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MODIFIED LAGRANGIAN DUALITY FOR THE SUPREMUM OF CONVEX FUNCTIONS*

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Abstract: In this paper we associate with the problem (P) consisting of minimizing the supremum h of a family of extended real-valued convex functions on a given closed convex set C a modified Lagrange dual problem (D) . Exploiting recent developments of the authors on convex duality, we provide min-sup and inf-max duality theorems for the pair $(P)-(D)$, we characterize the subdifferentials of the objective function g of the reformulation of (P) as an unconstrained problem (nothing else but the sum of h with the indicator of C) and its conjugate g^*, and finally we give an explicit expression for the optimal set of (P).

Key words: *Convex infinite programming, duality*

Mathematics Subject Classification: *90C25; 90C48, 46N10*

1 Introduction

It is well-known that minimax optimization problems arise in a huge number of applied fields as engineering design [12], computer-aided-design [13], circuit design [18], optimal control [16], etc. In the convex setting, and besides the duality for the sum of convex functions (recently considered in [17]), the duality for the supremum of a family of convex functions is in the heart of convex optimization and has many applications, specially in zero sum game theory (see [9] for instance) and min-max convex optimization (see [1] and references therein). In this note we present updated versions of more or less classical results by using recent improvements established in [4] and [5].

Having X a locally convex separated topological vector space, T a possibly infinite index set, $(h_t)_{t\in T}$ a family of extended real-valued ($h_t : X \to \mathbb{R} \cup \{+\infty\}$, $t \in T$) lower semicontinuous (lsc in short) proper convex functions on X, and C a non-empty closed convex subset of X, we are concerned with the problem

$$(P) \ \inf_x h(x), \ \text{s.t. } x \in C,$$

where h is the *supremum function* given by

$$h := \sup_{t\in T} h_t.$$

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A classical way to associate a Lagrangian dual problem to (P) is to transform it in a cone-constrained convex optimization problem (see e.g. [2, page 80]). To this end, let us consider the locally convex product space $\mathbb{R}^T$ partially ordered by the closed convex cone $\mathbb{R}_+^T$ and to which we add a greatest element denoted by ∞. A representation of the topological dual of $\mathbb{R}^T$ is furnished by the space $\mathbb{R}^{(T)}$ of mappings $\xi : T \to \mathbb{R}$ whose *support* $\operatorname{supp} \xi := \{t \in T : \xi_t := \xi(t) \neq 0\}$ is finite. The standard bilinear coupling is then as follows:

$$\left\langle \xi, (\eta_t)_{t\in T} \right\rangle = \textstyle\sum_{t\in T} \xi_t \eta_t, \quad \left(\xi, (\eta_t)_{t\in T}\right) \in \mathbb{R}^{(T)} \times \mathbb{R}^T.$$

We denote by $\mathbb{R}_+^{(T)}$ the positive cone of $\mathbb{R}^{(T)}$ and by

$$S_T := \left\{ \xi \in \mathbb{R}_+^{(T)} : \textstyle\sum_{t\in T} \xi_t = 1 \right\}$$

the *unit simplex* in the same space. Let us introduce the set

$$M := \bigcap_{t\in T} \operatorname{dom} h_t,$$

where $\operatorname{dom} h_t := \{x \in X : h_t(x) < +\infty\}$ is the *effective domain* of h_t. We now associate with (P) the problem

$$(P_0) \ \inf_{x,r} f(x,r), \ \text{ s.t. } \ (x,r) \in C \times \mathbb{R}, g(x,r) \in -\mathbb{R}_+^T,$$

where, for each $(x,r) \in X \times \mathbb{R}$,

$$f(x,r) := r, \ \ g(x,r) := \begin{cases} (h_t(x) - r)_{t\in T}, & \text{if } (x,r) \in M \times \mathbb{R}, \\ \infty, & \text{else.} \end{cases}$$

It holds that (P) and (P_0) have the same optimal value, which may be $\pm\infty$.

The usual Lagrange dual problem associated with the cone-constrained convex problem (P_0) is

$$(D_0) \ \sup_{\xi} \inf_{(x,r)\in(C\cap M)\times\mathbb{R}} \left(f(x,r) + \langle \xi, g(x,r)\rangle\right), \ \text{ s.t. } \xi \in \mathbb{R}_+^{(T)}.$$

The optimal value of (D_0) satisfies

$$\begin{aligned} \sup(D_0) &= \sup\nolimits_{\xi\in\mathbb{R}_+^{(T)}} \inf\nolimits_{(x,r)\in(C\cap M)\times\mathbb{R}} \left(r + \textstyle\sum_{t\in T}\xi_t (h_t(x) - r)\right) \\ &= \sup\nolimits_{\xi\in\mathbb{R}_+^{(T)}} \inf\nolimits_{x\in C\cap M} \left(\textstyle\sum_{t\in T}\xi_t h_t(x) + \inf\nolimits_{r\in\mathbb{R}} r\left(1 - \textstyle\sum_{t\in T}\xi_t\right)\right) \\ &= \sup\nolimits_{\xi\in S_T} \inf\nolimits_{x\in C\cap M} \textstyle\sum_{t\in T}\xi_t h_t(x), \end{aligned}$$

with the rule $\inf \emptyset = +\infty$.

Setting

$$L_0(x,\xi) := \textstyle\sum_{t\in T}\xi_t h_t(x), \ \text{ for } \ (x,\xi) \in (C\cap M) \times S_T,$$

we have

$$-\infty \leq \sup(D_0) = \sup_{\xi\in S_T} \inf_{x\in C\cap M} L_0(x,\xi) \leq \inf_{x\in C\cap M} \sup_{\xi\in S_T} L_0(x,\xi) = \inf(P) \leq +\infty.$$

We intend to get a simpler version of the above Lagrangian L_0 by introducing

$$L(x,\xi) := \textstyle\sum_{t\in T}\xi_t h_t(x), \ \text{ for } \ (x,\xi) \in X \times S_T,$$

where

$$\textstyle\sum_{t\in T}\xi_t h_t(x) := \sum_{t\in \operatorname{supp}\xi}\xi_t h_t(x), \text{ for } (x,\xi)\in X\times S_T. \tag{1.1}$$

It is worth noting that (1.1) enables to define $\sum_{t\in T}\xi_t h_t(x)$ for the elements x of X that do not belong to M, and it is equivalent to adopt the convention $0\times(+\infty)=0$ (see [15, p. 24]; another possibility is to set $0\times(+\infty)=+\infty$, which is the choice made for instance in [19]).

