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ON SECOND-ORDER CONVERSE DUALITY IN NONLINEAR PROGRAMMING PROBLEMS*

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Abstract: In this paper, converse duality theorems for scalar and multiobjective second order dual problems in nonlinear programming are established. Our results fill some gaps in works of Husain et al. [I. Husain, N.G. Rueda and Z. Jabeen, Fritz John second-order duality for nonlinear programming , Applied Mathematics Letters 14 (2001) 513-518], Yang et al. [X.M. Yang, X.Q. Yang and K.L. Teo, Huard type second-order converse duality for nonlinear programming, Applied Mathematics Letters 18 (2005) 205-208], Gulati et al. [T.R. Gulati and Divya Agarwal, On Huard type second order converse duality in nonlinear programming problems, Applied Mathematics Letters 20(2007) 1057-1063] and Ahmad et al. [I. Ahmad, Z. Husain and S.A. Homidan, Second-order duality in nondifferentiable fractional programming, Nonlinear Analysis, 12 (2011) 1103-1110].

Key words: *nonlinear programming, second-order dual problems, converse duality theorems*

Mathematics Subject Classification: *90C30, 90C46*

1 Introduction

The study of second order duality is useful due to the computational advantage over first order duality as it gives bounds for the value of the objective function when approximations are used. Mangasarian [5] first introduced second-order dual for nonlinear mathematical programming problems and established duality results under the inclusion conditions. Based on the first order convexity, Mond [6] introduced the definition of second order convexity and gave duality results under the more simpler conditions than these of Mangasarian by using the generalized form of convexity. Later, Mond and Weir [7], Mond and Zhang [8] introduced another kinds of second-order duals and obtained duality results under more simpler conditions. In recent years, there has been an increasing interest in generating Monds original notion of second-order and higher-order convexity and utilized for establishing various duality results for several classes of nonlinear programming problems ([1–3, 9, 10]).

Husain et al. [3], Yang et al. [10] and Gulati et al. [2] consider the following nonlinear scalar programming problem.

$$\text{(NP1)} \qquad \text{Minimize } \ f(x)$$
$$\text{subject to } g(x) \leqq 0,$$

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where $f: R^n \to R, g: R^n \to R^m$ are twice differentiable functions.

And they formulated second order dual problem (ND1) for (NP1) as follows:

$$\text{(ND1)} \qquad \text{Maxmize } \ f(u) - \frac{1}{2}p^T\nabla^2 f(u)p$$
$$\text{subject to } r(\nabla f(u) + \nabla^2 f(u)p + \nabla y^T g(u) + \nabla^2 y^T g(u)p = 0,$$
$$y^T g(u) - \frac{1}{2}p^T\nabla^2(y^T g(u))p \geqq 0,$$
$$(r, y) \geqq 0,$$
$$(r, y) \neq 0.$$

where $r \in R, y \in R^m$ and $p, u \in R^n$.

Husain et al. [3] gave a weak duality, a strong duality, a strict converse duality and a Huard type converse duality for problems (NP1) and (ND1) . In particular, they prove the following Huard type converse duality theorem.

Theorem 1.1 (Converse Duality (see Theorem 2.4 in [3]). *Let* (r^*, x^*, y^*, p^*) *be a optimal solution of* (ND1)*). Assume that*

(A1) *the* $n \times n$ *Hessian matrix* $\nabla[r^*\nabla^2 f(x^*) + \nabla^2(y^{*T}g(x^*))]p^*$ *is positive or negative definite,*

(A2) $\nabla y^* g(x^*) + \nabla^2(y^{*T}g(x^*))p^* \neq 0,$

(A3) *the vectors* $\{[\nabla^2 f(x^*)]_j, [\nabla^2 y^* g(x^*)]_j, j = 1, \ldots, n\}$ *are linearly independent, where* $[\nabla^2 f(x^*)]_j$ *and* $[\nabla^2 y^* g(x^*)]_j$ *are jth rows of* $\nabla^2 f(x^*)$ *and* $\nabla^2 y^* g(x^*)$*, respectively. If the generalized convexity hypotheses of Theorem 2.1 in [3] are satisfied, then* x^* *is an optimal solution of* (NP1).

Yang et al. [10] note that the matrix $\nabla[r^*\nabla^2 f(x^*) + \nabla^2(y^{*T}g(x^*))]p^*$ is positive or negative definite in the assumption (A1) of Theorem 1.1, and the result of Theorem 1.1 implies $p^* = 0$. See the proof of Theorem 2.4 in [3]. It is obvious that the assumption and the result are inconsistent. Hence, they gave an appropriate modification for this deficiency contained in Theorem 1.1. And they established the following converse duality theorem.

Theorem 1.2 (Converse Duality (see Theorem 2 in [10]). *Let* (r^*, x^*, y^*, p^*) *be a optimal solution of* (ND1)*). Assume that*

(B1) $\nabla^2 y^{*T}g(x^*)$ *is positive define and* $y^{*T}g(x^*) \leqq 0$ *or* $\nabla^2 y^{*T}g(x^*)$ *is negative define and* $y^{*T}g(x^*) \geqq 0$.

