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RELAXATION IN NONCONVEX OPTIMAL CONTROL PROBLEMS FOR NONAUTONOMOUS FRACTIONAL EVOLUTION EQUATIONS*

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Abstract: We deal with the minimization problem of an integral functional with an integrand that is not convex in the control, on solutions of a control system described by a nonautonomous fractional evolution equation with mixed nonconvex constraints on the control. A relaxation problem is treated along with the original problem. It is proved that the relaxation problem has an optimal solution and that for each optimal solution there is a minimizing sequence of the original problem that converges to the optimal solution with respect to the trajectory, the control and the functional in appropriate topologies simultaneously.

Key words: *nonautonomous problems, fractional evolution equation, optimal control, relaxation property, nonconvex constraint, feedback control*

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1 Introduction

For the last decades, the theory of fractional derivatives or integrals has undergone rapid development. To a large extent this was due to the great applications of the theory to problems in mathematical physics. As stated in [13, 23], both the ordinary and the partial differential equations of fractional order have become an excellent tools for the description of memory and hereditary properties of various materials and processes, such as viscoelasticity, electrodynamics and heat conduction. For more details on these topics, one can see [3, 14–20] and the references therein.

Nonautonomous problems are of interest in theory as well as in practice. The reason for studying the nonautonomous equation is that it and various variants of it appear in mathematical models of viscoelasticity. In fact, they typically appear in mathematical physics by some constitutive laws of the memory type when combined with the usual conservation laws and arise in the theory of aging of materials with memory. Significant progresses have been made for the integer order nonautonomous differential equations. However, due to the difficulty of the time variance and the memory property of the fractional order, there are still few topics in the literature to study the nonautonomous fractional differential equations. For more details, one can see [4, 7, 8].

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Recently, existence and relaxation problems for nonconvex differential inclusions have been received great attentions by many researchers from different points of view. Since Bogolyubov first proved a theorem on relaxation for a class of problems of the classical calculus of variations in 1930. Thereby, this theorem has been extended in several directions by many authors including Ekeland and Temam [6], Ioffe and Tikhomirov [12] and Mcshane [21]. Among more recent generalizations are the works by De Blasi, ect. [5] and Tolstonogov [24–27]. Very recently, Liu et al. [15] considered the relaxation in nonconvex optimal problems described by fractional differential equations. In their work, $A(t) \equiv A$ and A generates a compact semigroup. However, their results clearly cannot apply to the equations with a nonautonomous $A(t)$ which is a more general and maybe more important case.

The main purpose of this article is to extend and develop the above work, that is, we shall discuss an analogue of Bogolyubov's theorem for the following problem.

For a numerical function $g : J \times X \times Y \to R$, we consider the problem (P):

$$\mathcal{J}(x,u) = \int_J g(t,x(t),u(t))dt \to \inf,$$

on the solution set of the following control system described by the following nonautonomous fractional evolution equation:

$$\begin{cases} {}^C D_t^\alpha x(t) + A(t)x(t) = B(t)u(t), \quad t \in J = [0,b],\ 0 < \alpha < 1, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

subject to the mixed nonconvex constraints on the control

$$u(t) \in U(t,x(t)), \quad \text{a.e. on } J, \tag{1.2}$$

where ${}^C D_t^\alpha$ is the Caputo fractional derivative of order α with the lower limit zero and $b > 0$ is a finite real number. $\{A(t) : t \in J\}$ is a family of linear closed densely defined operators on Banach space X such that the domain of $A(t)$ does not depend on t. $B : J \to \mathcal{L}(Y,X)$, where $\mathcal{L}(Y,X)$ is the space of continuous linear operators from Y into X and Y is a separable reflexive Banach space modeling the control space. $U : J \times X \to 2^Y \backslash \{\emptyset\}$ is a multivalued map with closed values that is not necessarily convex.

Let $\overline{R} = (-\infty, +\infty]$ and $g_U : J \times X \times Y \to \overline{R}$ be the function defined by

$$g_U(t,x,u) = \begin{cases} g(t,x,u), & u \in U(t,x), \\ +\infty, & u \notin U(t,x), \end{cases}$$

and let $g^{**}(t,x,u)$ be the bipolar of the function $u \to g_U(t,x,u)$ (see 1.4.2 of [6]).

Along with the problem (P), we also consider the relaxation problem (RP):

$$\mathcal{J}^{**}(x,u) = \int_J g_U^{**}(t,x(t),u(t))dt \to \inf,$$

on solutions of the control system (1.1) with the convexified constraints

$$u(t) \in \overline{\text{co}}U(t,x(t)), \quad \text{a.e. on } J, \tag{1.3}$$

on the control. Here $\overline{\text{co}}$ stands for the closed convex hull of a set.

Our aim of this paper is to explore an interrelation between the solutions of the problems (P) and (RP). Under sufficiently general conditions, we show that for every solution $(x_*(\cdot), u_*(\cdot))$ of the control system (1.1) with constraints (1.3), there exists a sequence $(x_n(\cdot), u_n(\cdot))$ $(n \geq 1)$ of the control system (1.1) with constraints (1.2) such that

$$x_n(\cdot) \to x(\cdot) \quad \text{in} \quad C(J,X), \tag{1.4}$$

$$\int_J g(t, x_n(t), u_n(t))dt \to \int_J g_U^{**}(t, x_*(t), u_*(t))dt. \tag{1.5}$$

By using this result, we get that the problem (RP) has a solution and for any solution of (RP), there is a minimizing sequence of (P) converging to the solution of (RP) which takes place simultaneously with respect to the trajectory, the control and the functional in the appropriate topologies. Usually, this property is called the relaxation (cf. [6]). Relations (1.4) and (1.5) are an analogue of Bogolyubov's theorem [15, 24–26] in the calculus of variations being the set of solutions of the control systems (1.1), (1.2) and (1.1), (1.3).

The rest of this paper is organized as follows. Next, we will present some basic definitions and preliminary facts, such as definitions, lemmas and theorems, which will be used throughout the following sections. In section 3, some auxiliary results needed in the proof of our main results are presented. We consider the existence result of the control systems described by nonautonomous fractional evolution equations in section 4. In section 5, we establish an interrelation between the solutions of the problems (P) and (RP) and prove our main result. Finally, a concrete application is given to illustrate our main result.

2 Preliminaries

Let $J = [0, b]$ be the closed interval of the real line with the Lebesgue measure μ and the σ-algebra Σ of μ measurable sets. The norm of the Banach space X (or Y) will be denoted by $\|\cdot\|_X$ (or $\|\cdot\|_Y$). Let $C(J, X)$ denote the Banach space of all continuous functions from J into X with the norm $\|x\|_C = \sup_{t\in J}\|x(t)\|_X$. For a Banach space X, the symbol w-X stands for X equipped with the weak $\sigma(X, X^*)$ topology. The same notation will be used for subsets of X. In all other cases we assume that X and its subsets are equipped with the strong (normed) topology.

Firstly, let us recall the following definitions. For more details, one can see [13] and [23].

Definition 2.1. The integral

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \qquad t > 0,\ 0 < \alpha < 1,$$

is called Riemann-Liouville fractional integral of order α, where Γ is the gamma function.

Definition 2.2. The Caputo fractional derivative for a function f of order α is defined by

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha}[f(s) - f(0)]ds, \qquad t > 0,\ 0 < \alpha < 1.$$

Remark 2.3. (i) The Caputo fractional derivative of a constant is equal to zero.

(ii) If the function $f \in AC[0, \infty)$, then we can get

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)ds = I_t^{1-\alpha} f'(t), \quad t > 0,\ 0 < \alpha < 1.$$

(iii) If f is an abstract function with values in Banach space E, then integrals which appear in Definition 2.1 and 2.2 are taken in Bochner's sense.

Now we introduce some basic definitions and results from multivalued analysis. For more details on multivalued analysis, see the books [1, 11].

We use the following notations: $\mathcal{P}_f(Y)$ is the set of all nonempty closed subsets of Y, $\mathcal{P}_{fb}(Y)$ is the set of all nonempty, closed and bounded subsets of Y.

On $\mathcal{P}_{fb}(Y)$, we consider $H_d : \mathcal{P}_{fb}(X)\times\mathcal{P}_{fb}(X) \to R_+ \cup \{\infty\}$, then we can have a metric, known as the "Hausdorff metric" and defined by

$$H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\},$$

where $d(x,C)$ is the distance from a point x to a set C. We say a multivalued map is h-continuous if it is continuous in the Hausdorff metric $H_d(\cdot,\cdot)$.

