiobjective optimization. In section 3, we firstly transform the lower level multiobjective optimization problem into a scalar optimization problem by weighted method, so the bilevel multiobjective optimization problem turn into an auxiliary semivectorial bilevel programming problem whose upper level is a multiobjective optimization problem, and lower level is a scalar optimization problem. Then, we discuss the relationship between BMOP and the auxiliary semivectorial bilevel programming problem. In section 4, we replace the lower level scalar optimization problem of the auxiliary semivectorial bilevel programming problems by its KKT conditions, so the auxiliary semivectorial bilevel programming problem is transformed into a multiobjective optimization problem with equilibrium constraints. Then we give a necessary conditions for the existence of efficient solution for BMOP via this multiobjective optimization problem with equilibrium constraints. In section 5, we consider the optimal value function of the lower level scalar optimization problem of auxiliary bilevel optimization problem as a penalization, therefore the auxiliary semivectorial bilevel programming problem is transformed into a multiobjective optimization problem with a feasible region which satisfy MFCQ. Then the necessary conditions for the existence of efficient solution is considered via a direct approach.

2 Preliminaries

In this section, we mainly recall some basic definitions and results about variational analysis and multiobjective optimization problem, which play an important role in getting our main results.

2.1 Variational analysis

Definition 2.1 ([29]). Let Ω be a nonempty subset of a finite dimensional space Z, given $z \in \Omega$, the convex cone

$$N^{\pi}(z;\Omega) := \{\xi \in Z : \exists \sigma \geq 0, \text{ such that } \langle \xi, z' - z\rangle \leq \sigma \parallel z' - z \parallel^2 \ \ \forall z' \in \Omega\},$$

is called the proximal normal cone to set Ω at point z. The cone

$$\hat{N}(z;\Omega) = \{\xi : \langle \xi, z' - z\rangle \leq o(\|z' - z\|)\ \forall z' \in \Omega\},$$

is called regularity normal cone. The cone

$$N(z;\Omega) = \{\xi : \exists \xi_k \to \xi, z_k \to z(z_k \in \Omega) : \xi_k \in \hat{N}(z_k;\Omega)\},$$

is called the limiting (Mordukhovich) normal cone to Ω at point z.

Definition 2.2 ([27, 29])**.** Given a point $\bar{z}$, $\limsup_{z\to\bar{z}} \Xi(z)$ is said to be the Kuratowski-Painlevée outer upper limit of a set-valued mapping $\Xi : R^n \rightrightarrows R^m$ at $\bar{z}$, if

$$\limsup_{z\to\bar{z}} \Xi(z) := \{v \in R^m : \exists z_k \to \bar{z}, v_k \to \bar{v}\ with\ v_k \in \Xi(z_k)\ as\ k \to \infty\}.$$

Its graph gphΞ is denoted as follow:

$$gph\Xi := \{(u, v) \in R^n \times R^m : v \in \Xi(u)\}.$$

The coderivative of Ξ at $(\bar{u}, \bar{v}) \in ghp\Xi$ is a positively homogeneous mapping $D^*\Xi(\bar{u}, \bar{v}) : R^m \rightrightarrows R^n$ which is defined as follows:

$$D^*\Xi(\bar{u}, \bar{v})(v^*) := \{u^* \in R^n | (u^*, v^*) \in N((\bar{u}, \bar{v}), gph\Xi)\}, \quad \text{for } v^* \in R^m.$$

Definition 2.3 ([27, 29])**.** For an extended real-valued function $\psi : R^n \to \bar{R}$, $\hat{\partial}\psi(\bar{z})$ is said to be the Fréchet subdifferential of ψ at a point $\bar{z}$ of it's domain, if

$$\hat{\partial}\psi(\bar{z}) = \left\{ v \in R^n : \liminf_{z\to\bar{z}} \frac{\psi(z) - \psi(\bar{z}) - \langle v, z - \bar{z}\rangle}{\|z - \bar{z}\|} \geq 0 \right\},$$

given a point $\bar{z}$, $\partial\psi(\bar{z})$ is said to be the basic/Mordukovich subdifferential of ψ at $\bar{z}$, if

$$\partial\psi(\bar{z}) = \limsup_{z\to\bar{z}} \hat{\partial}\psi(z).$$

If ψ is convex, $\psi(\bar{z}) \neq \emptyset$, then $\partial\psi(\bar{z})$ reduce to the subdifferential in the sense of convex analysis:

$$\partial\psi(\bar{z}) = \{v \in R^n : \psi(z) - \psi(\bar{z}) \geq \langle v, z - \bar{z}\rangle, \forall z \in R^n\},$$

the two subdifferentials coincide in this case.

$\partial\psi(\bar{z})$ is nonempty and compact when ψ is local Lipschitz continuous, its convex hull is the Clarke subdifferential $\bar{\partial}\psi(\bar{z})$:

$$\bar{\partial}\psi(\bar{z}) = co\partial\psi(\bar{z}).$$

here,"co" stands for the convex hull of the set in question. Via this link between the basic and Clarke subdifferential, we have the following convex hull property which plays a important role in this paper:

$$co\partial(-\psi)(\bar{z}) = -co\partial\psi(\bar{z}). \tag{2.1}$$

Definition 2.4 ([29, 32])**.** Let Z, W be two finite dimensional spaces, $S : Z \rightrightarrows W$ be a set-valued mapping, and $(r, s) \in gphS$. Then

$$N_+(z; S(r)) := \{\lim_{k\to\infty} \xi_k \in N^\pi(z_k; S(r_k)), z_k \in S(r_k), z_k \to z, r_k \to r\},$$

is the extended normal cone to $S(r)$ at z. The mapping S is normally semicontinuous at (r, s) if

$$N_+(z; S(r)) = N(z; S(r)).$$

For a lower semicontinuous function $\psi : R^n \to \bar{R}$, the Mordukhvich subdifferential of ψ is defined by:

$$\partial\psi(\bar{z}) := \{z' \in R^n : (z', -1) \in N((\bar{z}, \psi(\bar{z})); epi\psi)\},$$

here, $epi\psi$ is the epigraph of ψ. If ψ is continuous differentiable, then $\partial\psi(\bar{z}) = \{\nabla\partial(\bar{z})\}$.

Let N_Ω denotes the set-valued mapping $x \mapsto N(x; \Omega)$. The coderivative $D^*N_\Omega(\bar{x}, \bar{y})(v^*)$ satisfied the following equation.

$$\xi \in D^*N_\Omega(\bar{x}, \bar{y})(v^*) \Leftrightarrow (\xi, -v^*) \in N((\bar{x}, \bar{y}); gphN_\Omega).$$

Proposition 2.5 ([29])**.** *Let $X \subset R^n$ and $D \subset R^m$ be two closed sets, $F : R^n \to R^m$ be a continuously differentiable mapping. Here $F(x) = (f_1(x), \cdots, f_m(x))$. Let $C = \{x \in X : F(x) \in D\}$, at any $\bar{x} \in C$ one has*

$$\hat{N}(\bar{x}; C) \supset \{\sum_{i=1}^m y_i \nabla f_i(\bar{x}) + z : y \in \hat{N}(F(\bar{x}); D), z \in \hat{N}(\bar{x}; X)\},$$

where $y = (y_1, y_2, \cdots, y_m)$. On the other hand, one has

$$N(\bar{x}; C) \subset \{\sum_{i=1}^m y_i \nabla f_i(\bar{x}) + z : y \in N(F(\bar{x}); D), z \in N(\bar{x}; X)\},$$

at any $\bar{x}$ satisfying the constraint qualification that, the only vector $y \in N(F(\bar{x}); D)$ for which

$$-\sum_{i=1}^m y_i \nabla f_i(\bar{x}) \in N(\bar{x}; X),$$

is $y = (0, \cdots, 0)$.

$\boxed{2.2}$ Multiobjective optimization

Definition 2.6 ([20, 24])**.** Let M be a finite dimensional Banach space $K \subseteq M$ be a closed convex cone with nonempty interior. K is said to be pointed convex cone if $K \cap -K = \{0\}$. We say the partial order relation for two vector $x, y \in W$ is defined by $x \preceq_K y$ if and only if $y - x \in K$ and $x \neq y$.

Remark 2.7. In this paper the term " $\preceq_K$ " stands for the partial order induced by K.

Definition 2.8 ([34])**.** Let $l(r) := \{t \in W : t \preceq_K r\}$ denotes the level set at $r \in W$ with respect to the given partial order $\preceq_K$, $B(\delta, r)$ with radius $\delta > 0$ sufficiently small denotes the neighborhood of r. We say that a partial order $\preceq_K$ is closed around $\bar{r} \in W$ provided that:

(i) for any $r \in W$, there exist a $\delta > 0$, such that $r \in \overline{l(r)}$, for all $r \in B(\delta, \bar{r})$, here $\overline{l(r)}$ denotes the closure of the level set $l(r)$;

(ii) for any $r \preceq_K s$, $t \in \overline{l(r)}$ implies that $t \preceq_K s$.

