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A FULL-NEWTON STEP INFEASIBLE INTERIOR-POINT METHOD BASED ON A NEW SEARCH DIRECTION

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Abstract: Here, we analyze a full-Newton step infeasible interior-point method for linear optimization based on a new search direction. The method is based on an algebraic transformation of the central path for finding the search directions. Each iteration of the algorithm consists of only a feasibility step. The new algorithm admits a simple analysis of complexity bound. For an infeasible interior-point algorithm based on the new search directions, the complexity bound is the best available.

Key words: *linear optimization, infeasible interior-point methods, new search directions, polynomial complexity*

Mathematics Subject Classification: *90C51*

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1 Introduction

Interior-point methods (IPMs) for solving linear optimization (LO) problems were initiated by Karmarkar [5]. They not only have polynomial complexity but are also highly efficient in practice. A comprehensive treatment of IPMs can be found in [15, 19, 20].

The primal-dual full-Newton step feasible IPM for LO was first analyzed by Roos et al. [15] and was later extended to infeasible version by Roos [13]. Subsequently, Gu et al. [4] proposed an improved version of full-Newton step infeasible IPM (IIPM) for LO where the convergence analysis of the algorithm was simplified. Both versions of the method were extended by Kheirfam and Mahdavi-Amiri [10] to linear complementarity problem over symmetric cones using Euclidean Jordan algebras, which the search direction is based on the Nesterov-Todd (NT) scheme. Zhang et al. [21] extended both versions of the method for semidefinite optimization (SDO) based on a new proximity measure and simplified the complexity analysis. Subsequently, Kheirfam [6] designed and analyzed the method for horizontal linear complementarity problem "(HLCP)" based on the proximity measure introduced in [21]. Some extensions of Roos' algorithm were carried out by Kheirfam [7, 8], Kheirfam and Hasani [11] and Liu et al. [12].

Darvay [3] proposed a full-Newton step feasible interior-point algorithm for LO based on an algebraic equivalent form of the central-path and then applying Newton's approach for the resulting system. Later on, Achache [1], Wang and Bai [16, 17] and Wang et al. [18] respectively extended Darvay's algorithm for LO to convex quadratic optimization (CQO), second-order cone optimization (SOCO), symmetric optimization (SO) and $P_*(\kappa)$-LCP. Ahmadi et al. [2] and Kheirfam [9] respectively adapted the technique proposed in [3] to IIPM. Very recently, Roos [14] proposed a new method for LO by improving the full-Newton step

IIPMs so that the centering steps not be needed, whereas the above-mentioned IIPMs require a few centering steps in each main iteration. An interesting question here is whether a new algebraic transformation of the central path to finding a search direction can be found that the improved version of IIPM based on this search direction is well-defined?.

The key idea of the method presented in this paper is to apply a new algebraic reformulation of the central path along with finding the search directions. The purpose of the paper is mainly theoretical, which we present an improved version of IIPM for LO based on the new search directions. The algorithm reduces the number of iterations and tendering a simple analysis, while the best complexity known for these types of methods is still maintained.

The remainder of our work is organized as follows. In the next section, we recall the perturbed problems and their central paths. In section 3, we derive new search directions and describe an iteration of our algorithm. We then present the algorithm. Section 4 is devoted to the analysis of the algorithm. In subsection 4.1, we derive an upper bound for the proximity measure after a full-Newton step. Subsection 4.2 serves to derive an upper bound for $\omega(v)$. In subsection 4.3, we fix values for the parameters τ and θ in the algorithm. Here, τ is a uniform upper bound for the values of the proximity measure $\delta(x, s; \mu)$ occurring during the course of the algorithm, and θ determines the progress to feasibility and optimality of the iterates. As a result, we realize the algorithm to be well-defined for the chosen values of θ and τ. We obtain the complexity of the algorithm coinciding with the best complexity known for IIPMs, whereas the iteration bound improves one in [13]. Finally, some conclusions and remarks follow in section 5.