We say that a multivalued map $F : J \to \mathcal{P}_f(X)$ is measurable if $F^{-1}(E) = \{t \in J : F(t)\cap E \neq \emptyset\} \in \Sigma$ for every closed set $E \subseteq X$. If $F : J\times X \to \mathcal{P}_f(X)$, then measurability of F means that $F^{-1}(E) \in \Sigma\otimes\mathcal{B}_X$, where $\Sigma\otimes\mathcal{B}_X$ is $\sigma-$algebra of subsets in $J\times X$ generated by the set $A\times B, A\in\Sigma, B\in\mathcal{B}_X$ and $\mathcal{B}_X$ is the $\sigma-$algebra of the Borel sets in X.

Let $U : J\times X \to 2^Y\setminus\{\emptyset\}$ be a multifunction, for $1\le p\le +\infty$, we define

$$N^p_{U,x} = \{u \in L^p(J,Y) : u(t) \in U(t,x(t)) \text{ a.e. on } \ t\in J\}.$$

Besides the standard norm on $L^p(J,Y)(1<p<\infty)$, we also consider the following weak norm

$$\|u(\cdot)\|_w = \sup_{0\le t_1\le t_2\le b}\left\|\int_{t_1}^{t_2} u(s)ds\right\|_Y, \qquad \text{for } \ u\in L^p(J,Y). \tag{2.1}$$

The space $L^p(J,Y)$ furnished with this norm will be denoted by $L^p_w(J,Y)$. The following result establishes a relation between convergence in w-$L^p(J,Y)$ and convergence in $L^p_w(J,Y)$.

Lemma 2.4 ([24])**.** *If a sequence $\{u_n\}_{n\ge1}\subseteq L^p(J,Y)$, is bounded and converges to u in $L^p_w(J,Y)$, then it converges to u in w-$L^p(J,Y)$.*

The following notation of solution for our problems is natural.

Definition 2.5. A solution of the control system (1.1), (1.2) is defined to be a pair $(x(\cdot),u(\cdot))$ consisting of a trajectory $x\in C(J,X)$ and a control $u\in L^1(J,Y)$ satisfying the equation (1.1) and the inclusion (1.2) almost everywhere.

A solution of the control system (1.1), (1.3) is defined similarly. If $(x(\cdot),u(\cdot))$ is a solution of system (1.1), (1.2), then $x(\cdot)$ is called a trajectory and $u(\cdot)$ is called a control.

We denote $\mathcal{R}_U(x_0)$, $\mathcal{T}r_U(x_0)$ $(\mathcal{R}_{\overline{co}U}(x_0),\mathcal{T}r_{\overline{co}U}(x_0))$ be the sets of all solutions, all trajectories of the control system (1.1), (1.2) (the control system (1.1), (1.3)).

To obtain our main results in this paper, we make the following hypotheses:

$H(A)$ The closed linear operator $A(t)$ satisfies the following:

(1) the domain $D(A(t))$ of $\{A(t) : t\in J\}$ is dense in X and independent of t, that is $D(A(t))\equiv D(A)$;

(2) for each $t\in J$, the resolvent $R(\lambda,A(t)) = [A(t)+\lambda I]^{-1}$ exists in $\mathcal{L}(X)$ for all λ with $Re\lambda\ge 0$ and

$$\|R(\lambda,A(t))\| \le \frac{C_1}{|\lambda|+1},$$

where C_1 is a positive constant independent both of t and λ;

(3) for any $t,s,\tau\in J$,

$$\|[A(t)-A(s)]A^{-1}(\tau)\| \le C_2|t-s|^\beta,$$

where $0<\beta\le 1,\ C_2>0$ and the constants β and C_2 are independents of t,s and τ;

(4) for each $t\in J$ and some $\lambda\in\rho(A(t))$ (the resolvent set of $A(t)$), the resolvent $R(\lambda,A(t))$ is a compact operator.

$H(B)$ $B \in L^\infty(J, \mathcal{L}(Y,X))$ and $\|B\|$ stands for $\|B\|_{L^\infty(J,\mathcal{L}(Y,X))}$.

$H(U)$ $U : J \times X \to \mathcal{P}_f(Y)$ is the multivalued map such that:

(1) $t \to U(t,x)$ is measurable for all $x \in X$;

(2) $H_d(U(t,x), U(t,y)) \leq k_1(t)\|x-y\|_X$ a.e. on J with $k_1 \in L^\infty(J, R^+)$,

(3) there exist a positive function $m \in L^p(J, R^+)$ ($p > \max\{\frac{1}{\alpha}, \frac{1}{\beta}\}$) (where α is given in system (1.1) and β appears in condition $H(A)(3)$) and a constant $\gamma > 0$ such that

$$\|U(t,x)\|_Y = \sup\{\|v\|_Y, v \in U(t,x)\} \leq m(t) + \gamma\|x\|_X, \quad \text{for a.e.} \quad t \in J.$$

$H(g)$ $g : J \times X \times Y \to R$ is a function such that:

(1) the map $t \to g(t,x,u)$ is measurable for all $(x,u) \in X \times Y$;

(2) $|g(t,x,u) - g(t,y,v)| \leq k_2(t)\|x-y\|_X + \rho\|u-v\|_Y$ a.e., $k_2 \in L^\infty(J,R^+)$ and $\rho > 0$;

(3) $|g(t,x,u)| \leq a_1(t) + b_1(t)\|x\|_X + c_1\|u\|_Y$ a.e. $t \in J$ with $a_1,\ b_1 \in L^p(J,R^+)$ and $c_1 > 0$.

In what follows, we assume that hypotheses H(A), H(B), H(U) and H(g) are satisfied.

Remark 2.6. (1) We remark that the conditions H(A)(1) and (2) imply that for each $s \in J,\ -A(s)$ is the infinitesimal generator of an analytic semigroup $e^{-tA(s)}(t > 0)$ and there exists a constant $C \geq 1$ such that $\|e^{-tA(s)}\|_X \leq C$ ([22, Theorem 2.5.2]). Moreover, it follows from Lemma 2.4.2 and Theorem 2.5.2 of [22] that the semigroup $e^{-tA(s)}(t > 0)$ is continuous in the uniform operator topology, then the assumption $H(A)(4)$ insures that $e^{-tA(s)}(t > 0)$ is compact ([22, Theorem 2.3.3]).

(2) Since $D(A(t)) \equiv D(A)$ is dense in X and $0 \in \rho(A(t))$, then $D(A)$ with the graph norm $\|x\|_1 = \|A(0)x\|_X$ is a Banach space ([29, Proposition 2.10.1]).

If conditions H(A)(1)-(3) are satisfied, then according to the paper [7], we have:

Definition 2.7. A function $x \in C(J,X)$ is said to be a mild solution of the system (1.1), (1.2) if $x(0) = x_0 \in D(A)$ and there exists $u \in L^p(J,Y)$ ($p > \max\{\frac{1}{\alpha}, \frac{1}{\beta}\}$) (where α is given in system (1.1) and β appears in condition H(A)(3)) such that $u(t) \in U(t,x(t))$ a.e. on $t \in J$ and

$$\begin{aligned} x(t) \quad &= \quad x_0 - \int_0^t \psi(t-\eta,\eta)A(\eta)x_0 d\eta - \int_0^t \int_0^\eta \psi(t-\eta,\eta)\phi(\eta,s)A(s)x_0 ds d\eta \\ &\quad + \int_0^t \psi(t-\eta,\eta)B(\eta)u(\eta)d\eta + \int_0^t \int_0^\eta \psi(t-\eta,\eta)\phi(\eta,s)B(s)u(s)dsd\eta. \end{aligned}$$

where

$$\psi(t,s) = \alpha \int_0^\infty t^{\alpha-1}\theta\xi_\alpha(\theta)e^{-t^\alpha\theta A(s)}d\theta,$$

$$\phi_1(t,s) = [A(t) - A(s)]\psi(t-s,s),$$

$$\phi(t,s) = \phi_1(t,s) + \int_s^t \phi_1(t,\tau)\phi(\tau,s)d\tau,$$

and

$$\xi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\varpi_\alpha(\theta^{-\frac{1}{\alpha}}) \geq 0,$$

$$\varpi_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \quad \theta \in (0,\infty),$$

ξ_α is a probability density function defined on $(0,\infty)$, that is

$$\xi_\alpha(\theta) \geq 0,\ \theta \in (0,\infty), \quad \text{and} \quad \int_0^\infty \xi_\alpha(\theta)d\theta = 1.$$

Due to the paper [7], we can obtain the following:

Lemma 2.8. *If the conditions* H(A)*(1)-(3) are satisfied, then we have:*

(i) *The operator-valued function $\psi(t-s,s)$ is uniformly continuous in the uniform topology in the varaibles $t,\ s$, where $0 \leq s \leq t-\epsilon,\ 0 \leq t \leq b$ for any $\epsilon > 0$. And*

$$\|\psi(t-s,s)\| \leq C_\psi(t-s)^{\alpha-1},$$

where C_ψ is a positive constant independent of $t,\ s$.