Definition 2.9 ([32]). We say that a partial order $\preceq_K$ is regular at $\bar{r} \in W$, provided that
(i) for any $r \in W$, $r \in \overline{l(r)}$;
(ii) for any $r \preceq_K s$, $t \in \overline{l(r)}$ implies that $t \preceq_K s$;
(iii) for any sequences r_k, $\theta_k \to \bar{r}$ in W,

$$\{\lim_{k\to\infty} v_k^* : v_k^* \in N(\theta_k; \overline{l(r_k)})\} \subset N(\bar{r}; \overline{l(\bar{r})}).$$

Proposition 2.10. *If a partial order $\preceq_K$ is induced by R^n_+, then it is a closed and regular partial order.*

Proof. Combing Definition 2.8 and Definition 2.9 it is easy to get the result. □

Proposition 2.11. *If $W = R^n$ and the partial order $\preceq$ is induced by R^n_+, then the following equality hold:*

$$N_+(r^*; \overline{l(r^*)}) = N(r^*; \overline{l(r^*)}).$$

Proof. Combing Definition 2.8 and Proposition 2.10, we see that $\preceq_{R^n_+}$ is closed and regular. By Example 3.8 in [38], we can get the results easily. □

Let M be a finite dimensional Banach space, $K \subseteq M$ be a closed convex point cone with nonempty interior, $\preceq_K$ be a partial order for vectors in M. Considering the the following multiobjective optimization problem with abstract constraint (See Chapter 4 in [23])

$$\min_z f(z) \quad s.t.\ z \in Z \tag{2.2}$$

where $f : R^n \to M$ is a vector-valued function and Z is a nonempty feasible set.

Definition 2.12 ([25]). The point $z^* \in Z$ is said to be an efficient (resp. weakly efficient) solution of problem (2.2) if there is no other feasible point $z \in Z$ such that $f(z) \preceq_K f(z^*)$ (resp. $f(z) \preceq_{\text{int}K} f(z^*)$).

Definition 2.13 ([25]). The point $z^* \in Z$ is said to be an local efficient (resp. weakly efficient) solution of problem (2.2) if there exists a neighborhood $B(\delta, z^*)$ of z^* with radius $\delta > 0$ sufficiently small such that there is no other feasible point $z \in Z \cap B$ such that $f(z) \preceq_K f(z^*)$ (resp. $f(z) \preceq_{\text{int}K} f(z^*)$).

Definition 2.14. [25] We said a vector-valued function $f : R^n \to R^m$ is called K-convex with respect to a partial order $\preceq_K$ induced by a closed convex pointed cone K, if we have

$$f(\lambda z_1 + (1-\lambda)z_2) \preceq_K \lambda f(z_1) + (1-\lambda)f(z_2), \quad \forall z_1, z_2 \in R^n, \forall \lambda \in (0,1).$$

Next we will define the efficient (local efficient) solution of BMOP (1.1)-(1.2) the same as Definition 2.12 and Definition 2.13. Let

$$\Upsilon := \{(x,y) : x \in R^n, y \in R^m, G(x) \le 0, H(x) = 0, y \in \psi_{we}(x)\}$$

be the feasible set of bilevel level multiobjective optimization problem.

Definition 2.15. The vector $(x^*, y^*) \in \Upsilon$ is said to be an efficient solution of BMOP (1.1)-(1.2), if there is no other vector $(x,y) \in \Upsilon$ such that $F(x,y) \preceq_{R^l_+} F(x^*, y^*)$.

Definition 2.16. The vector $(x^*, y^*) \in Z$ is said to be an local efficient solution of problem BMOP (1.1)-(1.2), if there exists a neighborhood $B(\delta, (x^*, y^*))$ of (x^*, y^*) with radius $\delta > 0$ sufficiently small such that there is no vector $(x,y) \in \Upsilon \cap B$ such that $F(x,y) \preceq_{R^l_+} F(x^*, y^*)$.

Remark 2.17. We pointed out that even though the lower level problem is solved weakly, the solution of the BMOP (1.1)-(1.2) is said to be efficient.

Consider the multiobjective optimization problem with equilibrium constraints (MOPEC) defined as follows:

$$(MOPEC) \quad \min_z F(z) \tag{2.3}$$

$$\begin{aligned} s.t. \quad & g(z) \leq 0,\ h(z) = 0, \\ & G(z) \geq 0,\ H(z) \geq 0,\ G(z)^\top H(z) = 0, \end{aligned} \tag{2.4}$$

here, M is a finite dimensional Banach space, $K \in M$ is a closed convex point cone with nonempty interior. $\preceq_K$ is a partial order for vectors in M. $F : R^n \to M$, $g : R^n \to R^p$, $h : R^n \to R^q$, $G : R^n \to R^m$, $H : R^n \to R^m$. We assume that F is Lipschitz continuous near z^* and all of other functions are continuously differentiable. Let $C = \{z \in R^n : g(z) \leq 0, h(z) = 0, G(z) \geq 0, H(z) \geq 0, G(z)^T H(z) = 0\}$. Then, MOPEC (2.3)-(2.4) can be described as the following multiobjective optimization problem with abstract constraint:

$$(MOP) \quad \min_z F(z) \tag{2.5}$$

$$s.t. \quad z \in C. \tag{2.6}$$

Proposition 2.18 ([32])**.** *Let z^* be a local solution of the multiobjective optimization problem (2.5)-(2.6). Supposed that F is Lipschitz continuous at a neighborhood $B(\delta, z^*)$ with radius $\delta > 0$ sufficiently small of z^* and the partial order $\preceq_K$ is regular at $F(z^*)$. Then there exists a unit vector $\lambda \in N_+(F(z^*), \overline{l(F(z^*))})$, and $\rho_0 \in \{0, 1\}$ such that*

$$0 \in \rho_0 \partial \langle \lambda, F \rangle (z^*) + N(z^*; C). \tag{2.7}$$

If $C = \{z \in R^n : g(z) \leq 0, h(z) = 0\}$ condition (2.7) also holds. In this papaer unit vector λ means $\|\lambda\| = 1$

In the next section we will translate the BMOP (1.1)-(1.2) into an auxiliary semivector bilevel programming problem (3.4)-(3.5), and consider the relationship between the two bilevel programming problems.

3 The transformation of BMOP

In this section, we will discuss the reformulation of BMOP (1.1)-(1.2). Firstly, we transform the lower level parametric multiobjective optimization problem (1.3)-(1.4) into a parametric scalar optimization problem by using scalar technique.

$$\min_y \bar{f}(x, y, z) := \langle z, f(x, y) \rangle \tag{3.1}$$

$$s.t. \quad y \in K(x), \tag{3.2}$$

where $K(x)$ was defined in Section 1, and the parameter z is a nonnegative vector of the unit space:

$$z \in Z := \{z \in R^r : z \geq 0, z^\top e^r = 1\}, \tag{3.3}$$

here e^r means a r dimensional vector with ith component equal to 1.

Since it is very difficult to find the best $y(x)$ on the Pareto front for a given upper level strategy x, we consider the set Z in (3.3) as a constraint set for the upper-level problem, the same as [16]. Let $\varphi : X \times Z \to R$ denotes the optimal valued function of optimization problem (3.1)-(3.2):

$$\varphi(x, z) := \min_y \{\langle z, f(x, y)\rangle : y \in K(x)\},$$

and $\psi : X \times Z \rightrightarrows Y$ denotes the solution set mapping of this problem:

$$\psi(x, z) := \{y \in K(x) : \langle z, f(x, y)\rangle \leq \varphi(x, z)\}.$$

In order to built a bridge between the solution set of the parametric multiobjective optimization problem and the solution set of its parametric scalar optimization problem, we give the following theorem.

Theorem 3.1 ([16]). *Assume that the functions $y \mapsto g(x, y)$ and $y \mapsto f(x, y)$ are R^q_+-convex and R^r_+-convex, respectively. Then, we have*

$$\psi_{we}(x) = \psi(x, Z) := \bigcup\{\psi(x, z) : z \in Z\}.$$

According to the above analysis, the bilevel multiobjective programming problem (1.1)-(1.2) can be transformed into the following auxiliary semivectorial bilevel programming problem:

$$\min_{x,y,z} F(x, y) \tag{3.4}$$

$$s.t. \quad x \in X, z \in Z, y \in \psi(x, z). \tag{3.5}$$

Next, we will discuss the relationship between auxiliary semivectorial bilevel programming problem (3.4)-(3.5) and BMOP (1.1)-(1.2). In order to show it more clearly, we defined another set-valued mapping $\Gamma : X \times Y \rightrightarrows R^r$ as follows:

$$\Gamma(x, y) := \{z \in Z : y \in \psi(x, z)\}.$$

Before showing the main theorem of this section, we need to firstly prove the following lemma which plays an important role in the main theorem.