2 Preliminaries

Consider the LO problem in the standard form

$$(P) \qquad \min \ \{c^T x: \ Ax = b, \ x \geq 0\},$$

with its dual problem

$$(D) \qquad \max \ \{b^T y: \ A^T y + s = c, \ s \geq 0\}.$$

Here $x, s, c \in R^n, b, y \in R^m$ and $A \in R^{m\times n}$ with rank$(A) = m$. In accordance with the available results on IIPMs (e.g., see [13]), it is assumed that there exists an optimal solution (x^*, y^*, s^*) such that $\|(x^*; s^*)\|_\infty \leq \xi$, where ξ is a (positive) number. In our algorithm, the initial iterates will be $(x^0, y^0, s^0) = \xi(e, 0, e)$, where e is the all-one vector. Note that, since x^* and s^* are feasible, $\|(x^*; s^*)\|_\infty \leq \xi$ holds if and only if

$$0 \leq x^* \leq \xi e, \quad 0 \leq s^* \leq \xi e. \tag{2.1}$$

In the case of an infeasible method, a triple (x, y, s) is called an ϵ-solution of (P) and (D) if

$$\max \left\{x^T s, \|b - Ax\|, \|c - A^T y - s\|\right\} \leq \epsilon,$$

where ϵ is a accuracy parameter. We start by choosing arbitrary $x^0 > 0$ and $s^0 > 0$ such that $x^0 s^0 = \mu^0 e$, for some (positive) number μ^0. For any ν, with $0 < \nu \leq 1$, we consider the perturbed problem (P_ν), defined by

$$(P_\nu) \qquad \min \ \{(c - \nu r_c^0)^T x: \ Ax = b - \nu r_b^0, \ x \geq 0\},$$

and its dual problem (D_ν), which is given by

$$(D_\nu) \qquad \max \ \{(b - \nu r_b^0)^T y: \ A^T y + s = c - \nu r_c^0, \ s \geq 0\},$$

where $r_b^0 := b - Ax^0$ and $r_c^0 := c - A^T y^0 - s^0$. In general, $r_b^0 \neq 0$ and $r_c^0 \neq 0$, i.e., the initial iterates are not feasible for (P) and (D). However, it is clear that $x = x^0$ is a strictly feasible solution of (P_ν), and $(y, s) = (y^0, s^0)$ is a strictly feasible solution of (D_ν), when $\nu = 1$. This means that the perturbed problems (P_ν) and (D_ν) satisfy the interior point condition (IPC), for $\nu = 1$, which then straightforwardly leads to the following result.

Lemma 2.1 (Theorem 5.13 in [20]). *The original problems,* (P) *and* (D)*, are feasible if and only if for each ν satisfying $0 < \nu \leq 1$ the perturbed problems* (P_ν) *and* (D_ν) *satisfy the IPC.*

Let (P) and (D) be feasible and $0 < \nu \leq 1$. Then, Lemma 2.1 implies that the perturbed problems (P_ν) and (D_ν) satisfy the IPC, and therefore the following system

$$\begin{aligned} b - Ax &= \nu r_b^0, \quad x \geq 0, \\ c - A^T y - s &= \nu r_c^0, \quad s \geq 0, \\ xs &= \mu e, \end{aligned} \tag{2.2}$$

has a unique solution, for every $\mu > 0$, as the μ-centers of the perturbed problems (P_ν) and (D_ν). Let $\psi(t)$ be a real valued function on $(0, \infty)$ such that $\psi(0) = 0$ and differentiable on $(0, \infty)$ such that $\psi'(t) > 0$, for each $t > 0$. The system of equations (2.2) can be written in the following equivalent form:

$$\begin{aligned} b - Ax &= \nu r_b^0, \quad x \geq 0, \\ c - A^T y - s &= \nu r_c^0, \quad s \geq 0, \\ \psi\left(\tfrac{x_i s_i}{\mu}\right) &= \psi(1), \quad i = 1, \ldots, n. \end{aligned} \tag{2.3}$$

In what follows, the parameters μ and ν always satisfy the relation $\mu = \mu^0 \nu$.