(ii) *For any $\epsilon > 0$ and $0 \leq s \leq t-\epsilon,\ 0 \leq t \leq b$, there exists a constant $C_\phi > 0$ such that*

$$\|\phi(t,s)\| \leq C_\phi(t-s)^{\beta-1},$$

where $0 < \beta \leq 1$ is given in condition H$(A)(3)$.

$\boxed{3}$ Auxiliary Results

In this section, we shall give some auxiliary results needed in the proof of our main result.

Lemma 3.1. *For any admissible trajectory x of the control system (1.1), (1.3), there exists a constant $\omega > 0$ such that*

$$\|x\|_C \leq \omega.$$

Proof. From definition 2.7, we know that there exists a $u(t) \in \overline{\text{co}}U(t,x(t))$ such that

$$\begin{aligned} x(t) \quad &= \quad x_0 - \int_0^t \psi(t-\eta,\eta)A(\eta)x_0 d\eta - \int_0^t\int_0^\eta \psi(t-\eta,\eta)\phi(\eta,s)A(s)x_0 ds d\eta \\ &\quad + \int_0^t \psi(t-\eta,\eta)B(\eta)u(\eta)d\eta + \int_0^t\int_0^\eta \psi(t-\eta,\eta)\phi(\eta,s)B(s)u(s)dsd\eta. \end{aligned}$$

Firstly, for $x_0 \in D(A)$, from H$(A)(3)$, we have

$$\begin{aligned} \|A(t)x_0\| &= \|A(t)A^-(0)A(0)x_0\| = \|[A(t)-A(0)+A(0)]A^-(0)A(0)x_0\| \\ &\leq \|[A(t)-A(0)]A^-(0)\|\|A(0)x_0\| + \|A(0)x_0\| \\ &\leq C_1 t^\beta\|A(0)x_0\| + \|A(0)x_0\| \\ &\leq (C_1 b^\beta + 1)\|A(0)x_0\| := C_A\|A(0)x_0\|. \end{aligned}$$

Then for $t \in J$, we obtain

$$\begin{aligned} \|x(t)\|_X \quad &\leq \quad \|x_0\| + \left\|\int_0^t \psi(t-\eta,\eta)A(\eta)x_0 d\eta\right\| \\ &\quad + \left\|\int_0^t\int_0^\eta \psi(t-\eta,\eta)\phi(\eta,s)A(s)x_0 dsd\eta\right\| \end{aligned}$$

$$
\begin{aligned}
&+\left\|\int_0^t \psi(t-\eta,\eta)B(\eta)u(\eta)d\eta\right\| \\
&+\left\|\int_0^t\int_0^\eta \psi(t-\eta,\eta)\phi(\eta,s)B(s)u(s)dsd\eta\right\| \\
\leq\ & \|x_0\|+\int_0^t C_\psi(t-\eta)^{\alpha-1}C_A\|A(0)x_0\|d\eta \\
&+\int_0^t\int_0^\eta C_\psi(t-\eta)^{\alpha-1}C_\phi(\eta-s)^{\beta-1}C_A\|A(0)x_0\|dsd\eta \\
&+\|B\|\int_0^t C_\psi(t-\eta)^{\alpha-1}[m(\eta)+\gamma\|x(\eta)\|_X]d\eta \\
&+\|B\|\int_0^t\int_0^\eta C_\psi(t-\eta)^{\alpha-1}C_\phi(\eta-s)^{\beta-1}[m(s)+\gamma\|x(s)\|_X]dsd\eta \\
\leq\ & \kappa+\varrho\int_0^t (t-s)^{\alpha-1}\|x(s)\|_X ds,
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa\ =\ & \|x_0\|+\frac{1}{\alpha}C_\psi C_A b^\alpha\|A(0)x_0\| \\
&+\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(1+\alpha+\beta)}C_\psi C_\phi C_A b^{\alpha+\beta}\|A(0)x_0\|+C_\psi\|B\|\left[\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}b^{\alpha-\frac{1}{p}}\right. \\
&\left.+\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}C_\phi\left(\frac{p-1}{p(\alpha+\beta)-1}\right)^{\frac{p-1}{p}}b^{\alpha+\beta-\frac{1}{p}}\right]\|m\|_{L^p(J,R^+)},
\end{aligned}
$$

$$
\varrho=\gamma C_\psi\|B\|\left[1+\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}C_\phi b^\beta\right].
$$

For the last inequality, it follows from Corollary 2 of [30] that

$$
\|x(t)\|_X\leq\kappa E_\alpha(\varrho\Gamma(\alpha)t^\alpha),
$$

where E_β is the Mittag-Leffler function defined by

$$
E_\beta(z)=\sum_{k=0}^\infty\frac{z^k}{\Gamma(k\beta+1)}.
$$

Therefore, $\|x\|_C=\sup_{t\in J}\|x(t)\|_X\leq E_\alpha(\varrho\Gamma(\alpha)b^\alpha):=\omega$. The proof is completed. □

Let $\mathrm{pr}_\omega:X\to X$ be the ω-radial retraction as follows

$$
\mathrm{pr}_\omega(x)=\begin{cases} x, & \|x\|_X\leq\omega, \\ \frac{\omega x}{\|x\|_X}, & \|x\|_X>\omega. \end{cases}
$$

This map is Lipschitz continuous (cf. [9]). Now, define $U_1(t,x)=U(t,\mathrm{pr}_\omega x)$. Evidently, U_1 satisfies H$(U)(1)$ and H$(U)(2)$. Moreover, by the properties of pr_ω, we have for a.e. $t\in J$, all $x\in X$ and all $u(t)\in U_1(t,x)$ such that

$$
\|u\|_Y\leq m(t)+\gamma\|x\|_X\leq m(t)+\gamma\omega.
$$

Hence, Lemma 3.1 is still valid with $U(t,x)$ substituted by $U_1(t,x)$. Consequently, without loss of generality, we can assume that for a.e. $t \in J$ and all $x \in X$,

$$\sup\{\|v\|_Y, v \in U(t,x)\} \leq m(t) + \gamma\omega := \varphi(t), \quad \varphi \in L^p(J, R^+) \ (p > \max\{\frac{1}{\alpha}, \frac{1}{\beta}\}). \tag{3.1}$$

Now we consider the following auxiliary problem:

$$\begin{cases} {}^C D_t^\alpha x(t) + A(t)x(t) = h(t), \quad t \in J, \ 0 < \alpha < 1,, \\ x(0) = x_0 \in D(A), \end{cases} \tag{3.2}$$

It is clear that for every $h \in L^p(J,X)$, equation (3.2) has a unique solution $S(h) \in C(J,X)$ which is given by

$$\begin{aligned} (Sh)(t) \quad = \quad & x_0 - \int_0^t \psi(t-\eta,\eta)A(\eta)x_0 d\eta - \int_0^t \int_0^\eta \psi(t-\eta,\eta)\phi(\eta,s)A(s)x_0 ds d\eta \\ & + \int_0^t \psi(t-\eta,\eta)h(\eta)d\eta + \int_0^t \int_0^\eta \psi(t-\eta,\eta)\phi(\eta,s)h(s)dsd\eta. \end{aligned}$$

Let φ be defined by (3.1), we put

$$Y_\varphi = \{h \in L^p(J,X) : \|h(t)\|_X \leq \varphi(t) \ \text{ a.e. } t \in J\}. \tag{3.3}$$

The following property of the solution map S is crucial in our main result.

Lemma 3.2. *The solution map $S : Y_\varphi \to C(J,X)$ is continuous from w-Y_φ to $C(J,X)$.*

Proof. Consider the operator $F : L^p(J,X) \to C(J,X)$ defined by

$$(Fh)(t) \quad = \quad \int_0^t \psi(t-\eta,\eta)h(\eta)d\eta + \int_0^t \int_0^\eta \psi(t-\eta,\eta)\phi(\eta,s)h(s)dsd\eta.$$

It is clear that F is linear. Moreover, from simple calculation, one has

$$\begin{aligned} \|Fh\|_C \leq C_\psi \Bigg[& \left(\frac{p-1}{p\alpha - 1}\right)^{\frac{p-1}{p}} b^{\alpha - \frac{1}{p}} \\ & + \frac{\Gamma(\alpha)\Gamma(\beta)C_\phi}{\Gamma(\alpha+\beta)} \left(\frac{p-1}{p(\alpha+\beta)-1}\right)^{\frac{p-1}{p}} b^{\alpha+\beta-\frac{1}{p}} \Bigg] \|h\|_{L^p(J,X)}. \end{aligned} \tag{3.4}$$

Hence, the operator F is continuous from $L^p(J,X)$ to $C(J,X)$. Next, let us prove the continuity of the operator F from w-$L^p(J,X)$ to $C(J,X)$.