Lemma 3.2. *Let f be a continuous function, $K(x)$ be a lower semicontinuous and closed mapping at $\bar{x}$. Then φ is upper semicontinuous at $(\bar{x}, z)$ where $z \in Z$, and ψ is closed at $(\bar{x}, z)$.*

Proof. Since f is a continuous function, it follows that, for any $z \in Z$, $\bar{f}(x, y, z) = \langle z, f(x, y)\rangle$ is also a continuous function. According to Theorem 4.2.1 and Theorem 4.2.2 in [5] we can get the results easily. □

Theorem 3.3. (i) *Let $(\bar{x}, \bar{y})$ be a local efficient solution (efficient solution) of BMOP (1.1)-(1.2). Assume that the functions $y \mapsto g(x, y)$ and $y \mapsto f(x, y)$ are R^q_+-convex and R^r_+-convex, respectively. Then for any $\bar{z} \in \Gamma(x, y)$, the point $(\bar{x}, \bar{y}, \bar{z})$ is a local efficient solution (efficient solution) of auxiliary semivectorial bilevel programming problem (3.4)-(3.5).*

(ii) *Let $(\bar{x}, \bar{y}, \bar{z})$ be an efficient solution of auxiliary semivectorial bilevel programming problem (3.4)-(3.5). Assume that the functions $y \mapsto g(x, y)$ and $y \mapsto f(x, y)$ are R^q_+-convex and R^r_+-convex, respectively. Then $(\bar{x}, \bar{y})$ is an efficient solution of BMOP (1.1)-(1.2).*

(iii) *Let $(\bar{x}, \bar{y}, z)$ be a local efficient solution of auxiliary semivectorial bilevel programming problem (3.4)-(3.5) for all $z \in \Gamma(\bar{x}, \bar{y})$. Assume that the functions $y \mapsto g(x, y)$ and $y \mapsto f(x, y)$ are R^q_+-convex and R^r_+-convex, respectively. If $K(x)$ is lower semicontinuous and closed mapping at $\bar{x}$, then $(\bar{x}, \bar{y})$ is a local efficient solution of BMOP (1.1)-(1.2).*

Proof. (i) We only need to prove the case of local efficient solution, and the case of efficient solution can be got by choosing $\delta = \infty$.

Supposed that $(\bar{x}, \bar{y})$ is a local efficient solution of BMOP (1.1)-(1.2), then we can find a $\delta > 0$, and there is no vector $(x, y) \in B(\delta, (\bar{x}, \bar{y}))$ feasible to BMOP (1.1)-(1.2), such that

$$F(x, y) \preceq_{R^l_+} F(\bar{x}, \bar{y}).$$

On the contrary, supposed that there exists a $\bar{z} \in \Gamma(\bar{x}, \bar{y})$ such that $(\bar{x}, \bar{y}, \bar{z})$ is not a local efficient solution of auxiliary semivectorial bilevel programming problem (3.4)-(3.5), then for δ, there exists at least one vector $(x^*, y^*, z^*) \in B(\delta, (\bar{x}, \bar{y}, \bar{z}))$ feasible to auxiliary semivectorial bilevel programming problem (3.4)-(3.5), such that $F(x^*, y^*) \preceq_{R^l_+} F(\bar{x}, \bar{y})$. Combining Theorem 3.1 and $y^* \in \psi(x^*, z^*)$, we can get that $y^* \in \psi_{we}(x^*)$. This contradicts the fact that $(\bar{x}, \bar{y})$ is a local efficient solution of BMOP (1.1)-(1.2).

(ii) Let $(\bar{x}, \bar{y}, \bar{z})$ be a global efficient solution of auxiliary semivectorial bilevel programming problem (3.4)-(3.5). On the contrary, supposed that $(\bar{x}, \bar{y})$ is not an efficient solution of BMOP (1.1)-(1.2), then we can find at least one vector $(x, y) \in X \times Y$ which is a feasible vector of BMOP (1.1)-(1.2), such that

$$F(x, y) \preceq_{R^l_+} F(\bar{x}, \bar{y}).$$

Combining $y \in \psi_{we}(x)$ and Theorem 3.1 we can derive that, there exist a $z \in Z$ such that (x, y, z) is a feasible vector of auxiliary semivectorial bilevel programming problem (3.4)-(3.5). This contradicts the fact that $(\bar{x}, \bar{y}, \bar{z})$ is an efficient solution problem (3.4)-(3.5).

(iii) Let $(\bar{x}, \bar{y}, \bar{z})$ be a local efficient solution of auxiliary semivectorial bilevel programming problem (3.4)-(3.5), then there exist a $\delta > 0$, and there is no vector $(x, y, z) \in B(\delta, (\bar{x}, \bar{y}, \bar{z}))$ feasible to problem (3.4)-(3.5), such that

$$F(x, y) \preceq_{R^l_+} F(\bar{x}, \bar{y}).$$

On the contrary, supposed that $(\bar{x}, \bar{y})$ is not a local efficient solution of BMOP (1.1)-(1.2), then we can find $x_k \in X$, $x_k \to \bar{x}$, $y_k \in Y$, $y_k \to \bar{y}$ such that (x_k, y_k) is a feasible vector of BMOP (1.1)-(1.2) and

$$F(x_k, y_k) \preceq_{R^l_+} F(\bar{x}, \bar{y}).$$

According to Theorem 3.1 we can see that there exist $z_k \in Z$ such that all $\{(x_k, y_k, z_k)\}$ are feasible vectors of problem (3.4)-(3.5) for every k. Since Z is a compact set, there exists a subsequence $\{z_{k_v}\}$ of $\{z_k\}$ such that $z_{k_v} \to \bar{z}$, $\bar{z} \in Z$. Combining the lower semicontinuous and closed property of $K(x)$ and Lemma 3.2 we can derived that ψ is closed at $(\bar{x}, \bar{z})$. Duo to the closed property of ψ at $(\bar{x}, \bar{z})$, we have $\bar{y} \in \psi(\bar{x}, \bar{z})$, so, $(\bar{x}, \bar{y}, \bar{z})$ is a feasible vector of auxiliary semivectorial bilevel programming problem (3.4)-(3.5), but it is not a local efficient solution. This is a contradiction. □

4 Necessary condition via KKT approach

In this part, we will firstly transform auxiliary semivectorial bilevel programming problem (3.4)-(3.5) into a single level multiobjective problem by using KKT approach. Then we will consider the necessary condition of the existence of efficient solution.

As we know that the KKT conditions of nonlinear programming are not always necessary and sufficient (See Chapter 3 in [7]), therefore we give the following definition. It can be used to guarantee that the bilevel programming problem which the lower level is a convex optimization problem is equal to its transformation problem, this can be see in [14].

Definition 4.1 ([29])**.** We say that $K(x)$ is satisfied Slater's constraint qualification (Slater's CQ) if $x \in R^n$ there exists $\bar{y}(x) \in R^m$ such that $g_i(x, \bar{y}(x)) < 0$, $i = 1, 2, \cdots, q$.

Since the KKT conditions of problem (3.1)-(3.2) are taken as follows:

$$z^\top \nabla_y f(x, y) + \mu^\top \nabla_y g(x, y) = 0, \quad \mu \geq 0, \quad \langle g(x, y), \mu \rangle = 0.$$

Therefore, auxiliary semivectorial bilevel programming problem (3.4)-(3.5) can be changed into the following single level multiobjective optimization problem with equilibrium constraint (4.1):

$$\begin{aligned} \min_{x,y,z,\mu} \ & F(x, y) \\ s.t. \ \ & G(x) \leq 0, \ H(x) = 0, \\ & -z \leq 0, \ z^\top e^r = 1, \\ & z^\top \nabla_y f(x, y) + \mu^\top \nabla_y g(x, y) = 0, \\ & \mu \geq 0, \ \langle g(x, y), \mu \rangle = 0, \\ & g(x, y) \leq 0. \end{aligned} \tag{4.1}$$

Next we will consider the relationship between auxiliary semivectorial bilevel programming problem (3.4)-(3.5) and multiobjective optimization problem (4.1).