3 New Search Direction and the Algorithm

Let (x, y, s) be a strictly feasible solution of (P_ν) and (D_ν) for some $\mu > 0$. Our aim is to define search direction $(\Delta x, \Delta y, \Delta s)$ such that the new iterate $(x + \Delta x, y + \Delta y, s + \Delta s)$ is feasible for (P_{ν^+}) and (D_{ν^+}), where $\nu^+ := (1 - \theta)\nu$ with $\theta \in (0, 1)$. Applying Newton's approach to the system (2.3) and linearizing the third equation, by some elementary calculations, we get

$$\begin{aligned} A\Delta x &= \theta \nu r_b^0, \\ A^T \Delta y + \Delta s &= \theta \nu r_c^0, \\ s\Delta x + x\Delta s &= \mu \big(\psi'(\tfrac{xs}{\mu})\big)^{-1} \big(\psi(e) - \psi(\tfrac{xs}{\mu})\big). \end{aligned} \tag{3.1}$$

Defining

$$v := \sqrt{\frac{xs}{\mu}}, \quad d_x := \frac{v\Delta x}{x}, \quad d_s := \frac{v\Delta s}{s}, \tag{3.2}$$

the system (3.1) turns to

$$\begin{aligned} \bar{A} d_x &= \theta \nu r_b^0, \\ \bar{A}^T \tfrac{\Delta y}{\mu} + d_s &= \theta \nu v s^{-1} r_c^0, \\ d_x + d_s &= p_v, \end{aligned} \tag{3.3}$$

where $\bar{A} := AV^{-1}X, V := \mathrm{diag}(v), X := \mathrm{diag}(x)$ and $p_v := \frac{\psi(e) - \psi(v^2)}{v\psi'(v^2)}$. By choosing $\psi(t)$ appropriately, the system (3.3) can be used to define a class of search directions. For example:

(i) $\psi(t) = t$ yields $p_v = v^{-1} - v$ which gives the search directions of the feasibility step in [13].

(ii) $\psi(t) = \sqrt{t}$ yields $p_v = 2(e - v)$ which gives the search directions of the feasibility step in [2].

Here, we choose $\psi(t) = \frac{\sqrt{t}}{2(1+\sqrt{t})}$. In this case, $\psi^{'}(t) = \frac{1}{4\sqrt{t}(1+\sqrt{t})^2}$ and

$$\frac{\psi(1) - \psi(t^2)}{t\psi'(t^2)} = \frac{\frac{1}{4} - \frac{t}{2(1+t)}}{t\big(\frac{1}{4t(1+t)^2}\big)} = \frac{\frac{1-t}{4(1+t)}}{\frac{1}{4(1+t)^2}} = (1+t)(1-t) = 1 - t^2.$$

Therefore, we get $p_v = e - v^2$. A solution of (3.3) with $p_v = e - v^2$ returns d_x and d_s, and then Δx and Δs compute via (3.2). After a full-Newton step, the iterates given by

$$x_+ := x + \Delta x, \quad y_+ := y + \Delta y, \quad s_+ := s + \Delta s. \tag{3.4}$$

We measure proximity to the μ-centers of the perturbed problems (P_ν) and (D_ν) by the quantity, which has been considered for SDO for the first time in [21],

$$\delta(v) := \delta(x, s; \mu) := \|p_v\| = \|e - v^2\|. \tag{3.5}$$

As a consequence, we have the following lemma.

Lemma 3.1 (Lemma 2.1 in [6]). *If $\delta := \delta(v)$, then*

$$\sqrt{1-\delta} \leq v_i \leq \sqrt{1+\delta}, \; i = 1, 2, \ldots, n.$$

We just established that if $\nu = 1$ and $\mu = \mu^0$, then $x = x^0$ is the μ-center of the perturbed problem (P_ν) and $(y, s) = (y^0, s^0)$ the μ-center of (D_ν). Initially, we have $\delta(x, s; \mu) = \delta(x^0, s^0; \mu^0) = 0$. In what follows, we assume that at start of each iteration, just before the μ-update, $\delta(x, s; \mu) \leq \tau$, where $\tau > 0$ is a threshold value. So, this is certainly true at the start of the first iteration.