Let $\Xi \in \mathcal{P}_{fb}(L^p(J,X))$ and suppose that for any $h \in \Xi$, $\|h\|_{L^p(J,X)} \leq M$ ($M > 0$ is a constant). Next we will show that F is completely continuous.

(a). From (3.4), we know that F maps bounded subsets into bounded subsets in $C(J,X)$.

(b). $\{(Fh)(t) : h \in \Xi\}$ is equicontinuous. Let $0 \leq \tau_1 < \tau_2 \leq b$, we obtain

$$\begin{aligned} & \Big\|(Fh)(\tau_2) - (Fh)(\tau_1)\Big\| \\ \leq \quad & \Big\| \int_0^{\tau_2} \psi(\tau_2 - \eta,\eta)h(\eta)d\eta - \int_0^{\tau_1} \psi(\tau_1-\eta,\eta)h(\eta)d\eta \Big\| \\ & + \Big\| \int_0^{\tau_2} \int_0^\eta \psi(\tau_2-\eta,\eta)\phi(\eta,s)h(s)dsd\eta - \int_0^{\tau_1}\int_0^\eta \psi(\tau_1-\eta)\phi(\eta,s)h(s)dsd\eta \Big\| \end{aligned}$$

$$\begin{aligned}
&\leq \Big\|\int_0^{\tau_1}[\psi(\tau_2-\eta,\eta)-\psi(\tau_1-\eta,\eta)]h(\eta)d\eta\Big\|+\Big\|\int_{\tau_1}^{\tau_2}\psi(\tau_2-\eta,\eta)h(\eta)d\eta\Big\|\\
&\quad+\Big\|\int_0^{\tau_1}\int_0^{\eta}[\psi(\tau_2-\eta,\eta)-\psi(\tau_1-\eta)]\phi(\eta,s)h(s)dsd\eta\Big\|\\
&\quad+\Big\|\int_{\tau_1}^{\tau_2}\int_0^{\eta}\psi(\tau_2-\eta,\eta)\phi(\eta,s)h(s)dsd\eta\Big\|\\
&\leq Q_1+Q_2+Q_3+Q_4.
\end{aligned}$$

By the assumption $\|h\|_{L^p(J,X)}\leq M$, Lemma 2.8 and the Holder's inequality, we have

$$\begin{aligned}
Q_2 &\leq C_\psi\int_{\tau_1}^{\tau_2}(\tau_2-\eta)^{\alpha-1}\|h(\eta)\|d\eta\leq C_\psi\Big(\frac{p-1}{p\alpha-1}\Big)^{1-\frac{1}{p}}M(\tau_2-\tau_1)^{\alpha-\frac{1}{p}},\\
Q_4 &\leq C_\psi C_\phi\int_{\tau_1}^{\tau_2}\int_0^{\eta}(\tau_2-\eta)^{\alpha-1}(\eta-s)^{\beta-1}\|h(\eta)\|dsd\eta\\
&\leq \frac{1}{\alpha}C_\psi C_\phi\Big(\frac{p-1}{p\beta-1}\Big)^{1-\frac{1}{p}}Mb^{\beta-\frac{1}{p}}(\tau_2-\tau_1)^{\alpha}.
\end{aligned}$$

For $\tau_1=0,\ 0<\tau_2\leq b$, it is easy to see that $Q_1=Q_3=0$. For $\tau_1>0$ and $\delta>0$ small enough, we have

$$\begin{aligned}
Q_1 &\leq \Big\|\int_0^{\tau_1-\delta}[\psi(\tau_2-\eta,\eta)-\psi(\tau_1-\eta,\eta)]h(\eta)d\eta\Big\|\\
&\quad+\Big\|\int_{\tau_1-\delta}^{\tau_1}[\psi(\tau_2-\eta,\eta)-\psi(\tau_1-\eta,\eta)]h(\eta)d\eta\Big\|\\
&\leq \sup_{s\in[0,\tau_1-\delta]}\|\psi(\tau_2-\eta,\eta)-\psi(\tau_1-\eta,\eta)\|b^{1-\frac{1}{p}}M\\
&\quad+C_\psi\int_{\tau_1-\delta}^{\tau_1}[(\tau_2-\eta)^{\alpha-1}+(\tau_1-\eta)^{\alpha-1}]\|h(\eta)\|d\eta\\
&\leq \sup_{s\in[0,\tau_1-\delta]}\|\psi(\tau_2-\eta,\eta)-\psi(\tau_1-\eta,\eta)\|b^{1-\frac{1}{p}}M\\
&\quad+C_\psi(\frac{p-1}{p\alpha-1})^{1-\frac{1}{p}}M\Big[(\tau_2+\delta-\tau_1)^{\alpha-\frac{1}{p}}-(\tau_2-\tau_1)^{\alpha-\frac{1}{p}}+\delta^{\alpha-\frac{1}{p}}\Big].
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
Q_3 &\leq \Big\|\int_0^{\tau_1-\delta}\int_0^{\eta}[\psi(\tau_2-\eta,\eta)-\psi(\tau_1-\eta,\eta)]\phi(\eta,s)h(s)dsd\eta\Big\|\\
&\quad+\Big\|\int_{\tau_1-\delta}^{\tau_1}\int_0^{\eta}[\psi(\tau_2-\eta,\eta)-\psi(\tau_1-\eta,\eta)]\phi(\eta,s)h(s)dsd\eta\Big\|\\
&\leq \sup_{s\in[0,\tau_1-\delta]}\|\psi(\tau_2-\eta,\eta)-\psi(\tau_1-\eta,\eta)\|\\
&\quad\times C_\psi\int_0^{\tau_1-\delta}\int_0^{\eta}(\eta-s)^{\beta-1}\|h(s)\|dsd\eta\\
&\quad+C_\psi C_\phi\int_{\tau_1-\delta}^{\tau_1}\int_0^{\eta}[(\tau_2-\eta)^{\alpha-1}+(\tau_1-\eta)^{\alpha-1}](\eta-s)^{\beta-1}\|h(s)\|dsd\eta\\
&\leq \sup_{s\in[0,\tau_1-\delta]}\|\psi(\tau_2-\eta,\eta)-\psi(\tau_1-\eta,\eta)\|\\
&\quad\times C_\psi(\frac{p-1}{p\beta-1})^{1-\frac{1}{p}}b^{1+\beta-\frac{1}{p}}M(\tau_1-\delta)
\end{aligned}$$

$$+\frac{1}{\alpha}C_\psi\Big(\frac{p-1}{p\beta-1}\Big)^{1-\frac{1}{p}}b^{\beta-\frac{1}{p}}M\Big[(\tau_2+\delta-\tau_1)^\alpha-(\tau_2-\tau_1)^\alpha+\delta^\alpha\Big].$$

From Lemma 2.8, we know that $\psi(t-\eta,\eta)$ is uniformly continuous in the uniform topology in the varaibles $t,\ \eta$, where $0\le\eta\le t-\delta\le b$, for any $\delta>0$. So it can be easily seen that Q_1 and Q_3 tend to zero independently of $h\in\Xi$ as $\tau_2\to\tau_1,\ \delta\to0$. It is also clear that $Q_i(i=2,4)$ tend to zero as $\tau_2\to\tau_1$ does not depend on particular choice of h. Thus, we get that $\|(Fh)(\tau_2)-(Fh)(\tau_1)\|$ tends to zero independently of $h\in\Xi$ as $\tau_2\to\tau_1$. Therefore, we can get $\{(Fh)(t):h\in\Xi\}$ is an equicontinuous subset in $C(J,X)$.