Theorem 4.2. (i) *Let $(\bar{x}, \bar{y}, \bar{z})$ be a local efficient solution (efficient solution) of auxiliary semivectorial bilevel programming problem (3.4)-(3.5). Supposed that the functions $y \mapsto g(x, y)$ and $y \mapsto f(x, y)$ are R^q_+-convex continuously differentiable, and R^r_+-convex continuously differentiable respectively. Also supposed that Slater's CQ is satisfied for the lower level parametric problem at $\bar{x}$. Then for each*

$$\bar{\mu} \in \Lambda(\bar{x}, \bar{y}, \bar{z}) := \{\mu \geq 0 : \bar{z}^\top \nabla_y f(\bar{x}, \bar{y}) + \mu^\top \nabla_y g(\bar{x}, \bar{y}) = 0, \quad \mu \geq 0, \quad \langle g(\bar{x}, \bar{y}), \mu \rangle = 0\}$$

the point $(\bar{x}, \bar{y}, \bar{z}, \bar{\mu})$ is a local efficient solution (efficient solution) of single level multiobjective optimization problem (4.1).

(ii) *Let $(\bar{x}, \bar{y}, \bar{z}, \bar{\mu})$ be an efficient solution of multiobjective optimization problem (4.1). Supposed that the functions $y \mapsto g(x, y)$ and $y \mapsto f(x, y)$ are R^q_+-convex continuously differentiable, and R^r_+-convex continuously differentiable respectively. Also assumed that Slater's CQ is satisfied for the lower level parametric problem for all $x \in X$. Then $(\bar{x}, \bar{y}, \bar{z})$ is an efficient solution of auxiliary semivectorial bilevel programming problem (3.4)-(3.5).*

(iii) *Let $(\bar{x}, \bar{y}, \bar{z}, \bar{\mu})$ be a local efficient solution of multiobjective optimization problem (4.1) for all $\bar{\mu} \in \Lambda(\bar{x}, \bar{y}, \bar{z})$. Supposed that the functions $y \mapsto g(x, y)$ and $y \mapsto f(x, y)$ are R^q_+-convex continuously differentiable, and R^r_+-convex continuously differentiable respectively. Also assumed that Slater's CQ is satisfied for the lower level parametric problem at $\bar{x}$. Then $(\bar{x}, \bar{y}, \bar{z})$ is a local efficient solution of auxiliary semivectorial bilevel programming problem (3.4)-(3.5).*

Proof. For the proof of condition (i) and (ii). Since the function $y \mapsto f(x, y)$ is R^r_+-convex, it follows that $\bar{f}(x, y, z) = \langle z, f(x, y)\rangle$ is convex function w.r.t y, for any $z \in Z$, $x \in X$. From the R^q_+-convexity of $y \mapsto g(x, y)$, we know that the lower level optimization problem (3.1)-(3.2) is a convex parametric optimization problem which satisfied Slater's CQ, so the KKT conditions of the lower level parametric optimization problem are sufficient and necessary. The remainder of the argument is analogous to that in proposition 3.1 [34], so we omit it here.

For the proof of condition (iii). Generalizing Theorem 3.2 in [14] we can get condition (iii) easily, so we also omit it here.

□

Remark 4.3. Since the Slater's CQ is used to guarantee the existence of KKT conditions, we can replace the Slater's CQ in Theorem 4.2 by more weaker CQ such as Cottle constraint qualification (definition see [10]). This also can see from (ii) of Remark 3.1 in [34].

Remark 4.4. The assumption of convexity of lower level problem is used to guarantee the sufficient of KKT conditions, KKT point is also a globally optimal solution under this assumption.

In the next of this section, we will discuss the necessary optimality condition of BMOP (1.1)-(1.2). Before doing that we firstly need to get the necessary conditions of the existence of efficient solution of multiobjective optimization problem with equilibrium constraint (4.1), this can be used as a bridge.

Let $(\bar{x}, \bar{y}, \bar{z}, \bar{\mu})$ be a feasible point of multiobjective optimization problem (4.1). For simplicity, we define the following index sets:

$$I_G = I_G(\bar{x}, \bar{y}, \bar{z}, \bar{\mu}) = \{i : G_i(\bar{x}) = 0\},$$

$$I_{\bar{z}} = I_z(\bar{x}, \bar{y}, \bar{z}, \bar{\mu}) = \{i : \bar{z}_i = 0\},$$

$$I_{0+} = I_{0+}(\bar{x}, \bar{y}, \bar{z}, \bar{\mu}) = \{i : g_i(\bar{x}, \bar{y}) = 0, \bar{\mu}_i > 0\},$$

$$I_{+0} = I_{+0}(\bar{x}, \bar{y}, \bar{z}, \bar{\mu}) = \{i : g_i(\bar{x}, \bar{y}) < 0, \bar{\mu}_i = 0\},$$

$$I_{00} = I_{00}(\bar{x}, \bar{y}, \bar{z}, \bar{\mu}) = \{i : g_i(\bar{x}, \bar{y}) = 0, \bar{\mu}_i = 0\}.$$

We now give a necessary optimality conditions for multiobjective optimization problem with equilibrium constraint (4.1)

Theorem 4.5. *Let $(\bar{x}, \bar{y}, \bar{z}, \bar{\mu})$ be a local efficient solution of problem (4.1). Assume that F, G, H are continuously differentiable. Also assume that f, g are continuously differentiable w.r.t x at a neighbourhood of $(\bar{x}, \bar{y})$, and twice continuously differentiable w.r.t y at a neighbourhood of $(\bar{x}, \bar{y})$. Then there exist $\mu_0 \in \{0, 1\}$, $\eta^{kkt} \in R^m$, $\eta^G \in R^{p_1}$, $\eta^H \in R^{p_2}$, $\eta^g \in R^q$, $\eta^{ze} \in R$, $\eta^z \in R^r$ not all zero and a unit vector $\lambda \in N(F(\bar{x}, \bar{y}, \bar{z}, \bar{\mu}), \overline{l(F(\bar{x}, \bar{y}, \bar{z}, \bar{\mu}))})$ such that*

$$\mu_0\lambda^\top\nabla_x F(\bar{x}, \bar{y}) + \nabla_x(\bar{z}^\top\nabla_y f + \bar{\mu}^\top\nabla_y g)(\bar{x}, \bar{y})^\top\eta^{kkt} + \nabla H(\bar{x})^\top\eta^H + \nabla G(\bar{x})^\top\eta^G + \nabla_x g(\bar{x}, \bar{y})^\top\eta^g = 0, \tag{4.2}$$

$$\mu_0\lambda^\top\nabla_y F(\bar{x}, \bar{y}) + \nabla_y(\bar{z}^\top\nabla_y f + \bar{\mu}^\top\nabla_y g)(\bar{x}, \bar{y})^\top\eta^{kkt} + \nabla_y g(\bar{x}, \bar{y})^\top\eta^g = 0, \tag{4.3}$$

$$\eta^{ze}e^r - \eta_z + \nabla_y f(\bar{x}, \bar{y})^\top\eta^{kkt} = 0, \tag{4.4}$$

$$\eta_i^g = 0, \forall i \in I_{+0}, \quad (\nabla_y g(\bar{x}, \bar{y})^\top \eta^{kkt})_i = 0, \forall i \in I_{0+}, \tag{4.5}$$

$$\eta_i^G \geq 0, \forall i \in I_G, \quad \eta_i^G = 0, \forall i \notin I_G, \quad \eta_i^z \geq 0, \forall i \in I_z, \quad \eta_i^z = 0, \forall i \notin I_z, \tag{4.6}$$

$$either\ \eta_i^g > 0, (\nabla_y g(\bar{x}, \bar{y})^\top \eta^{kkt})_i > 0\ or\ \eta_i^g (\nabla_y g(\bar{x}, \bar{y})^\top \eta^{kkt})_i = 0, \forall i \in I_{00}. \tag{4.7}$$

Proof. Noting that problem (4.1) is a multiobjective optimization problem with equilibrium constraint, it follows from Theorem 5.2 and Theorem 5.3 in [32] that, there exist $\mu_0 \in \{0, 1\}$, $\eta^{kkt} \in R^m$, $\eta^G \in R^{p_1}$, $\eta^H \in R^{p_2}$, $\eta^g \in R^q$, $\eta^{ze} \in R$, $\eta^z \in R^r$, $\eta^\mu \in R^q$ not all zero and a unit vector $\lambda \in N(F(\bar{x}, \bar{y}, \bar{z}, \bar{\mu}), \overline{l(F(\bar{x}, \bar{y}, \bar{z}, \bar{\mu}))})$ such that

$$\begin{aligned} &\mu_0 \lambda^\top \nabla_{x,y,z,\mu} F(\bar{x}, \bar{y}) + \nabla_{x,y,z,\mu} (\bar{z}^\top \nabla_y f + \bar{\mu}^\top \nabla_y g)(\bar{x}, \bar{y})^\top \eta^{kkt} + \nabla_{x,y,z,\mu} H(\bar{x})^\top \eta^H + \\ &\quad \nabla_{x,y,z,\mu} G(\bar{x})^\top \eta^G + \nabla_{x,y,z,\mu} g(\bar{x}, \bar{y})^\top \eta^g + \nabla_{x,y,z,\mu} (-z)^\top \eta^z + \nabla_{x,y,z,\mu} (-\mu)^\top \eta^z = 0, \end{aligned} \tag{4.8}$$

$$\eta_i^G \geq 0, \forall i \in I_G, \quad \eta_i^G = 0, \forall i \notin I_G,$$

$$\eta_i^z \geq 0, \forall i \in I_z, \quad \eta_i^z = 0, \forall i \notin I_z,$$

$$(-\eta_i^\mu, \eta_i^g) \in N((\bar{\mu}, g(\bar{x}, \bar{y})); gphN_{R_+^q}).$$