For each $x\in X$ we have

$$h(x) = \sup_{\xi\in S_T} \textstyle\sum_{t\in T}\xi_t h_t(x)$$

and, consequently,

$$\inf(P) = \inf_{x\in C}\sup_{\xi\in S_T} L(x,\xi).$$

We thus introduce another Lagrange dual of (P) by setting

$$(D)\ \sup_{\xi}\inf_{x\in C} L(x,\xi), \text{ s.t. } \xi\in S_T.$$

It holds that

$$-\infty\le \sup(D)\le \sup(D_0)\le \inf(P)\le +\infty. \tag{1.2}$$

Let us give a simple example for which the two Lagrangian dual problems (D_0) and (D) are different.

Example 1.1. Take $X=C=\mathbb{R}^2$, $T=\{1,2\}$, $S_2:=S_T$, $h_1(x_1,x_2)=e^{x_2}$, $h_2(x_1,x_2)=x_1$ if $(x_1,x_2)\in\mathbb{R}\times\mathbb{R}_+$, and $h_2(x_1,x_2)=+\infty$, otherwise. We have $M=\mathbb{R}\times\mathbb{R}_+$, and for $(\xi_1,\xi_2)\in S_2$, $\xi_2\ne 0$, one gets

$$\inf_{\mathbb{R}^2}(\xi_1h_1+\xi_2h_2)=\inf_{M}(\xi_1h_1+\xi_2h_2)=-\infty,$$

while, for $(\xi_1,\xi_2)=(1,0)$, one has

$$\inf_{\mathbb{R}^2}(\xi_1h_1+\xi_2h_2)=0\ne 1=\inf_{M}(\xi_1h_1+\xi_2h_2).$$

Consequently,

$$0=\sup(D)<\sup(D_0)=1=\inf(P).$$

The paper is organized as follows. Section 2 is devoted to the *converse strong duality* $\min(P)=\sup(D)$, which is established in Theorem 2.4 by using a recent result about consistency of infinite convex systems [5, Corollary 3]. This property entails the equality of both optimal values and the existence of optimal solutions for the primal problem (P). In Section 3 we get a formula for the Legendre-Fenchel conjugate $(\mathrm{i}_C+h)^*$ of i_C+h, where i_C denotes the indicator function of C, in connection with the property $\inf(P)=\max(D)$ (Theorem 3.3 and subsequent corollaries). Finally, Section 4 is concerned with the subdifferential $\partial(\mathrm{i}_C+h)$ of the objective function i_C+h (Theorem 4.4, Corollary 4.5) and also with the subdifferential $\partial(\mathrm{i}_C+h)^*$ of $(\mathrm{i}_C+h)^*$ (Theorem 4.6, Proposition 4.7), which gives a formula for the optimal set $S(P)$ of the problem (P) (Corollaries 4.8 and 4.9).

2 The Min-Sup Property

In order to establish the relation $\min(P)=\sup(D)$ we need to recall some lemmas. Having a non-empty closed convex subset A of X and $f\in\Gamma(X)$ (the class of lsc proper convex functions on X), we denote by A_∞ (resp. f_∞) the recession cone of A (resp., the recession function of f). It holds that $(\mathrm{i}_A)_\infty = \mathrm{i}_{A_\infty}$, $\operatorname{epi} f_\infty = (\operatorname{epi} f)_\infty$, $f_\infty = \mathrm{i}^*_{\operatorname{dom} f^*}$, where $\operatorname{epi} f$ is the epigraph of f and $\operatorname{dom} f^*$ the effective domain of f^*, and $[f \leq r]_\infty = [f_\infty \leq 0]$ for all $r \in \mathbb{R}$ such that $[f \leq r] \neq \emptyset$ (here $[f \leq r] := \{x \in X : f(x) \leq r\}$). For any family of closed convex sets $(A_t)_{t\in T}$ with non-empty intersection, one has (see e.g. [8, p. 375])

$$\left(\bigcap_{t\in T} A_t\right)_\infty = \bigcap_{t\in T} (A_t)_\infty . \tag{2.1}$$

We also have (see, e.g., [8, p. 377]) that, for any $(x, \overline{x}) \in X \times \operatorname{dom} f$,

$$f_\infty (x) = \lim_{s\to+\infty} \frac{f(\overline{x} + sx) - f(\overline{x})}{s},$$

and this formula straightforwardly leads us to the following equality for any pair $f, g \in \Gamma(X)$ such that $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$:

$$(f + g)_\infty = f_\infty + g_\infty. \tag{2.2}$$

Moreover, the definition $\operatorname{epi} f_\infty = (\operatorname{epi} f)_\infty$, together with (2.1), yield the following property for any family $(f_t)_{t\in T} \subset \Gamma(X)$ such that $f := \sup_{t\in T} f_t$ is proper:

$$f_\infty = \sup_{t\in T} (f_t)_\infty . \tag{2.3}$$

Having $B \subset X^*$, $B \neq \emptyset$, we denote by B^- the *negative polar cone* of B, that is, $B^- = [\mathrm{i}^*_B \leq 0] \equiv \{x \in X : \mathrm{i}^*_B (x) \leq 0\}$. Recall that, by the Hanh-Banach Theorem, the negative bipolar of B coincides with the w^*-closure of the convex cone generated by B, i.e., $B^{--} = \operatorname{cl}^{w^*} \operatorname{cone} B$. In the sequel we denote by τ^* the Mackey topology on X^*. Recall that a convex function $\varphi : X^* \to \overline{\mathbb{R}} := [-\infty, +\infty]$ is said to be τ^*-*quasicontinuous* ([6], [7]) when the affine hull of $\operatorname{dom}\varphi$ is w^*-closed (or, equivalently, τ^*-closed) and has finite codimension, the relative interior of $\operatorname{dom}\varphi$ with respect to the topology induced by τ^* is nonempty (i.e. $\operatorname{rint}(\operatorname{dom}\varphi) \neq \emptyset$), and the restriction of φ to $\operatorname{rint}(\operatorname{dom}\varphi)$ is continuous. If φ is τ^*-quasicontinuous, then φ^* is weakly-inf-locally compact (meaning that, for each $r \in \mathbb{R}$, the *sublevel set* $[\varphi^* \leq r]$ is w-locally compact) [11, Corollary II.3]. The converse is true whenever $\varphi \in \Gamma(X^*)$ [6, Proposition 5.4]. We will use the fact that any extended real-valued convex function on X^* which is majorized by a convex τ^*-quasicontinuous one is τ^*-quasicontinuous too [11, Theorem II.3].

Lemma 2.1 ([11, Theorem III.3]). *Let $\varphi : X^* \to \overline{\mathbb{R}}$ be a convex function which is τ^*-quasicontinuous and such that $\varphi(0_{X^*}) \neq -\infty$ and $\operatorname{cl}^{w^*} \operatorname{cone}\operatorname{dom}\varphi$ is a linear subspace. Then $\partial\varphi(0_{X^*})$ is the sum of a non-empty w-compact convex set and a finite dimensional linear subspace.*

Lemma 2.2. *Let $g \in \Gamma(X)$ be weakly-inf-locally compact and such that $[g_\infty \leq 0]$ is a linear subspace. Then $\inf_X g \in \mathbb{R}$ and $\operatorname{argmin} g$ is the sum of a non-empty w-compact convex set and a finite dimensional linear subspace.*

Proof. Let us apply Lemma 2.1 for $\varphi = g^*$. Since g is w-inf-locally compact, g^* is τ^*-quasicontinuous. Since $g \in \Gamma(X)$, g^* is proper, and so, $g^*(0_{X^*}) \neq -\infty$. Finally,

$$\operatorname{cl}^{w^*} \operatorname{cone}\operatorname{dom} g^* = (\operatorname{dom} g^*)^{--} = \left[\mathrm{i}^*_{\operatorname{dom} g^*} \leq 0\right]^- = [g_\infty \leq 0]^-$$

is a linear subspace. We conclude the proof by noting that $\operatorname{argmin} g = \partial g^*(0_{X^*})$ (see, for instance, [19, Theorem 2.4.2(iii)]). $\square$

Lemma 2.3 ([5, Corollary 3]). *Let $(f_t)_{t\in T}$ be a family of functions from $\Gamma(X)$, and C be a non-empty closed convex subset of X. Assume that*

$$\exists\overline{\lambda}\in\mathbb{R}_+^{(T)}:\quad \mathrm{i}_C+\sum\nolimits_{t\in T}\overline{\lambda}_t f_t \text{ is } w\text{-inf-locally compact} \tag{2.4}$$

and

$$\bigcap_{t\in T}[(f_t)_\infty\le 0]\cap C_\infty \text{ is a linear subspace.} \tag{2.5}$$

Then the following statements are equivalent:
(i) The system $\{x\in C; f_t(x)\le 0, t\in T\}$ is consistent.
(ii) $\inf_C\sum\nolimits_{t\in T}\lambda_t f_t\le 0$, $\forall\lambda\in P_T:=\mathbb{R}_+^{(T)}\diagdown\{0_T\}$.
(iii) $\inf_C\sum\nolimits_{t\in T}\lambda_t f_t\le 0$, $\forall\lambda\in S_T$.

We are now in position to state and prove our first result.

Theorem 2.4. *Assume that*

$$\exists\overline{\lambda}\in\mathbb{R}_+^{(T)}:\quad \mathrm{i}_C+\sum\nolimits_{t\in T}\overline{\lambda}_t h_t \text{ is } w\text{-inf-locally compact} \tag{2.6}$$

and

$$\bigcap_{t\in T}[(h_t)_\infty\le 0]\cap C_\infty \text{ is a linear subspace.} \tag{2.7}$$

Then, either $\inf(P)=\sup(D)=+\infty$, *or* $\min(P)=\sup(D)\in\mathbb{R}$.
Moreover, if $\sup(D)<+\infty$ *then the optimal set $S(P)$ is the sum of a non-empty w-compact convex set and a finite dimensional linear subspace.*

Proof. Assume first that $\sup(D)=+\infty$. By (1.2) we have $\inf(P)=+\infty$.

Assume now that $\sup(D)<+\infty$ and let r be an arbitrary real number such that $r\ge\sup(D)$. By definition of (D) we have

$$\inf_C\sum\nolimits_{t\in T}\xi_t h_t\le r,\ \forall\xi\in S_T,$$

or, equivalently,

$$\inf_C\sum\nolimits_{t\in T}\xi_t(h_t-r)\le 0,\ \forall\xi\in S_T.$$

Setting $f_t:=h_t-r$, $t\in T$, in Lemma 2.3 we quote from (2.6) and (2.7) that the conditions (2.4) and (2.5) are satisfied whatever r may be. We infer that the system $\{x\in C;(h_t-r)(x)\le 0,t\in T\}$ is consistent and so there exists $\overline{x}\in C$ such that $\inf(P)\le h(\overline{x})\le r$. Since $r\ge\sup(D)$ is arbitrary, we get that $\sup(D)\ge\inf(P)$ and, by (1.2), $\sup(D)=\inf(P)\in[-\infty,+\infty[$. It remains to prove that $\inf(P)\neq-\infty$ and (P) does admit optimal solutions. To this end let us introduce the function $g:=\mathrm{i}_C+h$.