(c). Let $t\in J$ be fixed. We show that the set $\Pi(t)=\{(Fh)(t):h\in\Xi\}$ is relatively compact in X. Clearly, $\Pi(0)=\{0\}$ is compact. So it is only necessary to consider $t>0$. For each $\epsilon\in(0,t),\ t\in(0,b],\ h\in\Xi$ and any $\delta>0$, we define

$$\Pi_{\epsilon,\delta}(t)=\{F_{\epsilon,\delta}(h)(t):h\in\Xi\},$$

where

$$\begin{aligned}
&F_{\epsilon,\delta}(h)(t)\\
=\ &\alpha\int_0^{t-\epsilon}\int_\delta^\infty(t-\eta)^{\alpha-1}\theta\xi_\alpha(\theta)e^{-(t-\eta)^\alpha\theta A(\eta)}h(\eta)d\theta d\eta\\
&+\alpha\int_0^{t-\epsilon}\int_0^\eta\int_\delta^\infty(t-\eta)^{\alpha-1}\theta\xi_\alpha(\theta)e^{-(t-\eta)^\alpha\theta A(\eta)}\phi(\eta,s)h(s)dsd\theta ds\\
=\ &\alpha e^{-\epsilon^\alpha\delta A(\eta)}\Big[\int_0^{t-\epsilon}\int_\delta^\infty(t-\eta)^{\alpha-1}\theta\xi_\alpha(\theta)e^{-[(t-\eta)^\alpha\theta-\epsilon^\alpha\delta]A(\eta)}h(\eta)d\theta d\eta\\
&+\int_0^{t-\epsilon}\int_0^\eta\int_\delta^\infty(t-\eta)^{\alpha-1}\theta\xi_\alpha(\theta)e^{-[(t-\eta)^\alpha\theta-\epsilon^\alpha\delta]A(\eta)}\phi(\eta,s)h(s)dsd\theta ds\Big].
\end{aligned}$$

It is not difficult to get that the last formula of above are bounded. Thus, from the compactness of $e^{-\epsilon^\alpha\delta A(\eta)}$ $(\epsilon^\alpha\delta>0)$ (Remark 2.6), we obtain that $\Pi_{\epsilon,\delta}(t)=\{F_{\epsilon,\delta}(h)(t):h\in\Xi\}$ is a relatively compact subset in X for each $\epsilon\in(0,t)$ and $\delta>0$. Moreover, we have

$$\begin{aligned}
&\|F(h)(t)-F_{\epsilon,\delta}(h)(t)\|\\
=\ &\alpha\Big\|\int_0^t\int_0^\infty(t-\eta)^{\alpha-1}\theta\xi_\alpha(\theta)e^{-(t-\eta)^\alpha\theta A(\eta)}h(\eta)d\theta d\eta\\
&-\int_0^{t-\epsilon}\int_\delta^\infty(t-\eta)^{\alpha-1}\theta\xi_\alpha(\theta)e^{-(t-\eta)^\alpha\theta A(\eta)}h(\eta)d\theta d\eta\Big\|\\
&+\alpha\Big\|\int_0^t\int_0^\eta\int_0^\infty(t-\eta)^{\alpha-1}\theta\xi_\alpha(\theta)e^{-(t-\eta)^\alpha\theta A(\eta)}\phi(\eta,s)h(s)d\theta dsd\eta\\
&-\int_0^{t-\epsilon}\int_0^\eta\int_\delta^\infty(t-\eta)^{\alpha-1}\theta\xi_\alpha(\theta)e^{-(t-\eta)^\alpha\theta A(\eta)}\phi(\eta,s)h(s)d\theta dsd\eta\Big\|\\
\le\ &\alpha\Big\|\int_0^t\int_0^\delta(t-\eta)^{\alpha-1}\theta\xi_\alpha(\theta)e^{-(t-\eta)^\alpha\theta A(\eta)}h(\eta)d\theta d\eta\Big\|\\
&+\alpha\Big\|\int_{t-\epsilon}^t\int_0^\infty(t-\eta)^{\alpha-1}\theta\xi_\alpha(\theta)e^{-(t-\eta)^\alpha\theta A(\eta)}h(\eta)d\theta d\eta\Big\|\\
&+\alpha\Big\|\int_0^t\int_0^\eta\int_0^\delta(t-\eta)^{\alpha-1}\theta\xi_\alpha(\theta)e^{-(t-\eta)^\alpha\theta A(\eta)}\phi(\eta,s)h(s)d\theta dsd\eta\Big\|
\end{aligned}$$

$$
\begin{aligned}
&+\alpha\Big\|\int_{t-\epsilon}^{t}\int_0^{\eta}\int_0^{\infty}(t-\eta)^{\alpha-1}\theta\xi_\alpha(\theta)e^{-(t-\eta)^\alpha\theta A(\eta)}\phi(\eta,s)h(s)d\theta ds d\eta\\
\leq\ & \alpha C\Big(\frac{p-1}{p\alpha-1}\Big)^{1-\frac{1}{p}}M\Big[b^{\alpha-\frac{1}{p}}\int_0^{\delta}\theta\xi_\alpha(\theta)d\theta+\frac{1}{\Gamma(1+\alpha)}\epsilon^{\alpha-\frac{1}{p}}\Big]\\
&+\alpha CC_\phi M\Big[\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\Big(\frac{p-1}{p(\alpha+\beta)-1}\Big)^{1-\frac{1}{p}}b^{\alpha+\beta-\frac{1}{p}}\int_0^{\delta}\theta\xi_\alpha(\theta)d\theta\\
&+\frac{b^{\beta-\frac{1}{p}}}{\Gamma(1+\alpha)}\Big(\frac{p-1}{p\beta-1}\Big)^{1-\frac{1}{p}}\epsilon^{\alpha}\Big].
\end{aligned}
$$

Since $\int_0^\infty \theta\xi_\alpha(\theta)d\theta=\frac{1}{\Gamma(1+\alpha)}$, the last inequality tends to zero when $\epsilon\to 0$ and $\delta\to 0$. Therefore, there are relatively compact sets arbitrarily close to the set $\Pi(t)$ $(t>0)$. Hence the set $\Pi(t)$ $(t>0)$ is also relatively compact in X.

Therefore, we can get the operator F is continuous from w-$L^p(J,X)$ to $C(J,X)$. Finally, we show that the map S is continuous from w-Y_φ to $C(J,X)$.

Since Y_φ is a convex compact metrizable subset of w-$L^p(J,X)$, it suffices to prove the sequential continuity of the map S (Theorem 4.1 of [25]). Now let $\{h_n\}_{n\geq 1}\subseteq Y_\varphi,\ h\in Y_\varphi$ such that

$$h_n\to h \ \text{ in } w\text{-}L^p(J,X). \tag{3.5}$$

From the boundedness of $\{h_n\}_{n\geq1}$ and since $\{F(h_n)\}_{n\geq1}$ is relatively compact in $C(J,X)$, there exists a subsequence $\{h_{nk}\}_{k\geq1}$ of the sequence $\{h_n\}_{n\geq1}$ such that $F(h_{nk})\to z$ in $C(J,X)$ for some $z\in C(J,X)$. Now, since $F(h_{nk})\to z$ in $C(J,X)$ and $h_{nk}\to h$ in w-$L^p(J,X)$ (by (3.5)), it is not difficult to obtain

$$F(h_n)\to F(h) \quad \text{in} \quad C(J,X).$$

Thus, by the definitions of the operators S and F, we know $S(h)(t)=x_0-(FA)(t)x_0+(Fh)(t)$. From the above analysis, we have $S(h_n)\to S(h)$ in $C(J,X)$. The proof is completed. □

Consider the space $\widetilde{Y}=Y\times R$. Elements of the space $\widetilde{Y}$ will be denoted by $\widetilde{u}=(u,r), u\in Y, r\in R$. Endow the space $\widetilde{Y}$ with the norm $\|\widetilde{u}\|_{\widetilde{Y}}=\max(\|u\|_Y,|r|)$. Then $\widetilde{Y}$ is a separable reflexive Banach space. In accordance with (2.1), the norm on the space $L^p_w(J,\widetilde{Y})$ becomes

$$\|\widetilde{u}\|_w=\sup\nolimits_{0\leq t_1\leq t_2\leq b}\Big\{\max\Big(\Big\|\int_{t_1}^{t_2}u(s)ds\Big\|_Y,\Big|\int_{t_1}^{t_2}r(s)ds\Big|\Big)\Big\}, \tag{3.6}$$

where $\widetilde{u}=(u,r),\ u\in L^p(J,Y),\ r\in L^p(J,R)$. Let the multivalued map $\widetilde{U}:J\times X\to\widetilde{Y}$ be defined by

$$\widetilde{U}(t,x)=\{(u,r)\in\widetilde{Y}: u\in U(t,x), r=g(t,x,u)\}. \tag{3.7}$$

Then we have the following properties of the multivalued map $\widetilde{U}(t,x)$:

Lemma 3.3. *The multivalued map $\widetilde{U}$ has bounded closed values and is such that:*

(1) *the map $t\to\widetilde{U}(t,x)$ is measurable;*

(2) *$H_d(\widetilde{U}(t,x),\widetilde{U}(t,y))\leq k(t)\|x-y\|_X$ a.e., with $k(t)=\max\{k_1(t),\ k_2(t)+\rho k_1(t)\}$;*

(3) *for any* $\widetilde{u} = (u, r) \in \widetilde{U}(t, x)$, *we have*

$$|r| \le a_1(t) + b_1(t)\|x\|_X + c_1(m(t) + \gamma\|x\|_X), \quad \|u\|_Y \le m(t) + \gamma\|x\|_X;$$

where the functions $k_1(\cdot), k_2(\cdot), a_1(\cdot), b_1(\cdot), m(\cdot)$ *and the constants* ρ, γ, c_1 *are the same as in assumptions* $H(U)$ *and* $H(g)$.