Further calculation of the partial derivatives we have

$$\begin{gathered} \mu_0 \lambda^\top \nabla_x F(\bar{x}, \bar{y}) + \nabla_x (\bar{z}^\top \nabla_y f + \bar{\mu}^\top \nabla_y g)(\bar{x}, \bar{y})^\top \eta^{kkt} \\ + \nabla H(\bar{x})^\top \eta^H + \nabla G(\bar{x})^\top \eta^G + \nabla_x g(\bar{x}, \bar{y})^\top \eta^g = 0, \end{gathered} \tag{4.9}$$

$$\mu_0 \lambda^\top \nabla_y F(\bar{x}, \bar{y}) + \nabla_y (\bar{z}^\top \nabla_y f + \bar{\mu}^\top \nabla_y g)(\bar{x}, \bar{y})^\top \eta^{kkt} + \nabla_y g(\bar{x}, \bar{y})^\top \eta^g = 0, \tag{4.10}$$

$$\eta^{ze} e^r - \eta_z + \nabla_y f(\bar{x}, \bar{y})^\top \eta^{kkt} = 0, \tag{4.11}$$

$$\eta^\mu - \nabla_y g(\bar{x}, \bar{y})^\top \eta^{kkt} = 0, \tag{4.12}$$

$$\eta_i^G \geq 0, \forall i \in I_G, \quad \eta_i^G = 0, \forall i \notin I_G, \tag{4.13}$$

$$\eta_i^z \geq 0, \forall i \in I_z, \quad \eta_i^z = 0, \forall i \notin I_z, \tag{4.14}$$

$$(-\eta_i^\mu, \eta_i^g) \in N((\bar{\mu}, g(\bar{x}, \bar{y})); gphN_{R_+^q}). \tag{4.15}$$

Equality (4.12) is the partial derivative of (4.8) with respect to μ. Since we can derive that $\eta^\mu = \nabla_y g(\bar{x}, \bar{y})^\top \eta^{kkt}$, so we can replace η^μ with $\nabla_y g(\bar{x}, \bar{y})^\top \eta^{kkt}$. This is the reason why there is no η^μ in (4.2)-(4.7). From Proposition 1.4 in [32] we can calculate $N((\bar{\mu}, g(\bar{x}, \bar{y})); gphN_{R_+^q})$ as follow

$$\begin{aligned} &N((\bar{\mu}, g(\bar{x}, \bar{y})); gphN_{R_+^q}) \\ &= \left\{ (-\eta_i^\mu, \eta_i^g) \in R^{2q} : \begin{array}{ll} \eta_i^\mu = 0 & if\ i \in I_{0+} \\ \eta_i^g = 0 & if\ i \in I_{+0} \\ ether\ \eta_i^g > 0\ and\ \eta_i^\mu > 0\ or\ \eta_i^g \eta_i^\mu = 0 & if\ i \in I_{00} \end{array} \right\}. \end{aligned} \tag{4.16}$$

Combining (4.9)-(4.15) and (4.16) we can obtain conditions (4.2)-(4.7) easily. ∎

If $\mu_0 = 0$, then there is no information about $F(x, y)$ in the conditions (4.2)-(4.7) of Theorem 4.5. To avoid this situation, we can add an assumption to guarantee $\mu_0 = 1$.

Corollary 4.6. *If there is no nonzero vector* $(\eta^{kkt}, \eta^G, \eta^H, \eta^g, \eta^{ze}, \eta^z) \in R^{m+p_1+p_2+q+1}$ *(here* $\eta^{kkt} \in R^m$*,* $\eta^G \in R^{p_1}$*,* $\eta^H \in R^{p_2}$*,* $\eta^g \in R^q$*,* $\eta^{ze} \in R$*,* $\eta^z \in R^r$ *), such that*

$$\nabla_x(\bar{z}^\top \nabla_y f + \bar{\mu}^\top \nabla_y g)(\bar{x}, \bar{y})^\top \eta^{kkt} + \nabla H(\bar{x})^\top \eta^H + \nabla G(\bar{x})^\top \eta^G + \nabla_x g(\bar{x}, \bar{y})^\top \eta^g = 0,$$

$$\nabla_y(\bar{z}^\top \nabla_y f + \bar{\mu}^\top \nabla_y g)(\bar{x}, \bar{y})^\top \eta^{kkt} + \nabla_y g(\bar{x}, \bar{y})^\top \eta^g = 0,$$

$$\eta^{ze} e^r - \eta_z + \nabla_y f(\bar{x}, \bar{y})^\top \eta^{kkt} = 0,$$

$$\eta_i^g = 0, \forall i \in I_{+0}, \quad (\nabla_y g(\bar{x}, \bar{y})^\top \eta^{kkt})_i = 0, \forall i \in I_{0+},$$

$$\eta_i^G \geq 0, \forall i \in I_G, \quad \eta_i^G = 0, \forall i \notin I_G, \quad \eta_i^z \geq 0, \forall i \in I_z, \quad \eta_i^z = 0, \forall i \notin I_z,$$

$$\textit{either } \eta_i^g > 0, (\nabla_y g(\bar{x}, \bar{y})^\top \eta^{kkt})_i > 0 \textit{ or } \eta_i^g (\nabla_y g(\bar{x}, \bar{y})^\top \eta^{kkt})_i = 0, \forall i \in I_{00}.$$

Then μ_0 *can be taken as* 1.

Proof. The proof of this corollary is not particularly difficult, we omit it here. □

With the help of the preceding Theorem 4.5, we now give a necessary condition for the existence of local efficient solution for BMOP (1.1)-(1.2).

Theorem 4.7. *Let* $(\bar{x}, \bar{y})$ *be a local efficient solution of BMOP (1.1)-(1.2). Assume that* F*,* G*,* H *are continuously differentiable. Also we assume that* f*,* g *are continuously differentiable w.r.t* x*, and twice continuously differentiable w.r.t* y*, in the neighbourhood of* $(\bar{x}, \bar{y})$*. Supposed that the functions* $y \mapsto g(x, y)$ *is* R_+^q*-convex and* $y \mapsto f(x, y)$ *is* R_+^r*-convex. Moreover supposed that Slater's CQ is satisfied for the lower level parametric problem at* $\bar{x}$*. Then there exist* $\bar{z} \in Z$*,* $\bar{\mu} \in \Lambda(\bar{x}, \bar{y}, \bar{z})$ *such that* $(\bar{x}, \bar{y}, \bar{z}, \bar{\mu})$ *is a local efficient solution of multiobjective optimization problem (4.1), and there exist* $\mu_0 \in \{0, 1\}$*,* $\eta^{kkt} \in R^m$*,* $\eta^G \in R^{p_1}$*,* $\eta^H \in R^{p_2}$*,* $\eta^g \in R^q$*,* $\eta^{ze} \in R$*,* $\eta^z \in R^r$ *not all zero and an unit vector* $\lambda \in N(F(\bar{x}, \bar{y}, \bar{z}, \bar{\mu}), \overline{l(F(\bar{x}, \bar{y}, \bar{z}, \bar{\mu}))})$ *such that conditions (4.2)-(4.7) hold.*

Proof. Combining condition (i) of Theorem 3.3, condition (i) of Theorem 4.2 and Theorem 4.5, we can obtain the results of this theorem easily. □

Remark 4.8. The Slater's CQ in Theorem 4.7 can be replaced by more weaker CQ such as Cottle CQ, this can be see from Remark 4.3.