Since $\inf(P)\neq+\infty$ we have that g is proper and, so, $g\in\Gamma(X)$. From (2.2) and (2.3), we have

$$g_\infty=(\mathrm{i}_C)_\infty+\sup_{t\in T}(h_t)_\infty=\mathrm{i}_{(C_\infty)}+\sup_{t\in T}(h_t)_\infty,$$

and consequently, $[g_\infty\le 0]=C_\infty\cap\bigcap_{t\in T}[(h_t)_\infty\le 0]$. In order to apply Lemma 2.2 to the function g it remains to be checked that g is w-inf-locally compact. If $\overline{\lambda}$ in (2.6) is equal to 0_T, then C is w-locally compact and, since h is w-lsc, $g=\mathrm{i}_C+h$ is w-inf-locally compact. If

$\overline{\lambda} \neq 0_T$, let us set $\overline{\xi} = \left(\sum_{t\in T}\overline{\lambda}_t\right)^{-1}\overline{\lambda}$. Then $\overline{\xi} \in S_T$ and $\mathrm{i}_C + \sum_{t\in T}\overline{\xi}_t h_t$ is a w-inf-locally compact minorant of g. Since g is w-lsc, it follows that g is w-inf-locally compact too. From Lemma 2.2 we infer that $\inf(P) = \inf_X g \in \mathbb{R}$ with $\operatorname{argmin} g \neq \emptyset$, that means that (P) admits optimal solutions.

The last assertion in Theorem 2.4 is also a consequence of Lemma 2.2. □

Remark 2.5. *If X is finite dimensional, condition (2.6) in Theorem 2.4 is automatically satisfied.*

Let us revisite Example 1.1. One has $[(h_1)_\infty \leq 0] = \mathbb{R}\times\mathbb{R}_-$, $[(h_2)_\infty \leq 0] = \mathbb{R}_-\times\mathbb{R}_+$, and $C_\infty = \mathbb{R}\times\mathbb{R}$. Thus, condition (2.7) in Theorem 2.4 fails since

$$[(h_1)_\infty \leq 0] \cap [(h_2)_\infty \leq 0] \cap C_\infty = \mathbb{R}_- \times \{0\}$$

is not a linear space.

Notice that the sufficient condition (2.7) for converse strong duality is not necessary as the following modification of Example 1.1 shows:

Example 2.6. Take $X = \mathbb{R}^2$, $C = \mathbb{R}\times\mathbb{R}_+$, $T = \{1,2\}$, $S_2 := S_T$, $h_1(x_1,x_2) = e^{x_2}$ and $h_2(x_1,x_2) = x_1$. We have $M = \mathbb{R}^2$, and

$$\max(D) = 1 = \min(P),$$

while

$$\begin{aligned} C_\infty \cap [(h_1)_\infty \leq 0] \cap [(h_2)_\infty \leq 0] &= (\mathbb{R}\times\mathbb{R}_+) \cap (\mathbb{R}\times\mathbb{R}_-) \cap (\mathbb{R}_-\times\mathbb{R}) \\ &= \mathbb{R}_- \times \{0\} \end{aligned}$$

is not a linear subspace.

In the case when $C\cap M$ is non-empty and closed we can apply Theorem 2.4 replacing C by $C \cap M$. We obtain a result involving the classical dual (D_0) instead of our dual (D).

Corollary 2.7. *Assume that $C\cap M$ is non-empty and closed, and the two conditions below hold:*

$$\exists\overline{\lambda} \in \mathbb{R}_+^{(T)} : \quad \mathrm{i}_{C\cap M} + \sum_{t\in T}\overline{\lambda}_t h_t \text{ is } w\text{-inf-locally compact} \tag{2.8}$$

and

$$\bigcap_{t\in T} [(h_t)_\infty \leq 0] \cap (C\cap M)_\infty \text{ is a linear subspace.} \tag{2.9}$$

Then, either $\inf(P) = \sup(D_0) = +\infty$, *or* $\min(P) = \sup(D_0) \in \mathbb{R}$. *Moreover, if* $\sup(D_0) < +\infty$ *then the optimal set $S(P)$ is the sum of a non-empty w-compact convex set and a finite dimensional linear subspace.*

3 The Inf-Max Property

Let us consider the function $\varphi : X^* \to \overline{\mathbb{R}}$ given by

$$\varphi = \inf_{\xi\in S_T}\left(\mathrm{i}_C + \sum_{t\in T}\xi_t h_t\right)^*$$

and the set

$$\mathfrak{A} := \bigcup_{\xi \in S_T} \operatorname{epi}\left(\mathrm{i}_C + \sum\nolimits_{t\in T}\xi_t h_t\right)^* . \tag{3.1}$$

It is not difficult to check that $\mathfrak{A}$ is convex and lies between the strict epigraph and the epigraph of φ, i.e.

$$\operatorname{epi}_s \varphi \subset \mathfrak{A} \subset \operatorname{epi} \varphi.$$

It follows that $\mathrm{cl}^{w^*} \mathfrak{A} = \mathrm{cl}^{w^*} \operatorname{epi}_s \varphi = \mathrm{cl}^{w^*} \operatorname{epi} \varphi$ and, for each $x^* \in X^*$,

$$\varphi(x^*) = \inf\{t \in \mathbb{R} : (x^*, t) \in \mathfrak{A}\}. \tag{3.2}$$

Lemma 3.1. (*See also [2, Proposition 12.1]*) *The function φ is convex and $\varphi^* = \mathrm{i}_C + h$. Moreover, if $C \cap \operatorname{dom} h \neq \emptyset$, then $\mathrm{cl}^{w^*} \mathfrak{A} = \operatorname{epi}(\mathrm{i}_C + h)^*$.*