Proof. Using the similar arguments as Lemma 3.1 of [26] (also see Lemma 3.3 of [15]), one obtains the Lemma 3.3. Here we omit it. □

Let dom $g^{**}(t, x)$ be the effective set and epi $g^{**}(t, x)$ the epigraph of the function $u \to g^{**}(t, x, u)$, i.e.,

$$\text{dom } g^{**}(t, x) = \{u \in Y : g^{**}(t, x, u) < +\infty\},$$

$$\text{epi } g^{**}(t, x) = \{(u, r) \in \widetilde{Y} : g^{**}(t, x, u) \le r\}.$$

The following lemma gives us the properties of the function $g^{**}(t, x, u)$.

Lemma 3.4. *The following assertions hold for a.e.* $t \in J$:

(1) *dom* $g^{**}(t, x) = \overline{co}U(t, x)$;

(2) *for any* $u \in$ *dom* $g^{**}(t, x)$,

$$g^{**}(t, x, u) = min\{r \in R : (u, r) \in \overline{co}\widetilde{U}(t, x)\},$$

and hence $(u, g^{**}(t, x, u)) \in \overline{co}\widetilde{U}(t, x)$, *where* $u \in \overline{co}U(t, x)$ *and* $x \in X$;

(3) *for any* $\varepsilon > 0$, *there exists a closed set* $J_\varepsilon \subseteq J$, $\mu(J \backslash J_\varepsilon) \le \varepsilon$, *such that the map* $(t, x, u) \to g^{**}(t, x, u)$ *is l.s.c. on* $J_\varepsilon \times X \times Y$.

Proof. Using Lemma 3.3 and the similar arguments as Lemma 5.1 in [26] (also see Lemma 3.4 of [15]), one obtains the Lemma 3.4. Here we omit it. □

4 Analogue of Bogolyubov's Theorem

In the present section, we shall be interested in the existence results of the control systems (1.1), (1.2) and (1.1), (1.3) and we also show an analogue of Bogolyubov's theorem with constraints given by the solution sets of the control systems (1.1), (1.2) and (1.1), (1.3).

Theorem 4.1 (Theorem 4.1 in [15])**.** *The set* $\mathcal{R}_U(x_0)$ *is nonempty and the set* $\mathcal{R}_{\overline{co}U}(x_0)$ *is a compact subset of the space* $C(J, X) \times w\text{-}L^p(J, Y)$.

Now, we are in the position to prove the analogue of Bogolyubov's theorem.

Theorem 4.2. *For any* $(x_*(\cdot), u_*(\cdot)) \in \mathcal{R}_{\overline{co}U}(x_0)$, *we can have that there exists a sequence* $(x_n(\cdot), u_n(\cdot)) \in \mathcal{R}_U(x_0) (n \ge 1)$ *such that*

$$x_n \to x_* \quad in \quad C(J, X), \tag{4.1}$$

$$u_n \to u_* \quad in\ L^p(J, Y)\ and\ w\text{-}L^p(J, Y), \tag{4.2}$$

$$\lim_{n\to\infty} \sup_{0 \le t_1 \le t_2 \le b} \left| \int_{t_1}^{t_2} (g^{**}(s, x_*(s), u_*(s)) - g(s, x_n(s), u_n(s)))ds \right| = 0, \tag{4.3}$$

Proof. Let $(x_*(\cdot), u_*(\cdot)) \in \mathcal{R}_{\overline{\text{co}}U}(x_0)$. From Lemma 3.4, the function $t \to g^{**}(t, x_*(t), u_*(t))$ is measurable and

$$(u_*(t), g^{**}(t, x_*(t), u_*(t))) \in \overline{\text{co}}\widetilde{U}(t, x_*(t)) \quad a.e. \tag{4.4}$$

From the hypotheses $H(U)(3)$, $H(g)(3)$, Lemma 3.4 and (3.7), we obtain the map $t \to \overline{\text{co}}\widetilde{U}(t, x_*(t))$ is measurable and there exists a function $\psi \in L^p(J, R^+)$ such that.

$$\|\overline{\text{co}}\widetilde{U}(t, x_*(t))\|_{\widetilde{Y}} \leq \psi(t) \quad a.e. \tag{4.5}$$

Let $n \geq 1$ be fixed. By (4.4), (4.5), (3.6), (3.7) and the Theorem 2.2 in [28], we know that there exists a measurable selection $v_n(t)$ of the map $t \to U(t, x_*(t))$ such that

$$\sup_{0 \leq t_1 \leq t_2 \leq b} \left\| \int_{t_1}^{t_2} (u_*(s) - v_n(s))ds \right\|_Y \leq \frac{1}{n}, \tag{4.6}$$

$$\sup_{0 \leq t_1 \leq t_2 \leq b} \left| \int_{t_1}^{t_2} (g^{**}(s, x_*(s), u_*(s)) - g(s, x_*(s), v_n(s)))ds \right| \leq \frac{1}{n}. \tag{4.7}$$

Hence, the sequence $v_n \to u_*$ in w-$L^p(J, Y)$. For each $n \geq 1$, by $H(U)(2)$, we have that for any $x \in X$, a.e. on $t \in J$, there is a $v(t) \in U(t, x(t))$ such that

$$\|v_n(t) - v(t)\|_Y \leq k_1(t)\|x_*(t) - x(t)\|_X + \frac{1}{n}. \tag{4.8}$$

Let the map $H_n : J \times X \to 2^Y \backslash \{\emptyset\}$ be defined by

$$H_n(t, x) = \{v \in Y;\ v \text{ satisfies the inequality (4.8)}\}. \tag{4.9}$$

It follows from (4.8) that $H_n(t, x)$ is well defined for a.e. on $t \in J$ and for all $x \in X$, and its values are open sets. Using Corollary 2.1 in [28] (since we can assume without loss of generality that $U(t, x)$ is $\Sigma \otimes \mathcal{B}_X$ measurable, see Proposition 2.7.9 in [11]), we obtain that for any $\epsilon > 0$ there is a compact set $J_\epsilon \subseteq J$ with $\mu(J \backslash J_\epsilon) \leq \epsilon$, such that the restriction of $U(t, x)$ to $J_\epsilon \times X$ is l.s.c and the restrictions of $v_n(t)$ and $k_1(t)$ to J_ϵ are continuous. So (4.8) and (4.9) imply that the graph of the restriction of $H_n(t, x)$ to $J_\epsilon \times X$ is an open set in $J_\epsilon \times X \times Y$. Let the map $H : J \times X \to 2^Y$ be defined by

$$H(t, x) = H_n(t, x) \cap U(t, x). \tag{4.10}$$

It is obvious that for a.e. $t \in J$, all $x \in X$, $H(t, x) \neq \emptyset$. From the above analysis and Proposition 1.2.47 in [11], we know that the restriction of $H(t, x)$ to $J_\epsilon \times X$ is l.s.c. and so does $\overline{H}(t, x) = \overline{H(t, x)}$, here the bar stands for the closure of a set in Y.

Now we consider the system (1.1) with the following constraints on the control,

$$u(t) \in \overline{H}(t, x(t)) \qquad \text{a.e. on } J. \tag{4.11}$$

Since $\overline{H}(t, x) \subseteq U(t, x)$, the priori estimate Lemma 3.1 also holds in this situation. Repeating the proof of Theorem 4.1, we obtain that there is a solution $(x_n(\cdot), u_n(\cdot))$ of the control system (1.1), (4.11). The definition of $\overline{H}$ implies that $(x_n(\cdot), u_n(\cdot)) \in \mathcal{R}_U(x_0)$ and

$$\|v_n(t) - u_n(t)\|_Y \leq k_1(t)\|x_*(t) - x_n(t)\|_X + \frac{1}{n}. \tag{4.12}$$

Since $(x_n(\cdot),u_n(\cdot))\in\mathcal{R}_U(x_0)$ and $(x_*(\cdot),u_*(\cdot))\in\mathcal{R}_{\overline{\mathrm{co}}U}(x_0)$, we have

$$\begin{aligned}&x_*(t)\\=&x_0-\int_0^t\psi(t-\eta,\eta)A(\eta)x_0d\eta-\int_0^t\int_0^\eta\psi(t-\eta,\eta)\phi(\eta,s)A(s)x_0dsd\eta\\&+\int_0^t\psi(t-\eta,\eta)B(\eta)u_*(\eta)d\eta+\int_0^t\int_0^\eta\psi(t-\eta,\eta)\phi(\eta,s)B(s)u_*(s)dsd\eta.\end{aligned}\tag{4.13}$$

$$\begin{aligned}&x_n(t)\\=&x_0-\int_0^t\psi(t-\eta,\eta)A(\eta)x_0d\eta-\int_0^t\int_0^\eta\psi(t-\eta,\eta)\phi(\eta,s)A(s)x_0dsd\eta\\&+\int_0^t\psi(t-\eta,\eta)B(\eta)u_n(\eta)d\eta+\int_0^t\int_0^\eta\psi(t-\eta,\eta)\phi(\eta,s)B(s)u_n(s)dsd\eta.\end{aligned}\tag{4.14}$$