Now, we briefly describe one (main) iteration of our algorithm. Suppose that for some $\mu \in (0, \mu^0]$, we have x, y and s satisfying the feasibility conditions, the first two equations of (2.3), for $\nu = \frac{\mu}{\mu^0}$ and such that $\delta(x, s; \mu) \leq \tau$. We reduce μ to $\mu_+ = (1 - \theta)\mu$, with $\theta \in (0, 1)$, and find new iterates x_+, y_+ and s_+ that satisfy the first two equations of (2.3), with μ replaced by μ_+ and ν by $\nu_+ = \frac{\mu_+}{\mu^0}$, and such that $\delta(x_+, s_+; \mu_+) \leq \tau$. Due to (3.1) and (3.4), it is clear that x_+, y_+ and s_+ satisfy the first two affine equations in (2.3), with $\nu = \nu_+$. The main part of the analysis is to guarantee that x_+ and s_+ are positive and satisfy $\delta(x_+, s_+; \mu_+) \leq \tau$.

A formal description of the algorithm is given in Figure 1.

Primal – Dual Infeasible IPM

Input :
  Accuracy parameter $\epsilon > 0$;
  barrier update parameter θ, $0 < \theta < 1$.
begin
  $x := \xi e$; $y := 0$; $s := \xi e$; $\mu := \nu\xi^2$; $\nu = 1$;
while $\max(n\mu, \|r_b\|, \|r_c\|) > \epsilon$ **do**
  begin
  solve the system (3.3) and use (3.2) to obtain $(\Delta x, \Delta y, \Delta s)$;
    $(x, s) := (x, s) + (\Delta x, \Delta s)$;
   update of μ and ν :
    $\mu := (1 - \theta)\mu$;
    $\nu := (1 - \theta)\nu$;
  end
end

Figure 1 : The algorithm.

4 Analysis of the Algorithm

Let x, y and s denote the iterates at the start of an iteration, and assume $\delta(x, s; \mu) \leq \tau$. Moreover, let $q_v = d_x - d_s$. Then

$$d_x = \frac{p_v + q_v}{2}, \quad d_s = \frac{p_v - q_v}{2}, \quad d_x d_s = \frac{p_v^2 - q_v^2}{4}. \tag{4.1}$$

It follows that

$$\frac{\|q_v\|^2}{4} = \frac{\|p_v\|^2}{4} - d_x^T d_s. \tag{4.2}$$

4.1 Upper bound for $\delta(v_+)$

Using (3.2) and (3.4), we may write

$$x_+ = \frac{x}{v}(v + d_x), \quad s_+ = \frac{s}{v}(v + d_s). \tag{4.3}$$

From (4.3), the third equation of (3.3) and (4.1), it follows that

$$x_+ s_+ = \frac{xs}{v^2}(v + d_x)(v + d_s) = \mu(v^2 + v p_v + d_x d_s) = \mu\Big(\Big(v + \frac{p_v}{2}\Big)^2 - \frac{q_v^2}{4}\Big). \tag{4.4}$$

In the sequel, we use the notation

$$\omega(v) := \frac{1}{2}\big(\|d_x\|^2 + \|d_s\|^2\big).$$

It follows that

$$-d_x^T d_s \leq \big|d_x^T d_s\big| \leq \|d_x\|\|d_s\| \leq \frac{1}{2}\big(\|d_x\|^2 + \|d_s\|^2\big) = \omega(v). \tag{4.5}$$

Lemma 4.1. *Let $x > 0$ and $s > 0$ be feasible solutions for (P_ν) and (D_ν), respectively, and let $\delta(v) := \delta(x, s; \mu)$ which satisfies $\delta(v) + \omega(v) < 1$. Then x_+ and s_+ are strictly feasible solutions of (P_{ν_+}) and (D_{ν_+}).*

Proof. Introduce a step length α, with $0 \le \alpha \le 1$, and define

$$x(\alpha) := x + \alpha\Delta x, \quad s(\alpha) := s + \alpha\Delta s.$$

Using (3.2), the third equation of system (3.3) and (4.1), we have

$$\begin{aligned} \frac{x(\alpha)s(\alpha)}{\mu} &= (v + \alpha d_x)(v + \alpha d_s) = v^2 + \alpha v(d_x + d_s) + \alpha^2 d_x d_s \\ &= (1-\alpha)v^2 + \alpha\Big(\big(v + \frac{p_v}{2}\big)^2 - (1-\alpha)\frac{p_v^2}{4} - \alpha\frac{q_v^2}{4}\Big). \end{aligned} \tag{4.6}$$