Proof. Since $\mathfrak{A}$ is convex, it follows from (3.2) that φ is convex. One has

$$\begin{aligned}
\varphi^* &= \left(\inf\nolimits_{\xi\in S_T}\left(\mathrm{i}_C + \sum\nolimits_{t\in T}\xi_t h_t\right)^*\right)^* \\
&= \sup\nolimits_{\xi\in S_T}\left(\mathrm{i}_C + \sum\nolimits_{t\in T}\xi_t h_t\right)^{**} \\
&= \sup\nolimits_{\xi\in S_T}\left(\mathrm{i}_C + \sum\nolimits_{t\in T}\xi_t h_t\right) \\
&= \mathrm{i}_C + \sup\nolimits_{\xi\in S_T}\sum\nolimits_{t\in T}\xi_t h_t = \mathrm{i}_C + h.
\end{aligned}$$

Finally, if $C \cap \operatorname{dom} h \neq \emptyset$, then $\operatorname{dom}\varphi^* \neq \emptyset$ and, consequently, $\varphi^{**} = \mathrm{cl}^{w^*} \varphi$. In other words,

$$\mathrm{cl}^{w^*} \mathfrak{A} = \operatorname{epi} \mathrm{cl}^{w^*} \varphi = \operatorname{epi}\varphi^{**} = \operatorname{epi}(\mathrm{i}_C + h)^* .$$

$\square$

In order to state the next theorem we have to recall the concept of closedness regarding a set which has been introduced and used in [2] (see also [3] and [14] for related approaches).

Definition 3.2. Having two subsets A and B of a topological space, one says that A is *closed regarding to B* if $B \cap \mathrm{cl}\, A = B \cap A$.

Clearly, a closed subset is closed regarding any subset. Also, A is closed regarding B if and only if A is closed regarding each subset of B.

Now we can state:

Theorem 3.3. *Assume that $C \cap \operatorname{dom} h \neq \emptyset$. Then, for any $x^* \in X^*$, the following statements are equivalent:*

(i) $(\mathrm{i}_C + h)^*(x^*) = \min_{\xi\in S_T}\left(\mathrm{i}_C + \sum\nolimits_{t\in T}\xi_t h_t\right)^*(x^*)$, *including the value $+\infty$.*

(ii) $\mathfrak{A}$ is w^*-closed regarding $\{x^*\}\times\mathbb{R}$.

Proof. Assume first that $(\mathrm{i}_C + h)^*(x^*) = +\infty$.

By Lemma 3.1 we have $\varphi(x^*) = \varphi^{**}(x^*) = +\infty$ and the statement (i) holds true. Since $\operatorname{dom}\varphi^* = C \cap \operatorname{dom} h \neq \emptyset$, we have that $\varphi^{**} = \mathrm{cl}^{w^*}\varphi$. So, $\mathrm{cl}^{w^*}\varphi(x^*) = +\infty$ and $\{x^*\}\times\mathbb{R}$ does not meet the set $\{x^*\}\times \mathrm{cl}^{w^*}\operatorname{epi}\varphi$. Recall that $\mathrm{cl}^{w^*}\operatorname{epi}\varphi = \mathrm{cl}^{w^*}\mathfrak{A}$. Therefore, $\{x^*\}\times\mathbb{R}$ does not meet $\mathrm{cl}^{w^*}\mathfrak{A}$ and this proves that $\mathfrak{A}$ is w^*-closed regarding $\{x^*\}\times\mathbb{R}$. So, the statements (i) and (ii) are both true in this case.

Assume now that $\alpha := (\mathrm{i}_C + h)^*(x^*) \neq +\infty$.

Since $\mathrm{i}_C + h \in \Gamma(X)$, we have that $(\mathrm{i}_C + h)^* \in \Gamma(X^*)$ and therefore, $\alpha \in \mathbb{R}$. By Lemma 3.1 it holds that

$$(x^*, \alpha) \in \operatorname{epi} \varphi^{**} = \operatorname{cl}^{w^*} \operatorname{epi} \varphi = \operatorname{cl}^{w^*} \mathfrak{A}. \tag{3.3}$$

If (ii) holds we get from (3.3) that $(x^*, \alpha) \in \mathfrak{A}$ and there exists $\overline{\xi} \in S_T$ such that

$$\inf_{\xi \in S_T} \left(\mathrm{i}_C + \sum\nolimits_{t \in T} \xi_t h_t\right)^* (x^*) \le \left(\mathrm{i}_C + \sum\nolimits_{t \in T} \overline{\xi}_t h_t\right)^* (x^*) \le \alpha. \tag{3.4}$$

Since $\mathrm{i}_C + \sum_{t \in T} \xi_t h_t \le \mathrm{i}_C + h$ for all $\xi \in S_T$, we have

$$\inf_{\xi \in S_T} \left(\mathrm{i}_C + \sum\nolimits_{t \in T} \xi_t h_t\right)^* \ge (\mathrm{i}_C + h)^*$$

and, by (3.4), $\alpha = \min_{\xi \in S_T} \left(\mathrm{i}_C + \sum_{t \in T} \xi_t h_t\right)^* (x^*)$ (the minimum α is attained at $\overline{\xi}$).

Conversely, if $\alpha = \min_{\xi \in S_T} \left(\mathrm{i}_C + \sum_{t \in T} \xi_t h_t\right)^* (x^*)$, let us prove that (ii) holds. So, let $r \in \mathbb{R}$ be such that $(x^*, r) \in \operatorname{cl}^{w^*} \mathfrak{A} = \operatorname{epi} \varphi^{**}$. By Lemma 3.1 we have $\alpha = (\mathrm{i}_C + h)^* (x^*) = \varphi^{**}\ (x^*) \le r$ and there exists $\overline{\xi} \in S_T$ such that $\left(\mathrm{i}_C + \sum_{t \in T} \overline{\xi}_t h_t\right)^* (x^*) \le r$, and this entails $(x^*, r) \in \mathfrak{A}$. □

Corollary 3.4. *Assume that* $C \cap \operatorname{dom} h \neq \emptyset$. *The following statements are equivalent:*
(i) $\inf(P) = \max(D)$, *including the value* $-\infty$.
(ii) $\mathfrak{A}$ *is* w^**-closed regarding* $\{0_{X^*}\} \times \mathbb{R}$.