5 Necessary condition via penalization

In order to derive the necessary optimality conditions for the auxiliary semivectorial bilevel programming problem (3.4)-(3.5), we translate the problem (3.4)-(3.5) into the following multiobjective optimization problem (5.1), by replacing the lower level programming problem with its optimal value function φ.

$$\min_{x,y,z} F(x, y) \tag{5.1}$$

$$
\begin{aligned}
s.t.\quad & \bar{f}(x,y,z)-\varphi(x,z)\le 0\\
& x\in X, z\in Z, y\in K(x).
\end{aligned}
$$

It is easy to show that multiobjective optimization problem (5.1) is equal to auxiliary semivectorial bilevel programming problem (3.4)-(3.5).

Theorem 5.1. *Let $(\bar{x},\bar{y},\bar{z})$ be a local efficient solution (efficient solution) of auxiliary semivectorial bilevel programming problem (3.4)-(3.5), then it's a local efficient solution (efficient solution) of multiobjective optimization problem (5.1). Reverse, if $(\bar{x},\bar{y},\bar{z})$ is a local efficient solution (efficient solution) of multiobjective optimization problem (5.1), then it's a local efficient solution (efficient solution) of auxiliary semivectorial bilevel programming problem (3.4)-(3.5).*

Proof. Since the proof is simple, we omit it here. □

As we know that, the MFCQ is not satisfied at the constraint set of multiobjective optimization problem (5.1) because of the constraint condition $\bar{f}(x,y,z)-\varphi(x,z)\le 0$. So we consider this constraint condition as a penalty, and discuss the property of partially calm for multiobjective optimization problem (5.1). We give the definition of partially calm property similar to the Definition 3.1 in [31] .

Definition 5.2. Let $(\bar{x},\bar{y},\bar{z})$ be a feasible point of problem (5.1). We said problem (5.1) is partially calm at $(\bar{x},\bar{y},\bar{z})$ if there is a $\tau\in R$, $\tau>0$ and a neighborhood $B(\delta,(\bar{x},\bar{y},\bar{z},0))$ with radius $\delta>0$ sufficiently small such that there is no vector $(x,y,z,\upsilon)\in B$ feasible to the following partially perturbed problem to

$$
\begin{aligned}
&\min_{x,y,z} F(x,y) \\
&s.t.\quad \bar{f}(x,y,z)-\varphi(x,z)+\upsilon=0,\\
&\qquad\ \ x\in X, z\in Z, y\in K(x),
\end{aligned}
\tag{5.2}
$$

such that

$$F(x,y)+\tau|\upsilon|e^l \preceq_{R^l_{\geq}} F(\bar{x},\bar{y}).$$

Theorem 5.3. *Let $(\bar{x},\bar{y},\bar{z})$ be a local efficient solution of multiobjective optimization problem (5.1), we say that problem (5.1) is partially calm at $(\bar{x},\bar{y},\bar{z})$ if there exist $\tau\in R$, $\tau>0$ such that $(\bar{x},\bar{y},\bar{z})$ is a local efficient solution of the following partially penalized problem (5.3):*

$$
\begin{aligned}
&\min_{x,y,z} F(x,y)+\tau(\bar{f}(x,y,z)-\varphi(x,z))e^l \\
&s.t.\quad x\in X,\ z\in Z,\ y\in K(x)
\end{aligned}
\tag{5.3}
$$

here, e^l is a l-dimensional unit vector, $F(x,y)+\tau(\bar{f}(x,y,z)-\varphi(x,z))e^l$ stands for the l-dimensional vector which the i th component equal to $F(x,y)_i+\tau(\bar{f}(x,y,z)-\varphi(x,z))$.

Proof. Since $\bar{x},\bar{y},\bar{z}$ is a local efficient solution of multiobjective optimization problem (5.1), so

$$\bar{f}(\bar{x},\bar{y},\bar{z})-\varphi(\bar{x},\bar{z}))=0.$$

Duo to $(\bar{x},\bar{y},\bar{z})$ is also a local efficient solution of problem (5.3) for $\bar{\tau}>0$, there exists a neighborhood $B(\delta,(\bar{x},\bar{y},\bar{z}))$ with radius $\delta>0$ sufficiently small such that there is no other vector $(x,y,z)\in B$ feasible to problem (5.3) such that

$$F(x,y)+\bar{\tau}(\bar{f}(x,y,z)-\varphi(x,z))e^l \preceq_{R^l_+} F(\bar{x},\bar{y})+\bar{\tau}(\bar{f}(\bar{x},\bar{y},\bar{z})-\varphi(\bar{x},\bar{z}))e^l=F(\bar{x},\bar{y}) \tag{5.4}$$

Let $(x, y, z, \upsilon) \in B(\delta, (\bar{x}, \bar{y}, \bar{z}, 0))$ feasible to (5.2), obviously, (x, y, z) is feasible vector of problem 5.3. Since $\bar{f}(x, y, z) - \varphi(x, z)) = -\upsilon$ and $\bar{f}(x, y, z) - \varphi(x, z)) \geq 0$ then

$$F(x, y) + \bar{\tau}|\upsilon|e^l = F(x, y) + \bar{\tau}(\bar{f}(x, y, z) - \varphi(x, z))e^l.$$

According to (5.4), it follows that there is no other vector $(x, y, z, \upsilon) \in B(\delta, (\bar{x}, \bar{y}, \bar{z}, 0))$ feasible to (5.2), such that

$$F(x, y) + \bar{\tau}|\upsilon|e^l \preceq_{R^l_+} F(\bar{x}, \bar{y}),$$

that is problem (5.1) is partially calm at $(\bar{x}, \bar{y}, \bar{z})$. □

Remark 5.4. It is easy to verify that the reverse of Theorem 5.3 also hold.

In the following part of this section, we will consider another necessary condition for the existence of local efficient solution of BMOP (1.1)-(1.2) via multiobjective optimization problem (5.3). We need to consider the necessary optimality condition of problem (5.3) firstly.

The following regularity conditions which are similar to [19] are necessary for the proof of main theorem of this section.

Definition 5.5. Let $(\bar{x}, \bar{y})$ be a feasible point of problem (1.1)-(1.2), if g is a continuously differentiable at $(\bar{x}, \bar{y})$. H, G are continuously differentiable at $\bar{x}$. Then the following conditions

$$\left[\sum_{j=1}^{p_1} \alpha_j \nabla G_j(\bar{x}) + \sum_{k=1}^{p_2} \nabla \beta_j H(\bar{x}) = 0, \quad \alpha_j \geq 0, \alpha_j G_j(\bar{x}) = 0\right] \Longrightarrow \left\{ \begin{array}{ll} \alpha_j = 0, & j = 1, 2, \cdots, p_1, \\ \beta_k = 0, & k = 1, 2, \cdots, p_2, \end{array} \right. \tag{5.5}$$

$$\left[\sum_{i=1}^{w} \mu_i \nabla_y g_i(\bar{x}, \bar{y}) = 0, \quad \mu_i \geq 0, \mu_i g_i(\bar{x}, \bar{y}) = 0\right] \Longrightarrow \mu_i = 0, \quad i = 1, 2, \cdots, q, \tag{5.6}$$

define the upper level regularity condition at $\bar{x}$ and the lower level regularity condition at $(\bar{x}, \bar{y})$ respectively. Here, the function $G_i, i = 1, \ldots, p_1$, $H_j, j = 1, \ldots, p_2$, are assumed to be differentiable at $\bar{x}$, and $g_i, i = 1, \ldots, q$ are assumed to be differentiable at $(\bar{x}, \bar{y})$.

The next lemma shows the method of calculating the basic subdifferential of φ which is essential in the proof of main theorem of this section. For simplicity, we first define the follow index set

$$I_g(x, y) = \{i : g_i(x, y) = 0\},$$
$$I_G(x) = \{i : G_i(x) = 0\},$$
$$I_z(z) = \{i : z_i = 0\}.$$

Lemma 5.6. *Assume that ψ is inner semicompact set at $(\bar{x}, \bar{z})$ and the lower level regularity condition (5.6) hold at $(\bar{x}, \bar{y})$. Then φ is Lipschitz continuous around $\bar{x}$. From Theorem 7 in [28] we have*

$$\partial \varphi(\bar{x}, \bar{z}) \subset \bigcup_{y \in \psi(\bar{x}, \bar{z})} \bigcup_{\beta \in \Lambda(\bar{x}, y, \bar{z})} \left\{ \begin{bmatrix} \bar{z}^\top \nabla_x f(\bar{x}, y) + \beta^\top \nabla_x g(\bar{x}, y) \\ f(\bar{x}, y) \end{bmatrix} \right\}$$

here $\Lambda(\bar{x}, y, \bar{z})$ is the lower level Lagrange multipliers set which is defined as:

$$\Lambda(\bar{x}, y, \bar{z}) := \{\beta \in R^q : \bar{z}^\top \nabla_y f(\bar{x}, y) + \beta^\top \nabla_y g(\bar{x}, y) = 0,$$
$$\beta_i \geq 0, \beta_i \in I_g(\bar{x}, y), \quad \beta_i = 0, \beta_i \notin I_g(\bar{x}, y)\}.$$

Proof. According to the proof process of Theorem 7 in [28], we can obtain this lemma easily, we omit it here. □

The next theorem shows the necessary optimality conditions of multiobjective optimization problem (5.3).