The inequality $x(\alpha)s(\alpha) > 0$ holds if

$$\Big\|(1-\alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\Big\|_\infty < \min_i \Big(v + \frac{p_v}{2}\Big)_i^2 = \min_i \Big(v_i + \frac{1 - v_i^2}{2}\Big)^2. \tag{4.7}$$

One can easily verify that $f(t) = -t^2 + 2t + 1$, for $\sqrt{1 - \delta(v)} \le t \le \sqrt{1 + \delta(v)}$, is concave. Thus, for each $i = 1, 2, \ldots, n$, we get

$$f(v_i) \ge \min\big\{f(\sqrt{1-\delta(v)}), f(\sqrt{1+\delta(v)})\big\} = f(\sqrt{1-\delta(v)}) \ge 2 - \delta(v).$$

Therefore,

$$\Big(\frac{2 - \delta(v)}{2}\Big)^2 \le \min_i \Big(\frac{f(v_i)}{2}\Big)^2. \tag{4.8}$$

On the other hand, using (4.2) and (4.5), we obtain

$$\begin{aligned} \Big\|(1-\alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\Big\|_\infty &\le (1-\alpha)\Big\|\frac{p_v^2}{4}\Big\|_\infty + \alpha\Big\|\frac{q_v^2}{4}\Big\|_\infty \le (1-\alpha)\frac{\|p_v\|^2}{4} + \alpha\frac{\|q_v\|^2}{4} \\ &= \frac{\delta(v)^2}{4} - \alpha d_x^T d_s \le \frac{\delta(v)^2}{4} + \alpha\omega(v) \le \frac{\delta(v)^2}{4} + \omega(v). \end{aligned} \tag{4.9}$$

Therefore, (4.7) holds if

$$\frac{\delta(v)^2}{4} + \omega(v) < \Big(\frac{2 - \delta(v)}{2}\Big)^2.$$

This implies $\delta(v) + \omega(v) < 1$. Thus, we obtain that $x(\alpha)s(\alpha) > 0$, for all $0 \le \alpha \le 1$. Hence, none the entries of $x(\alpha)$ and $s(\alpha)$ vanishes, for $0 \le \alpha \le 1$. Since $x(0) = x > 0$ and $s(0) = s > 0$, and $x(\alpha)$ and $s(\alpha)$ depend linearly on α, these imply that $x(\alpha)$ and $s(\alpha)$ are positive, for $0 \le \alpha \le 1$. Hence, $x(1) = x_+ > 0$ and $s(1) = s_+ > 0$. This completes the proof. □

We proceed by deriving an upper bound for $\delta(x_+, s_+; \mu_+)$. Recall from definition (3.5) that

$$\delta(v_+) = \delta(x_+, s_+; \mu_+) = \big\|e - v_+^2\big\|, \text{ where } v_+ = \sqrt{\frac{x_+ s_+}{\mu_+}}.$$

Lemma 4.2. *Let $x > 0$ and $s > 0$ be feasible solutions for* (P_ν) *and* (D_ν) *respectively, and let $\delta(v) + \omega(v) < 1$. Then, we have*

$$\delta(v_+) \leq \frac{1}{1-\theta}\Big(\theta\sqrt{n} + \frac{\delta(v)^2}{1+\sqrt{1-\delta(v)}} + \omega(v)\Big).$$

Proof. Using (4.6) with $\alpha = 1$ and $p_v = e - v^2$, after, by $\mu_+ = (1-\theta)\mu$, we get

$$\begin{aligned}\frac{x_+ s_+}{\mu_+} &= \frac{\big(v + \frac{p_v}{2}\big)^2 - \frac{q_v^2}{4}}{1-\theta} = \frac{v^2 + vp_v + \frac{p_v^2}{4} - \frac{q_v^2}{4}}{1-\theta} = \frac{e - (e-v)p_v + \frac{p_v^2}{4} - \frac{q_v^2}{4}}{1-\theta} \\ &= \frac{e - \frac{p_v^2}{e+v} + \frac{p_v^2}{4} - \frac{q_v^2}{4}}{1-\theta} = \frac{e - \big(\frac{3e-v}{e+v}\big)\frac{p_v^2}{4} - \frac{q_v^2}{4}}{1-\theta}.\end{aligned}$$