Proof. Apply Theorem 3.3 with $x^* = 0_{X^*}$, noting that $-\inf(P) = (\mathrm{i}_C + h)^* (0_{X^*})$. □

According to Corollary 3.4, applied again to Example 1.1, the set

$$\mathfrak{A} = \bigcup_{\xi \in S_2} \operatorname{epi} (\xi_1 h_1 + \xi_2 h_2)^*$$

is not closed regarding $\{0\} \times \mathbb{R}$. Moreover, the set

$$\bigcup_{\xi \in S_2} \operatorname{epi} (\mathrm{i}_M + \xi_1 h_1 + \xi_2 h_2)^*$$

is closed regarding $\{0\} \times \mathbb{R}$ since this fact corresponds to the strong duality for the usual Lagrangian L_0 :

$$\inf(P) = \max_{\xi \in S_2} \inf_{x \in M} (\xi_1 h_1(x) + \xi_2 h_2(x)) = 1,$$

the maximum being attained for $\xi = (1, 0)$.

In fact, as long as the standard Lagrangian dual (D_0) is concerned, we can replace again C by $C \cap M$ provided that $C \cap M$ is non-empty and closed. In this way we get from Corollary 3.4:

Corollary 3.5. *Assume that* $C \cap \operatorname{dom} h \neq \emptyset$ *and* $C \cap M$ *is closed. The following statements are equivalent:*
(i) $\inf(P) = \max(D_0)$, *including the value* $-\infty$.
(ii) $\bigcup_{\xi \in S_T} \operatorname{epi} \left(\mathrm{i}_{C \cap M} + \sum_{t \in T} \xi_t h_t\right)^*$ *is* w^**-closed regarding* $\{0_{X^*}\} \times \mathbb{R}$.

Finally we present a straightforward consequence of Theorem 3.3:

Corollary 3.6. *Assume that* $C \cap \operatorname{dom} h \neq \emptyset$. *The following statements are equivalent:*
(i) $(\mathrm{i}_C + h)^* = \min_{\xi \in S_T} \left(\mathrm{i}_C + \sum_{t \in T} \xi_t h_t\right)^*$.
(ii) $\mathfrak{A}$ *is* w^**-closed.*

$\boxed{4}$ Subdifferential and Argmin Calculus

Let us consider again the function $g = \mathrm{i}_C + h$. The set

$$\operatorname{argmin} g = \left\{ x \in C \cap \operatorname{dom} h : \ g(x) = \inf_X g \right\}$$

coincides with the optimal solution set $S(P)$ of (P), and we have

$$x \in S(P) \Leftrightarrow 0_{X^*} \in \partial g(x) \Leftrightarrow x \in \partial g^*(0_{X^*}).$$

Therefore, computing the subdifferential of the functions g and g^* is of crucial importance in our context. In this section we will apply Theorem 3.3 and its corollaries to obtain formulas for ∂g, ∂g^*, and $\operatorname{argmin} g$.

Given $x \in C \cap \operatorname{dom} h = \operatorname{dom} g$, let us consider the set

$$M(x) := \left\{ \xi \in S_T : \sum\nolimits_{t \in T} \xi_t h_t(x) = h(x) \right\}. \tag{4.1}$$

Next lemma furnishes another expression for $M(x)$.

Lemma 4.1. *We have* $M(x) = \{\xi \in S_T : h_t(x) = h(x) \ \ \forall t \in \operatorname{supp} \xi\}$.

Proof. One has $\xi \in M(x)$ if and only if $\sum_{t \in \operatorname{supp} \xi} \xi_t (h_t(x) - h(x)) = 0$. Since $h_t(x) \le h(x)$ for any $t \in T$, each term of the above sum is non-positive. Since this sum is equal to zero, each term of the sum must be equal to zero.

Conversely, if $h_t(x) = h(x)$ for each $t \in \operatorname{supp} \xi$, then $\sum_{t \in T} \xi_t h_t(x) = \sum_{t \in T} \xi_t h(x) = h(x)$, and $\xi \in M(x)$. $\square$

Let $x \in C \cap \operatorname{dom} h = \operatorname{dom} g$. Concerning ∂g let us quote two general facts in Proposition 4.2 and Theorem 4.4 below.

Proposition 4.2. $\partial g(x) \supset \bigcup\limits_{\xi \in M(x)} \partial \left(\mathrm{i}_C + \sum_{t \in T} \xi_t h_t \right)(x)$.

Proof. Let $x^* \in \bigcup\limits_{\xi \in M(x)} \partial \left(\mathrm{i}_C + \sum_{t \in T} \xi_t h_t \right)(x)$. Let $\overline{\xi} \in M(x)$ be such that $x^* \in \partial \left(\mathrm{i}_C + \sum_{t \in T} \overline{\xi}_t h_t \right)(x)$. Noting that $\sum_{t \in T} \overline{\xi}_t h_t \le h$, $\mathrm{i}_C + \sum_{t \in T} \overline{\xi}_t h_t \le g$, and $\left(\mathrm{i}_C + \sum_{t \in T} \overline{\xi}_t h_t \right)^* \ge g^*$, we have

$$\begin{aligned} \langle x^*, x \rangle - g(x) &\le g^*(x^*) \\ &\le \left(\mathrm{i}_C + \sum\nolimits_{t \in T} \overline{\xi}_t h_t \right)^* (x^*) \\ &= \langle x^*, x \rangle - \sum\nolimits_{t \in T} \overline{\xi}_t h_t(x) \\ &= \langle x^*, x \rangle - h(x) = \langle x^*, x \rangle - g(x). \end{aligned}$$

So, $\langle x^*, x \rangle - g(x) = g^*(x^*)$ and $x^* \in \partial g(x)$. $\square$

Proposition 4.3. *For any $x \in \operatorname{dom} g$, the convex set $\mathfrak{A}$ in (3.1) is w^*-closed regarding the set*

$$B(x) := \left(\bigcup_{\xi \in M(x)} \partial \left(\mathrm{i}_C + \sum\nolimits_{t \in T} \xi_t h_t \right)(x) \right) \times \mathbb{R}.$$