Theorem 4.1 and $\{(x_n(\cdot),u_n(\cdot))\}_{n\geq1}\subseteq\mathcal{R}_U(x_0)\subseteq\mathcal{R}_{\overline{\mathrm{co}}U}(x_0)$ imply that we can assume the sequence $(x_n(\cdot),u_n(\cdot))\to(\overline{x}(\cdot),\overline{u}(\cdot))\in\mathcal{R}_{\overline{\mathrm{co}}U}(x_0)$ in $C(J,X)\times w\text{-}L^p(J,Y)$. Subtracting (4.14) from (4.13), we have

$$\begin{aligned}&\|x_*(t)-x_n(t)\|_X\\\leq&\left\|\int_0^t\psi(t-\eta,\eta)B(\eta)[u_*(\eta)-v_n(\eta)]d\eta\right\|\\&+\left\|\int_0^t\psi(t-\eta,\eta)B(\eta)[v_n(\eta)-u_n(\eta)]d\eta\right\|\\&+\left\|\int_0^t\int_0^\eta\psi(t-\eta,\eta)\phi(\eta,s)B(s)[u_*(s)-v_n(s)]dsd\eta\right\|\\&+\left\|\int_0^t\int_0^\eta\psi(t-\eta,\eta)\phi(\eta,s)B(s)[v_n(s)-u_n(s)]dsd\eta\right\|\\\leq&\left\|\int_0^t\psi(t-\eta,\eta)B(\eta)[u_*(\eta)-v_n(\eta)]d\eta\right\|\\&+C_\psi\|B\|\int_0^t(t-\eta)^{\alpha-1}\left(\frac{1}{n}+k_1(\eta)\|x_*(\eta)-x_n(\eta)\|_X\right)d\eta\\&+\left\|\int_0^t\int_0^\eta\psi(t-\eta,\eta)\phi(\eta,s)B(s)[u_*(s)-v_n(s)]dsd\eta\right\|+C_\psi C_\phi\|B\|\\&\times\int_0^t\int_0^\eta(t-\eta)^{\alpha-1}(\eta-s)^{\beta-1}\left(\frac{1}{n}+k_1(s)\|x_*(s)-x_n(s)\|_X\right)dsd\eta\\\leq&\left\|\int_0^t\psi(t-\eta,\eta)B(\eta)[u_*(\eta)-v_n(\eta)]d\eta\right\|\\&+\frac{C_\psi\|B\|b^\alpha}{n\alpha}+C_\psi\|B\|\|k_1\|_{L^\infty}\int_0^t(t-\eta)^{\alpha-1}\|x_*(\eta)-x_n(\eta)\|_Xd\eta\\&+\left\|\int_0^t\int_0^\eta\psi(t-\eta,\eta)\phi(\eta,s)B(s)[u_*(s)-v_n(s)]dsd\eta\right\|\\&+\frac{C_\psi C_\phi\|B\|\Gamma(\alpha)\Gamma(\beta)b^\beta}{\Gamma(\alpha+\beta)}\left[\frac{b^\alpha}{n}+\|k_1\|_{L^\infty}\int_0^t(t-s)^{\alpha-1}\|x_*(s)-x_n(s)\|_Xds\right].\end{aligned}\tag{4.15}$$

By the property of the operator F defined in Lemma 3.2 and since $v_n \to u_*$ in w-$L^p(J,Y)$, we have that for any $t \in J$,

$$\left\| \int_0^t \psi(t-\eta,\eta)B(\eta)[u_*(\eta)-v_n(\eta)]d\eta \right\| \to 0, \quad \text{as } n \to \infty,$$

$$\left\| \int_0^t \int_0^\eta \psi(t-\eta,\eta)\phi(\eta,s)B(s)[u_*(s)-v_n(s)]dsd\eta \right\| \to 0, \quad \text{as } n \to \infty,$$

Since $\|x_*(t)\|_X \le \omega,\ \|x_n(t)\|_X \le \omega$ for any $n,\ t \in J$ and $x_n \to \overline{x}$ in $C(J,X)$, let $n \to \infty$ in (4.15), we obtain

$$\begin{aligned} &\|x_*(t)-\overline{x}(t)\| \\ \le\ & C_\psi \|B\| \|k_1\|_{L^\infty} \left(1 + \frac{C_\phi \Gamma(\alpha)\Gamma(\beta) b^\beta}{\Gamma(\alpha+\beta)}\right) \int_0^t (t-s)^{\alpha-1} \|x_*(s)-\overline{x}(s)\|_X ds. \end{aligned}$$

Then by Corollary 2 of [30], we get $x_* = \overline{x}$, i.e., $x_n \to x_*$ in $C(J,X)$. Hence from (4.10), we have $(v_n - u_n) \to 0$ in $L^p(J,Y)$. Therefore $u_n = u_n - v_n + v_n \to u_*$ in w-$L^p(J,Y)$ and $L^p_w(J,Y)$, i.e. (4.1) and (4.2) hold. Then we have

$$\begin{aligned} & \sup_{0 \le t_1 \le t_2 \le b} \left| \int_{t_1}^{t_2} (g^{**}(s,x_*(s),u_*(s)) - g(s,x_n(s),u_n(s)))ds \right| \qquad (4.16) \\ \le\ & \sup_{0 \le t_1 \le t_2 \le b} \left| \int_{t_1}^{t_2} (g^{**}(s,x_*(s),u_*(s)) - g(s,x_*(s),v_n(s)))ds \right| \\ & + \sup_{0 \le t_1 \le t_2 \le b} \left| \int_{t_1}^{t_2} (g^{**}(s,x_*(s),v_n(s)) - g(s,x_n(s),u_n(s)))ds \right|. \end{aligned}$$

It follows from $H(g)(2)$ and (4.10) that

$$|g(t,x_*(t),v_n(t)) - g(t,x_n(t),u_n(t))| \le (k_2(t) + \rho k_1(t)) \|x_*(t)-x_n(t)\|_X + \frac{\rho}{n}.$$

Then the last inequality, (4.7) and (4.16) imply that (4.3) holds. The Theorem is proved. □

5 Main Result

In this section, we present the following main result of this article.

Theorem 5.1. *The problem (RP) has a solution and*

$$\min_{(x,u)\in \mathcal{R}_{\overline{co}U}(x_0)} \mathcal{J}^{**}(x,u) = \inf_{(x,u)\in \mathcal{R}_U(x_0)} \mathcal{J}(x,u), \qquad (5.1)$$

For any solution (x_*,u_*) of problem (RP) there exists a minimizing sequence $(x_n,u_n) \in \mathcal{R}_U(x_0)(n \ge 1)$ for the problem (P) which converges to (x_*,u_*) in the spaces $C(J,X) \times w$-$L^p(J,Y)$ and $C(J,X) \times L^p_w(J,Y)$ and the following formula holds

$$\lim_{n\to\infty} \sup_{0 \le t_1 \le t_2 \le b} \left| \int_{t_1}^{t_2} (g^{**}(s,x_*(s),u_*(s)) - g(s,x_n(s),u_n(s)))ds \right| = 0. \qquad (5.2)$$

Conversely, if $(x_n, u_n)(n \geq 1)$ is a minimizing sequence for problem (P), then there is a subsequence $(x_{n_k}, u_{n_k})(k \geq 1)$ of the sequence $(x_n, u_n)(n \geq 1)$, and a solution (x_*, u_*) of problem (RP) such that the subsequence $(x_{n_k}, u_{n_k})(k \geq 1)$, converges to (x_*, u_*) in $C(J, X) \times w\text{-}L^p(J, Y)$ and relation (5.2) holds for this subsequence $(x_{n_k}, u_{n_k})(k \geq 1)$.