Hence, using Lemma 2.1, (3.5), (4.2) and (4.5), we may write

$$\begin{aligned}\delta(v_+) &= \big\|e - v_+^2\big\| = \Big\|e - \frac{e - \big(\frac{3e-v}{e+v}\big)\frac{p_v^2}{4} - \frac{q_v^2}{4}}{1-\theta}\Big\| \\ &= \frac{1}{1-\theta}\Big\| -\theta e + (\frac{3e-v}{e+v})\frac{p_v^2}{4} + \frac{q_v^2}{4}\Big\| \\ &\leq \frac{1}{1-\theta}\Big(\theta\sqrt{n} + \|\frac{3e-v}{e+v}\|_\infty \|\frac{p_v^2}{4}\| + \|\frac{q_v^2}{4}\|\Big) \\ &\leq \frac{1}{1-\theta}\Big(\theta\sqrt{n} + \frac{3-\sqrt{1-\delta(v)}}{1+\sqrt{1-\delta(v)}} \times \frac{\delta(v)^2}{4} + \frac{\delta(v)^2}{4} - d_x^T d_s\Big) \\ &\leq \frac{1}{1-\theta}\Big(\theta\sqrt{n} + \frac{\delta(v)^2}{1+\sqrt{1-\delta(v)}} + \omega(v)\Big).\end{aligned}$$

This completes the proof. □

$\boxed{4.2}$ Upper bound for $\omega(v)$

Let us denote the null space of the matrix $\bar{A}$ as $\mathcal{N}$. So,

$$\mathcal{N} := \{\zeta : \bar{A}\zeta = 0\}.$$

Then it is clear from the first equation of (3.3) that the affine space $\{\zeta : \bar{A}\zeta = \theta\nu r_b^0\}$ equals $\mathcal{N} + d_x$. Note that due to a well-known result from linear algebra the row space of $\bar{A}$ equals the orthogonal complement $\mathcal{N}^\perp$ of $\mathcal{N}$. Then, due to the second equation of (3.3), the affine space $\{\bar{A}^T z + \theta\nu v s^{-1} r_c^0\}$ equals $\mathcal{N}^\perp + d_s$. Also note that $\mathcal{N}^\perp \cap \mathcal{N} = \{0\}$, and as a consequence the affine spaces $\mathcal{N} + d_x$ and $\mathcal{N}^\perp + d_s$ meet in a unique point q.

Lemma 4.3. *Let q be the (unique) point in the intersection of the affine spaces $\mathcal{N} + d_x$ and $\mathcal{N}^\perp + d_s$. Then*

$$2\omega(v) \leq \|q\|^2 + \big(\|q\| + \delta(v)\big)^2.$$

Proof. To simplify the notation, we denote $r = e - v^2$. By the same way as in the proof of Lemma 3.5 in [14], we may derive

$$2\omega(v) \leq \|q\|^2 + \big(\|q\| + \|r\|\big)^2 = \|q\|^2 + \big(\|q\| + \delta(v)\big)^2,$$

proving the lemma. □

Lemma 4.4. (Lemma 3.6 in [14]) *One has*

$$\|q\| \leq \frac{\theta}{\xi v_{\min}}(\|x\|_1 + \|s\|_1).$$

Following section 3.4 in [14], we can prove that $\|x\|_1 + \|s\|_1 \leq \xi(n + \|v\|^2)$. Substitution into Lemma 4.4 yields that

$$\|q\| \leq \frac{\theta(n + \|v\|^2)}{v_{\min}}. \tag{4.10}$$

Using Lemma 3.1, we get

$$\|v\|^2 = \sum_{i=1}^{n} v_i^2 \leq n(1 + \delta(v)) \quad \text{and} \quad v_{\min} \geq \sqrt{1 - \delta(v)}.$$

Substitution of these two bounds into (4.10) yields

$$\|q\| \leq \frac{n\theta(2 + \delta(v))}{\sqrt{1 - \delta(v)}}. \tag{4.11}$$

It follows from (4.11) and Lemma 4.3 that

$$\omega(v) \leq \frac{n^2\theta^2(2 + \delta(v))^2}{1 - \delta(v)} + \frac{\delta(v)^2}{2} + \frac{n\theta(2 + \delta(v))\delta(v)}{\sqrt{1 - \delta(v)}}. \tag{4.12}$$