Proof. Let $x \in \operatorname{dom} g$ and $(x^*, r) \in (\mathrm{cl}^{w^*} \mathfrak{A}) \cap B(x)$. By Lemma 3.1 one has $g^*(x^*) \leq r$ and, by definition of $B(x)$, there exists $\overline{\xi} \in M(x)$ such that

$$x^* \in \partial \left(\mathrm{i}_C + \sum\nolimits_{t \in T} \overline{\xi}_t h_t \right)(x), \tag{4.2}$$

and so,

$$\begin{aligned} r & \geq g^*(x^*) \geq \langle x^*, x \rangle - g(x) \\ & = \langle x^*, x \rangle - h(x) \\ & = \langle x^*, x \rangle - \sum\nolimits_{t \in T} \overline{\xi}_t h_t(x) \\ & = \left(\mathrm{i}_C + \sum\nolimits_{t \in T} \overline{\xi}_t h_t \right)^* (x^*). \end{aligned}$$

The last equality comes from (4.2). Thus, $(x^*, r) \in \operatorname{epi} \left(\mathrm{i}_C + \sum\nolimits_{t \in T} \overline{\xi}_t h_t \right)^* \subset \mathfrak{A}$. □

Theorem 4.4. *For any $x \in \operatorname{dom} g$, the following statements are equivalent:*
(i) $\partial g(x) = \bigcup_{\xi \in M(x)} \partial \left(\mathrm{i}_C + \sum\nolimits_{t \in T} \xi_t h_t \right)(x)$.
(ii) $\mathfrak{A}$ *is w^*-closed regarding $\partial g(x) \times \mathbb{R}$.*

Proof. $(i) \Rightarrow (ii)$ comes from Proposition 4.3.

Let us prove that $(ii) \Rightarrow (i)$. By Proposition 4.2 it suffices to prove that the inclusion "$\subset$" in (i) is satisfied. So, let $x^* \in \partial g(x)$. By Lemma 3.1 we have $(x^*, g^*(x^*)) \in (\mathrm{cl}^{w^*} \mathfrak{A}) \cap (\partial g(x) \times \mathbb{R})$ and, by (ii), $(x^*, g^*(x^*)) \in \mathfrak{A}$. Thus, there exists $\overline{\xi} \in S_T$ such that

$$(x^*, g^*(x^*)) \in \operatorname{epi} \left(\mathrm{i}_C + \sum\nolimits_{t \in T} \overline{\xi}_t h_t \right)^*,$$

and

$$\begin{aligned} \langle x^*, x \rangle - h(x) & \leq \langle x^*, x \rangle - \sum\nolimits_{t \in T} \overline{\xi}_t h_t(x) \\ & \leq \left(\mathrm{i}_C + \sum\nolimits_{t \in T} \overline{\xi}_t h_t \right)^* (x^*) \\ & \leq g^*(x^*) = \langle x^*, x \rangle - g(x) = \langle x^*, x \rangle - h(x). \end{aligned}$$

It follows that $\sum\nolimits_{t \in T} \overline{\xi}_t h_t(x) = h(x)$ and

$$\left(\mathrm{i}_C + \sum\nolimits_{t \in T} \overline{\xi}_t h_t \right)^* (x^*) = \langle x^*, x \rangle - \sum\nolimits_{t \in T} \overline{\xi}_t h_t(x),$$

or, in other words, $\overline{\xi} \in M(x)$ and $x^* \in \partial \left(\mathrm{i}_C + \sum\nolimits_{t \in T} \overline{\xi}_t h_t \right)(x)$. □

Corollary 4.5. *[10, Theorem 2] If $\mathfrak{A}$ is w^*-closed then for any $x \in \operatorname{dom} g$, it holds*

$$\partial g(x) = \bigcup_{\xi \in M(x)} \partial \left(\mathrm{i}_C + \sum\nolimits_{t \in T} \xi_t h_t \right)(x).$$

We end this note by establishing a new formula for the subdifferential of the conjugate of the function $g = \mathrm{i}_C + h$. We shall also derive from this formula an expression for the optimal solution set $S(P)$, which furnishes necessary and sufficient optimality conditions.

We associate with $\xi \in S_T$ the set

$$N(\xi) := \left\{ x \in C \cap \operatorname{dom} h : \sum\nolimits_{t \in T} \xi_t h_t(x) = h(x) \right\}.$$

By (4.1) we have

$$x \in N(\xi) \Leftrightarrow \xi \in M(x)$$

and, by Lemma 4.1,

$$N(\xi) := \{x \in C \cap \operatorname{dom} h : h_t(x) = h(x) \ \ \forall t \in \operatorname{supp} \xi\}.$$

Theorem 4.6. *For any $x^* \in \operatorname{dom} g^*$ one has*

$$\partial g^*(x^*) \supset \bigcup_{\xi \in S_T} \left(N(\xi) \cap \partial \left(\mathrm{i}_C + \sum\nolimits_{t \in T} \xi_t h_t \right)^* (x^*) \right).$$

If, additionally, $\mathfrak{A}$ is w^*-closed regarding $\{x^*\} \times \mathbb{R}$, then

$$\partial g^*(x^*) = \bigcup_{\xi \in S_T} \left(N(\xi) \cap \partial \left(\mathrm{i}_C + \sum\nolimits_{t \in T} \xi_t h_t \right)^* (x^*) \right). \tag{4.3}$$

Proof. Let $\overline{\xi} \in S_T$ and $x \in N\left(\overline{\xi}\right) \cap \partial \left(\mathrm{i}_C + \sum_{t \in T} \overline{\xi}_t h_t \right)^* (x^*)$. One has

$$\begin{aligned} g^*(x^*) \quad &\leq \left(\mathrm{i}_C + \sum\nolimits_{t \in T} \overline{\xi}_t h_t \right)^* (x^*) \\ &= \langle x^*, x \rangle - \sum\nolimits_{t \in T} \overline{\xi}_t h_t(x) \\ &= \langle x^*, x \rangle - h(x) = \langle x^*, x \rangle - g(x) \end{aligned}$$

and, consequently, $g^*(x^*) = \langle x^*, x \rangle - g(x)$, that is, $x \in \partial g^*(x^*)$.