Proof. By the definition of the function $g_U(t, x, u)$, H$(U)(3)$, H$(g)(3)$ and the boundedness of the trajectories $\mathcal{T}r_{\overline{\text{co}}U}(x_0)$ of the control system (1.1), (1.3) (Lemma 3.1). We can get a function $\zeta \in L^p(J, R^+)$ such that

$$-\zeta(t) = -(a_1(t) + b_1(t)\omega + c_1(m(t) + \gamma\omega)) \leq g(t, x, u), \quad \text{a.e. } t \in J, \tag{5.3}$$

$$\text{with all} \quad x \in Q = \{h \in X : \|h\|_X \leq \omega\}, \quad u \in U(t, x).$$

The inequality (5.3) and the properties of the bipolar (see [6]) directly imply

$$-\zeta(t) \leq g_U^{**}(t, x, u) \leq g_U(t, x, u), \quad \text{a.e. } t \in J, \ x \in Q, \ u \in Y. \tag{5.4}$$

Hence, it follows from Lemma 3.3 item (3), (5.4) and Theorem 2.1 of [2] that the functional $\mathcal{J}^{**}$ is lower semicontinuous on $\mathcal{R}_{\overline{\text{co}}U}(x_0) \subseteq C(J, X) \times w\text{-}L^p(J, Y)$. Theorem 4.1 implies that $\mathcal{R}_{\overline{\text{co}}U}(x_0)$ is compact in $C(J, X) \times w\text{-}L^p(J, Y)$. Therefore, problem (RP) has a solution (x_*, u_*). By the assertion of item (1) in Lemma 3.4, we have

$$\mathcal{J}^{**}(x_*, u_*) \leq \inf_{(x,u) \in \mathcal{R}_U(x_0)} \mathcal{J}(x, u), \tag{5.5}$$

Now for this very solution (x_*, u_*) of problem (RP), using Theorem 4.2, we obtain that there exists a sequence $(x_n, u_n) \in \mathcal{R}_U(x_0)(n \geq 1)$, such that (4.1), (4.2) and (4.3) hold. Since

$$\begin{aligned} &\left| \int_J (g_U^{**}(s, x_*(s), u_*(s)) - g(s, x_n(s), u_n(s)))ds \right| \\ \leq &\sup_{0 \leq t_1 \leq t_2 \leq b} \left| \int_{t_1}^{t_2} (g_U^{**}(s, x_*(s), u_*(s)) - g(s, x_n(s), u_n(s)))ds \right|. \end{aligned} \tag{5.6}$$

By formulas (4.3), (5.5), (5.6), we get that (5.1), (5.2) hold and $(x_n(\cdot), u_n(\cdot)) \in \mathcal{R}_U(x_0)(n \geq 1)$ is a minimizing sequence for problem (P). Let $(x_n(\cdot), u_n(\cdot)) \in \mathcal{R}_U(x_0)(n \geq 1)$ be a minimizing sequence for problem (P). According to Theorem 4.1, without loss of generality we can assume that $(x_n, u_n) \to (x_*, u_*) \in \mathcal{R}_{\overline{\text{co}}U}(x_0)$ in the spaces $C(J, X) \times w\text{-}L^p(J, Y)$ and

$$\min(\text{RP}) = \lim_{n \to \infty} \int_J g(s, x_n(s), u_n(s))ds. \tag{5.7}$$

It follows from (5.4) and the properties of the function $g_U^{**}(t, x_*(t), u_*(t)$ that

$$\begin{aligned} \int_J (g_U^{**}(s, x_*(s), u_*(s))ds &\leq \underline{\lim}_{n \to \infty} \int_J g_U^{**}(s, x_n(s), u_n(s))ds \\ &\leq \lim_{n \to \infty} \int_J g(s, x_n(s), u_n(s))ds. \end{aligned} \tag{5.8}$$

From (5.7) and (5.8), we obtain

$$\min(\text{RP}) = \int_J (g_U^{**}(s, x_*(s), u_*(s))ds = \lim_{n \to \infty} \int_J g(s, x_n(s), u_n(s))ds. \tag{5.9}$$

Hence $(x_*(\cdot), u_*(\cdot)) \in \mathcal{R}_{\overline{\mathrm{co}}U}(x_0)$ is a solution of problem (RP). H(g)(3), H(U)(3) and Lemma 3.1 imply that $\{g(s, x_n(s), u_n(s))\}_{n\geq 1}$ is uniformly integrable. Therefore, by the Dunford-Pettis theorem, we have that there exists a subsequence $g(s, x_{n_k}(s), u_{n_k}(s))(k \geq 1)$ of the sequence $g(s, x_n(s), u_n(s))(n \geq 1)$ converging to a certain function $\lambda(t)$ in the topology of the space w-$L^1(J, Y)$. Since $(u_{n_k}(s), g(s, x_{n_k}(s), u_{n_k}(s))) \in \widetilde{U}(s, x_{n_k}(s))$ a.e. $s \in J$, Lemma 3.3 implies

$$(u_*(s), \lambda(s)) \in \overline{\mathrm{co}}\widetilde{U}(s, x_*(s)) \quad \text{a.e.} \quad s \in J.$$

Using this formula and Lemma 3.4, we obtain

$$g_U^{**}(s, x_*(s), u_*(s)) \leq \lambda(s), \quad \text{a.e. } . \tag{5.10}$$

Hence we have for any $t \in J$

$$\int_0^t g_U^{**}(s, x_*(s), u_*(s))ds \leq \int_0^t \lambda(s)ds \leq \lim_{k\to\infty} \int_0^t g(s, x_{n_k}(s), u_{n_k}(s)))ds. \tag{5.11}$$

Now we can obtain from (5.9), (5.10) and (5.11) that

$$g_U^{**}(t, x_*(t), u_*(t)) = \lambda(t), \quad \text{a.e.} \quad t \in J.$$

Hence the subsequence $g(s, x_{n_k}(s), u_{n_k}(s))) \to g_U^{**}(s, x_*(s), u_*(s))$, as $k \to \infty$ in w-$L^p(J, Y)$. This implies that

$$\lim_{k\to\infty} \sup_{0\leq t_1\leq t_2\leq b} \left| \int_{t_1}^{t_2} (g_U^{**}(s, x_*(s), u_*(s)) - g(s, x_{n_k}(s), u_{n_k}(s)))ds \right| = 0.$$

The Theorem is proved. $\square$

6 An Example

Let $J = [0, b]$ and $\Omega = [0, 1]$. We consider the following heat equation:

$$\begin{cases} {}^C D_t^\alpha x(t, y) = \frac{\partial^2}{\partial y^2} x(t, y) + a(t)x(t, y) + b(t)u(t, y), \ t \in J, \ y \in \Omega, \\ \quad x(t, 0) = x(t, 1) = 0, \quad t \in J, \\ \quad x(0, y) = x_0(y), \qquad y \in \Omega, \\ u(t, y) \in \widehat{U}(t, y, x(t, y)), \qquad a.e. \quad J \times \Omega, \end{cases} \tag{6.1}$$

where $0 < \alpha < 1$ and $x(t, y)$ represents the temperature at the point $y \in \Omega$ and time $t \in J$. It is supposed that $a(\cdot) : J \to R^+$ is a continuous function.

Next, take $X = L^2[0, 1]$ and the operator $A(t)$ be defined by $A(t)x = x'' + a(t)x$ with the common domain

$$D(A) = \{x \in X : x, \ x' \text{ are absolutely continuous}, x'' \in X, \ x(0) = x(1) = 0\}.$$

It follows that the operator $\{A(t) : t \in J\}$ satisfies conditions H(A)(1)-(4) and generates a compact evolution system $V(t, s)$ (cf. [10]) given by

$$V(t, s) = T(t - s)e^{-\int_s^t a(\tau)d\tau},$$

where $T(t)(t > 0)$ is the compact semigroup generated by the operator A with $Ax = x''$ for $x \in D(A)$.

Next, suppose that $\widehat{U} : J \times \Omega \times R \to 2^R \backslash \{\emptyset\}$ is a multivalued function with closed values satisfying the following conditions:

(1) the map $(t, y) \to \widehat{U}(t, y, x)$ is measurable;

(2) $H_d(\widehat{U}(t, y, x_1); \widehat{U}(t, y, x_2)) \le k_1(t)|x_1 - x_2|$ a.e. in $(t, y) \in J \times \Omega$ with k_1 in $L^\infty(J, R^+)$;

(3) $|\widehat{U}(t, y, x)| \le m(t, y) + \gamma|x|$ a.e. in $J \times \Omega$ with $m \in L^p(J, R^+)(p > \frac{1}{\alpha})$ and $\gamma > 0$.

Put $x(t) = x(t, y)$ that is $x(t)(y) = x(t, y)$, $t \in J$, $y \in \Omega$. Define a multivalued map $U : J \times X \to 2^X$ by

$$U(t, x) = \{u \ \text{ is measurable } : u(y) \in \widehat{U}(t, y, x(y)) \ \text{ a.e. in } \Omega\}, \quad x \in X. \tag{6.2}$$

From Lemma 7.3 of [27], when U is defined by (6.2), then the hypothesis $H(U)$ is satisfied. With $A(t)$ and U defined above, the fractional control system (6.1) can be rewritten in our abstract form (1.1), (1.2). Hence the abstract results obtained in the previous sections can be applied to the control system (6.1).

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