4.3 Fixing the parameters

By Lemma 4.1, the iterates x_+ and s_+ are strictly feasible if $\delta(v) + \omega(v) < 1$. It follows from (4.12) that if

$$\frac{n^2\theta^2(2 + \delta(v))^2}{1 - \delta(v)} + \frac{\delta(v)^2}{2} + \frac{n\theta(2 + \delta(v))\delta(v)}{\sqrt{1 - \delta(v)}} < 1 - \delta(v), \tag{4.13}$$

then inequality $\delta(v) + \omega(v) < 1$ certainly holds. It is easy to verify that the left-hand side of (4.13) is monotonically increasing with respect to $\delta(v)$, while the right-hand side is monotonically decreasing with respect to $\delta(v)$. Given a threshold parameter τ, with $\delta(v) \leq \tau$, we obtain

$$\frac{n^2\theta^2(2 + \tau)^2}{1 - \tau} + \frac{\tau^2}{2} + \frac{n\theta(2 + \tau)\tau}{\sqrt{1 - \tau}} < 1 - \tau. \tag{4.14}$$

In this stage, setting

$$\tau = \frac{1}{4}, \qquad \theta = \frac{1}{10\sqrt{2}n}, \tag{4.15}$$

an upper bound for the left-hand side of inequality (4.14) is 0.1109, while a lower bound for the right-hand side of inequality (4.14) is 0.7500. In this case, by Lemma 4.1, we conclude that the iterate (x_+, s_+) is strictly feasible. We proceed to derive an upper bound for $\delta(v_+)$. From Lemma 4.2, we have

$$\delta(v_+) \leq \frac{1}{1 - \theta}\Big(\theta\sqrt{n} + \frac{\delta(v)^2}{1 + \sqrt{1 - \delta(v)}} + \omega(v)\Big).$$

Using (4.15), it follows from the above inequality, with the right-hand side of the above inequality being monotonically increasing with respect to $\delta(v)$, that

$$\delta(v_+) \leq \frac{1}{1-\theta}\Big(\theta\sqrt{n} + \frac{\tau^2}{1+\sqrt{1-\tau}} + \omega(v)\Big) \leq 0.2315 < 0.2500 = \frac{1}{4}.$$

Therefore, the algorithm is well-defined in the sense that the property $\delta(x, s; \mu) \leq \tau$ is maintained in all iterations.

4.4 Complexity analysis

We have found that if at the start of an iteration the iterate satisfying $\delta(x, s; \mu) \leq \tau$ and τ and θ are defined as in (4.15), then after the full-Newton step, the new iterate is strictly feasible and satisfies $\delta(x_+, s_+; \mu_+) \leq \tau$. This establishes the algorithm to be well-defined.

In each main iteration, both the barrier parameter μ and $\|r_b\|$ and $\|r_c\|$ are reduced by the factor $1-\theta$. Hence, the total number of main iterations is bounded above by

$$\frac{1}{\theta} \log \frac{\max\{n\xi^2, \|r_b^0\|, \|r_c^0\|\}}{\epsilon}.$$

Since $\theta = \frac{1}{10\sqrt{2}n}$, this yields the following result.

Theorem 4.5. *Let* (P) *and* (D) *be feasible and* $\xi > 0$ *such that* $\|(x^*; s^*)\|_\infty \leq \xi$ *for some optimal solutions* x^* *of* (P) *and* (y^*, s^*) *of* (D)*. In this case, after at most*

$$10\sqrt{2}n \log \frac{\max\{n\xi^2, \|r_b^0\|, \|r_c^0\|\}}{\epsilon}$$

iterations the algorithm finds an ϵ*-solution of* (P) *and* (D)*.*

5 Conclusions

Based on a new reformulation of the central path for finding the search directions, we presented an improved full-Newton step IIPM for LO and derived the complexity results. As a result, the number of iterations of [2, 4, 11, 13] is improved by a factor 3. It is worth noting that Theorem 4.5 improves the iteration bound in [2,4,11,13] with a constant factor. We refer to [13] for a discussion on how to choose the number ξ in the algorithm, and on how infeasibility or unboundedness of the problems (P) and (D) can be established with the algorithm presented. An interesting topic for further research may be the development of the analysis to SO.

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