To prove (4.3), let $x \in \partial g^*(x^*)$. By Lemma 3.1 we have

$$(x^*, \langle x^*, x \rangle - g(x)) \in (\mathrm{cl}^{w^*} \mathfrak{A}) \cap \{x^*\} \times \mathbb{R}$$

and, since $\mathfrak{A}$ is w^*-closed regarding $\{x^*\} \times \mathbb{R}$, we have $(x^*, \langle x^*, x \rangle - g(x)) \in \mathfrak{A}$. So, there exists $\overline{\xi} \in S_T$ such that

$$(x^*, \langle x^*, x \rangle - g(x)) \in \operatorname{epi} \left(\mathrm{i}_C + \sum\nolimits_{t \in T} \overline{\xi}_t h_t \right)^*,$$

and

$$\begin{aligned} \langle x^*, x \rangle - \sum\nolimits_{t \in T} \overline{\xi}_t h_t(x) \quad &\leq \left(\mathrm{i}_C + \sum\nolimits_{t \in T} \overline{\xi}_t h_t \right)^* (x^*) \\ &\leq \langle x^*, x \rangle - g(x) \\ &\leq \langle x^*, x \rangle - \sum\nolimits_{t \in T} \overline{\xi}_t h_t(x). \end{aligned}$$

We thus have $\sum_{t \in T} \overline{\xi}_t h_t(x) = g(x) = h(x)$, $x \in N\left(\overline{\xi}\right)$, and

$$\langle x^*, x \rangle - \sum\nolimits_{t \in T} \overline{\xi}_t h_t(x) = \left(\mathrm{i}_C + \sum\nolimits_{t \in T} \overline{\xi}_t h_t \right)^* (x^*),$$

that means $x^* \in \partial \left(\mathrm{i}_C + \sum_{t \in T} \overline{\xi}_t h_t \right)(x)$ or, equivalently,

$$x \in \left(\mathrm{i}_C + \sum\nolimits_{t \in T} \overline{\xi}_t h_t \right)^* (x^*).$$

□

We can state a partial converse of Theorem 4.6.

Proposition 4.7. *Let $x^* \in X^*$ be such that $\partial g^*(x^*) \neq \emptyset$ and assume that (4.3) holds. Then $\mathfrak{A}$ is w^*-closed regarding $\{x^*\} \times \mathbb{R}$.*

Proof. Assume that $(x^*, r) \in \mathrm{cl}^{w^*} \mathfrak{A}$. We have to check that $(x^*, r) \in \mathfrak{A}$.

By Lemma 3.1 we have $g^*(x^*) \leq r$. Picking $x \in \partial g^*(x^*)$, which is non-empty, there exists, by (4.3), $\overline{\xi} \in S_T$ such that $x \in N(\overline{\xi}) \cap \partial \left(\mathrm{i}_C + \sum_{t\in T} \overline{\xi}_t h_t\right)^*(x^*)$. Then we have

$$\begin{aligned} g^*(x^*) &\leq \left(\mathrm{i}_C + \sum_{t\in T} \overline{\xi}_t h_t\right)^*(x^*) \\ &= \langle x^*, x\rangle - \sum_{t\in T} \overline{\xi}_t h_t(x) \\ &= \langle x^*, x\rangle - h(x) = \langle x^*, x\rangle - g(x) \leq g^*(x^*). \end{aligned}$$

Therefore, $g^*(x^*) = \left(\mathrm{i}_C + \sum_{t\in T} \overline{\xi}_t h_t\right)^*(x^*)$ and

$$(x^*, r) \in \mathrm{epi} \left(\mathrm{i}_C + \sum_{t\in T} \overline{\xi}_t h_t\right)^* \subset \mathfrak{A}.$$

□

Corollary 4.8. *Assume that* $\inf(P) \in \mathbb{R}$. *We have*

$$S(P) \supset \bigcup_{\xi \in S_T} \left(N(\xi) \cap \mathrm{argmin}\left(\mathrm{i}_C + \sum_{t\in T} \xi_t h_t\right)\right).$$

If, additionally, $\mathfrak{A}$ is w^*-closed regarding $\{0_{X^*}\} \times \mathbb{R}$, then

$$S(P) = \bigcup_{\xi \in S_T} \left(N(\xi) \cap \mathrm{argmin}\left(\mathrm{i}_C + \sum_{t\in T} \xi_t h_t\right)\right).$$

In other words, for any $x \in C \cap \mathrm{dom}\, h$ one has

$$x \in S(P) \Leftrightarrow \begin{cases} \exists \overline{\xi} \in S_T : h_t(x) = h(x) \ \ \forall t \in \mathrm{supp}\, \overline{\xi}, \\ \textit{and} \\ x \in \mathrm{argmin}\left(\mathrm{i}_C + \sum_{t\in T} \overline{\xi}_t h_t\right). \end{cases}$$

Proof. Apply Theorem 4.6 with $x^* = 0_{X^*}$, noting that $\partial g^*(0_{X^*}) = \mathrm{argmin}\, g$ and

$$\partial \left(\mathrm{i}_C + \sum_{t\in T} \xi_t h_t\right)^*(0_{X^*}) = \mathrm{argmin}\left(\mathrm{i}_C + \sum_{t\in T} \xi_t h_t\right)$$

for each $\xi \in S_T$. □

Corollary 4.9. *Assume that*

$$\emptyset \neq S(P) = \bigcup_{\xi \in S_T} \left(N(\xi) \cap \mathrm{argmin}\left(\mathrm{i}_C + \sum_{t\in T} \xi_t h_t\right)\right).$$

Then $\mathfrak{A}$ is w^*-closed regarding $\{0_{X^*}\} \times \mathbb{R}$.

Proof. Apply Proposition 4.7 with $x^* = 0_{X^*}$. □

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