Theorem 5.7. *Let $(\bar{x}, \bar{y}, \bar{z})$ be a local efficient solution of optimization problem (5.3), where F is Lipschitz continuous and differentiable at a neighborhood of $(\bar{x}, \bar{y})$, G, H, g are continuously differentiable, and the upper level regular condition (5.5) is satisfied at $\bar{x}$. Assume that ψ is an inner semicompact set at $(\bar{x}, \bar{z})$, and the lower level regular condition (5.6) is satisfied at $(\bar{x}, \bar{y})$. Then there exist an unit vector $\lambda \in N(F(\bar{x}, \bar{y}), \overline{l(F(\bar{x}, \bar{y}))})$, $\rho_0 \in \{0, 1\}$ and $\eta^G \in R^{p_1}$, $\eta^H \in R^{p_2}$, $\eta^g \in R^q$, $\eta^{ze} \in R$, $\eta^z \in R^r$, $\eta_k \in R^q$ $\xi_k \in R$, and $y_k \in \psi(\bar{x}, \bar{z})$, $k = 1, 2, \cdots, s$ with $\sum_{k=1}^s \xi_k = 1$ such that the following conditions hold:*

$$\rho_0 \lambda^\top \nabla_x F(\bar{x}, \bar{y}) + \rho_0 \tau \bar{z}^\top \nabla_x f(\bar{x}, \bar{y}) \lambda^\top e^l - \sum_{k=1}^s \rho_0 \xi_k \tau \bar{z}^\top \nabla_x f(\bar{x}, y_k) \lambda^\top e^l -$$
$$\sum_{k=1}^s \rho_0 \xi_k \tau \beta_k^\top \nabla_x g(\bar{x}, y_k) \lambda^\top e^l + \nabla G(\bar{x})^\top \eta^G + \nabla_x g(\bar{x}, \bar{y})^\top \eta^g + \nabla H(\bar{x})^\top \eta^H = 0,$$

$$\rho_0 \lambda^\top \nabla_y F(\bar{x}, \bar{y}) + \rho_0 \tau \bar{z}^\top \nabla_y f(\bar{x}, \bar{y}) \lambda^\top e^l + \nabla_y g(\bar{x}, \bar{y})^\top \eta^g = 0,$$

$$\rho_0 \tau f(\bar{x}, \bar{y}) \lambda^\top e^l - \sum_{k=1}^s \rho_0 \xi_k \tau f(\bar{x}, y_k) \lambda^\top e^l - (e^r)^\top \eta^z + \eta^{ze} = 0,$$

$$\bar{z}^\top \nabla_y f(\bar{x}, y_k) + \beta_k^\top \nabla_y g(\bar{x}, y_k) = 0,$$

$$\eta_i^G \geq 0, \forall i \in I_G(\bar{x}), \quad \eta_i^G = 0, \forall i \notin I_G(\bar{x}),$$

$$\eta_i^g \geq 0, \forall i \in I_g(\bar{x}, \bar{y}), \quad \eta_i^g = 0, \forall i \notin I_g(\bar{x}, \bar{y}),$$

$$\eta_i^z \geq 0, \forall i \in I_z(\bar{z}), \quad \eta_i^z = 0, \forall i \notin I_z(\bar{z}),$$

$$\beta_{ki}^g \geq 0, \forall i \in I_g(\bar{x}, y_k), \quad \beta_{ki}^z = 0, \forall i \notin I_g(\bar{x}, y_k).$$

Proof. According to Proposition 2.18, it follows that there exist an unit vector

$$\lambda \in N_+(F(\bar{x}, \bar{y}) + \tau(\bar{f}(\bar{x}, \bar{y}, \bar{z}) - \varphi(\bar{x}, \bar{z})), \overline{l(F(\bar{x}, \bar{y}) + \tau(\bar{f}(\bar{x}, \bar{y}, \bar{z}) - \varphi(\bar{x}, \bar{z})))}),$$

and $\rho_0 \in \{0, 1\}$ such that

$$0 \in \rho_0 \partial \langle \lambda, (F(\bar{x}, \bar{y}) + \tau(\bar{f}(\bar{x}, \bar{y}, \bar{z}) - \varphi(\bar{x}, \bar{z})) e^l) \rangle + N((\bar{x}, \bar{y}, \bar{z}), \Omega), \tag{5.7}$$

here $\Omega = \{(x, y, z) : G(x) \leq 0, g(x, y) \leq 0, -z \leq 0, H(x) = 0, z^\top e^r - 1 = 0\}$.

Since $(\bar{x}, \bar{y}, \bar{z})$ is a local efficient solution of optimization problem (5.3), it follows that

$$\bar{f}(\bar{x}, \bar{y}, \bar{z}) - \varphi(\bar{x}, \bar{z}) = 0,$$

so according to Definition 2.4 and Proposition 2.11, we have

$$\begin{aligned}&N_+(F(\bar{x},\bar{y})+\tau(\bar{f}(\bar{x},\bar{y},\bar{z})-\varphi(\bar{x},\bar{z})),\overline{l(F(\bar{x},\bar{y})+\tau(\bar{f}(\bar{x},\bar{y},\bar{z})-\varphi(\bar{x},\bar{z})))}\\&=N_+(F(\bar{x},\bar{y}),\overline{l(F(\bar{x},\bar{y}))})=N(F(\bar{x},\bar{y}),\overline{l(F(\bar{x},\bar{y}))}).\end{aligned}$$

We are now turn to the formula (5.7). Simply calculating, we have

$$\begin{aligned}0\in\ &\rho_0\nabla_{x,y,z}(\lambda^\top F)(\bar{x},\bar{y})+\rho_0\tau\nabla_{x,y,z}\bar{f}(\bar{x},\bar{y},\bar{z})\lambda^\top e^l\\&+\rho_0\tau\partial_{x,y,z}(-\varphi)(\bar{x},\bar{z})\lambda^\top e^l+N((\bar{x},\bar{y},\bar{z}),\Omega).\end{aligned}\tag{5.8}$$

Further calculating we can obtain that:

$$\rho_0\nabla_{x,y,z}(\lambda^\top F)(\bar{x},\bar{y})=\left\{\begin{bmatrix}\rho_0\lambda^\top\nabla_x F(\bar{x},\bar{y})\\ \rho_0\lambda^\top\nabla_y F(\bar{x},\bar{y})\\ 0\end{bmatrix}\right\}\tag{5.9}$$

and

$$\rho_0\tau\nabla_{x,y,z}\bar{f}(\bar{x},\bar{y},\bar{z})\lambda^\top e^l=\left\{\begin{bmatrix}\rho_0\tau\bar{z}^\top\nabla_x f(\bar{x},\bar{y})\lambda^\top e^l\\ \rho_0\tau\bar{z}^\top\nabla_y f(\bar{x},\bar{y})\lambda^\top e^l\\ \rho_0\tau f(\bar{x},\bar{y})\lambda^\top e^l\end{bmatrix}\right\}.\tag{5.10}$$

Next we will evaluate the basic normal cone Ω.

Applying Proposition 2.5, and through some calculations we know that there exist $\eta^G\in R^{p_1}$, $\eta^H\in R^{p_2}$, $\eta^g\in R^q$, $\eta^{ze}\in R$, $\eta^z\in R^r$ such that

$$N((\bar{x},\bar{y},\bar{z}),\Omega)\subset\left\{\begin{bmatrix}\nabla G(\bar{x})^\top\eta^G+\nabla_x g(\bar{x},\bar{y})^\top\eta^g+\nabla H(\bar{x})^\top\eta^H\\ \nabla_y g(\bar{x},\bar{y})^\top\eta^g\\ -(e^r)^\top\eta^z+\eta^{ze}\end{bmatrix}:\begin{array}{c}\eta_i^G\geq 0,i\in I_G(\bar{x});\eta_i^G=0,i\notin I_G(\bar{x}).\\ \eta_i^z\geq 0,i\in I_z(\bar{z});\eta_i^z=0,i\notin I_z(\bar{z}).\\ \eta_i^g\geq 0,i\in I_g(\bar{x},\bar{y});\eta_i^g=0,i\notin I_g(\bar{x},\bar{y}).\end{array}\right\}\tag{5.11}$$

and

$$\left\{\begin{array}{l}\nabla G(\bar{x})^\top\tau_1+\nabla_x g(\bar{x},\bar{y})^\top\tau_2+\nabla H(\bar{x})^\top\upsilon_1=0,\\ \nabla_y g(\bar{x},\bar{y})^\top\tau_2=0\\ -\tau_3+\upsilon_2 e^r=0\end{array}:\begin{array}{l}\tau_1\geq 0,\tau_1^\top G(\bar{x})\geq 0,\\ \tau_2\geq 0,\tau_2^\top g(\bar{x},\bar{y})=0\\ \tau_3\geq 0,\tau_3^\top z=0\end{array}\right\}\Rightarrow\left\{\begin{array}{l}\tau_1=0,\tau_2=0,\\ \tau_3=0,\\ \upsilon_1=0,\upsilon_1=0.\end{array}\right.\tag{5.12}$$

Here $\tau_1\in R^{p_1}$, $\tau_2\in R^q$, $\tau_3\in R^r$, $\upsilon_1\in R^{p_2}$, $\upsilon_2\in R$. It can easily be verified that, condition (5.12) can be derived by upper level regular condition (5.5) and lower level regular condition (5.6) easily.

Duo to Lemma 5.6, it follows that

$$\begin{aligned}&\rho_0\tau\partial_{x,y,z}\varphi(\bar{x},\bar{z})\lambda^\top e^l\\&\subset\bigcup_{y_k\in\psi(\bar{x},\bar{z})}\ \bigcup_{\beta_k\in\Lambda(\bar{x},y_k,\bar{z})}\left\{\begin{bmatrix}\rho_0\tau\bar{z}^\top\nabla_x f(\bar{x},y_k)\lambda^\top e^l+\rho_0\tau\beta_k^\top\nabla_x g(\bar{x},y_k)\lambda^\top e^l\\ 0\\ \rho_0\tau f(\bar{x},y_k)\lambda^\top e^l\end{bmatrix}\right\}.\end{aligned}\tag{5.13}$$

Here $\Lambda(\bar{x}, y_k, \bar{z})$ is the lower level Lagrange multipliers set which is defined as follows:

$$\Lambda(\bar{x}, y_k, \bar{z}) := \{\beta_k \in R^q : \bar{z}^\top \nabla_y f(\bar{x}, y_k) + \beta_k^\top \nabla_y g(\bar{x}, y_k) = 0,$$
$$\beta_{ki} \geq 0, \beta_{ki} \in I_g(\bar{x}, y_k), \quad \beta_{ki} = 0, \beta_{ki} \notin I_g(\bar{x}, y_k)\}.$$

According to (2.1), it is easy to verify that

$$\partial(-\varphi)(\bar{x}, \bar{z}) \subseteq \text{co}\partial(-\varphi)(\bar{x}, \bar{z}) = -\text{co}\partial\varphi(\bar{x}, \bar{z})$$

further, we have

$$\partial_{x,y,z}(-\varphi)(\bar{x}, \bar{z}) \subseteq -\text{co}\partial_{x,y,z}\varphi(\bar{x}, \bar{z})$$

Taking $\nu \in \text{co}\partial_{x,y,z}\varphi(\bar{x}, \bar{z})$ and we can find $\xi_k \in R$ and $\nu_k \in R^{m+n+q}$ with $k = 1, \dots, s$ such that

$$\nu = \sum_{k=1}^{s} \xi_k \nu_k, \quad \sum_{k=1}^{s} \xi_k = 1, \quad \xi_k \geq 0, \quad \nu_k \in \partial_{x,y,z}\varphi(\bar{x}, \bar{z}), \quad \text{for} k = 1, \dots, s.$$

Applying (5.13) we have $y_k \in \psi_{\bar{x},\bar{z}}$ and $\beta_k \in \Lambda(\bar{x}, y_k, \bar{z})$ such that

$$\rho_0 \tau \nu_k \lambda^\top e^l = \left[\begin{array}{c} \rho_0 \tau \bar{z}^\top \nabla_x f(\bar{x}, y_k) \lambda^\top e^l + \rho_0 \tau \beta_k^\top \nabla_x g(\bar{x}, y_k) \lambda^\top e^l \\ 0 \\ \rho_0 \tau f(\bar{x}, y_k) \lambda^\top e^l. \end{array} \right] \tag{5.14}$$

Combing (5.9), (5.10), (5.11), (5.14) and (5.8), we can get the results easily.

□

Corollary 5.8. *For Theorem 5.7, if constraint set Ω satisfies that, there is no nonzero vector $(\eta^G, \eta^H, \eta^g, \eta^{ze}, \eta^z) \in R^{p_1+p_2+q+1+r}$ (here $\eta^G \in R^{p_1}$, $\eta^H \in R^{p_2}$, $\eta^g \in R^q$, $\eta^{ze} \in R$, $\eta^z \in R^r$), such that*

$$\nabla G(\bar{x})^\top \eta^G + \nabla_x g(\bar{x}, \bar{y})^\top \eta^g + \nabla H(\bar{x})^\top \eta^H = 0,$$
$$\nabla_y g(\bar{x}, \bar{y})^\top \eta^g = 0,$$
$$-(e^r)^\top \eta^z + \eta^{ze} = 0,$$
$$\eta_i^G \geq 0, \forall i \in I_G(\bar{x}), \ \eta_i^G = 0, \forall i \notin I_G(\bar{x}),$$
$$\eta_i^z \geq 0, \forall i \in I_z(\bar{z}), \ \eta_i^z = 0, \forall i \notin I_z(\bar{z}),$$
$$\eta_i^g \geq 0, \forall i \in I_g(\bar{x}, \bar{y}), \ \eta_i^g = 0, \forall i \notin I_g(\bar{x}, \bar{y}).$$

Then ρ_0 can be taken as 1.

Proof. The proof of this theorem is easy, we omit it here. □

Next we will give another necessary condition of the existence of local efficient solution for BMOP (1.1)-(1.2).

Theorem 5.9. *Let $(\bar{x}, \bar{y})$ be a local efficient solution of BMOP (1.1)-(1.2). Assume that F is Lipschitz continuous and differentiable at a neighborhood of $(\bar{x}, \bar{y})$, G, H, g are continuously differentiable. The functions $y \mapsto g(x, y)$ and $y \mapsto f(x, y)$ are R_+^q-convex and R_+^r-convex respectively. Also assume that the upper level regular condition (5.5) is satisfied at $\bar{x}$, and the lower level regular condition (5.6) is satisfied at $(\bar{x}, \bar{y})$. Moreover supposed that ψ is an inner semicompact set at $(\bar{x}, \bar{z})$, and problem (5.1) is partially calm at $(\bar{x}, \bar{y}, \bar{z})$. Then , there exist an unit vector $\lambda \in N(F(\bar{x}, \bar{y}), \overline{l(F(\bar{x}, \bar{y}))})$, $\rho_0 \in \{0, 1\}$, $\tau > 0$, and $\eta^G \in R^{p_1}$, $\eta^H \in R^{p_2}$, $\eta^g \in R^q$, $\eta^{ze} \in R$, $\eta^z \in R^r$, $\eta_k \in R^q$ $\xi_k \in R$, and $y_k \in \psi(\bar{x}, \bar{z})$, $k = 1, 2, \cdots, s$ with $\sum_{k=1}^{s} \xi_k = 1$ such that all conditions of Theorem 5.7 hold.*

Proof. Combining condition (i) of Theorem 3.3, Theorem 5.1, Theorem 5.3, and Theorem 5.7 we can obtain the results easily. □

6 Conclusions

In this paper, we first considered the optimality conditions for bilevel multiobjective optimization problem(1.1)-(1.2). This BMOP is different from the bilevel problem proposed in [34] by Ye. After transforming the lower level multiobjective optimization problem into a scalar optimization problem (3.1)-(3.2) by weighted method, we get the auxiliary semivectorial bilevel programming problem (3.4)-(3.5). Then we considered the relationship between the two bilevel optimization problem. We got that they are equivalent when the have efficient solution, but they are equivalent only when K is an inner semicontinuous set-valued mapping in the case that, they only have local efficient solution. After replacing the lower-level scalar optimization problem by KKT conditions, we transformed the auxiliary semivectorial bilevel programming problem (3.4)-(3.5) into a MOPEC, and got a necessary optimality condition for BMOP which need the twice continuously differentiable property of f and g. Since the feasible region of multiobjective optimization problem (5.1) may not satisfy MFCQ when the constraint conditions contain optimal valued function, we transformed the problem (5.1) into a multiobjective optimization problem with penalization, and obtained the necessary conditions for the existence of solution which only need the first order continuously differentiable properties of f and g. We will discuss the relationship between stationary point sets and efficient solution set in our future work. The relationship between the two kinds of necessary optimality condition is also a meaningful issue.

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