(A2) $\nabla y^* g(x^*) + \nabla^2(y^{*T}g(x^*))p^* \neq 0,$

(A3) *the vectors* $\{[\nabla^2 f(x^*)]_j, [\nabla^2 y^* g(x^*)]_j, j = 1, \ldots, n\}$ *are linearly independent, where* $[\nabla^2 f(x^*)]_j$ *and* $[\nabla^2 y^* g(x^*)]_j$ *are jth rows of* $\nabla^2 f(x^*)$ *and* $\nabla^2 y^* g(x^*)$*, respectively. If the generalized convexity hypotheses of Theorem 2.1 in [3] are satisfied, then* x^* *is an optimal solution of* (NP1).

Gulati et al. [2] obtained an alternative proof of the converse duality theorem replacing assumption (A2) in Theorem 1.2 by a weaker assumption $\nabla f(x^*) + \nabla^2 f(x^*)p^* \neq 0$.

Observing that in the converse duality Theorems in [2,3,10], that is, Theorem 2.4 in [3], Theorem 2 in [10], Theorem 1 and Theorem 2 in [2], the assumptions that the vectors $\{[\nabla^2 f(x^*)]_j, [\nabla^2 y^* g(x^*)]_j, j = 1, \ldots, n\}$ are linearly independent, where $[\nabla^2 f(x^*)]_j$ and

$[\nabla^2 y^* g(x^*)]_j$ are jth rows of $\nabla^2 f(x^*)$ and $\nabla^2 y^* g(x^*)$, respectively, are erroneous. Since $\{[\nabla^2 f(x^*)]_j, [\nabla^2 y^* g(x^*)]_j, j = 1, \ldots, n\}$ are $2n-$ vectors in R^n, and they are linearly dependent.

In this work, Section 2 contains modified proofs for Theorem 2.4 in [3], Theorem 2 in [10], Theorem 1 and Theorem 2 in [2]. The deficiency in the second order converse duality theorems also appeared in the converse duality theorems for multiobjective programming problems in [3] and for nondifferentiable fractional programming problems in [1], and modified proofs are presented in Section 3 and Section 4, respectively.

2 Converse Duality for Nonlinear Scalar Programming Problem

In this section, we give the following modified converse duality theorem for problems (NP1) and (ND1).

Theorem 2.1 (Converse Duality for (ND1))**.** *Let (r^*, x^*, y^*, p^*) be a optimal solution of* (ND1)*). Assume that*

(i) $r^* \nabla f(x^*) + \nabla y^{*T} g(x^*) = 0.$

(ii) $\nabla^2 f(x^*)$ *is negative define or positive define.*

(iii) $r^*(\nabla^2 f(x^*) + \nabla(\nabla^2 f(x^*)p^*)) + \nabla^2(y^{*T} g(x^*)) + \nabla(\nabla^2(y^* g(x^{*T})p^*))$ *is nonsingular.*

(iv) $\nabla f(x^*) \neq 0.$

(v) $\nabla^2 y^{*T} g(x^*)$ *is positive define and* $y^{*T} g(x^*) \leqq 0$ *or* $\nabla^2 y^{*T} g(x^*)$ *is negative define and* $y^{*T} g(x^*) \geqq 0.$

Then $p^* = 0$, x^* is a feasible solution of (NP1) *and the two objective functions values are equal. Furthermore, if the generalized convexity hypotheses of Theorem 2.1 in [3] are satisfied, then x^* is an optimal solution of* (NP1)*.*

Proof. Since (r^*, x^*, y^*, p^*) is an optimal solution of (ND1), by Fritz John type necessary condition, there exist $\alpha \in R, \beta \in R^n, \theta \in R, \xi \in R$ and $\eta \in R^m$ such that

$$-\alpha[\nabla f(x^*) - \frac{1}{2}\nabla(p^{*T}\nabla^2 f(x^*)p^*)] - \theta[\nabla(y^{*T} g(x^*)) - \frac{1}{2}\nabla(p^{*T}\nabla^2(y^* g(x^*))p^*)]$$
$$+[r^*(\nabla^2 f(x^*) + \nabla(\nabla^2 f(x^*)p^*)) + \nabla^2(y^{*T} g(x^*)) + \nabla(\nabla^2(y^* g(x^{*T})p^*))]\beta = 0, \quad (2.1)$$

$$\beta^T[\nabla g_j(x^*) + \nabla^2 g_j(x^*)p^*] - \theta[g_j(x^*) - \frac{1}{2}p^{*T}\nabla^2 g_j(x^*)p^*] - \eta_j = 0, \quad (2.2)$$
$$j = 1, \ldots, m,$$

$$\beta^T[\nabla f(x^*) + \nabla^2 f(x^*)p^*] - \xi = 0, \quad (2.3)$$

$$(\alpha p^* + \beta r^*)^T[\nabla^2 f(x^*)] + (\theta p^* + \beta)^T[\nabla^2 y^{*T} g(x^*)] = 0, \quad (2.4)$$

$$\theta[y^{*T} g(x^*) - \frac{1}{2}p^*\nabla^2 y^{*T} g(x^*)p^*] = 0, \quad (2.5)$$

$$\xi r^* = 0, \quad (2.6)$$

$$\eta^T y^* = 0, \quad (2.7)$$

$$(\alpha, \theta, \xi, \eta) \geqq 0, \quad (2.8)$$

$$(\alpha, \beta, \theta, \xi, \eta) \neq 0. \quad (2.9)$$

First, we claim that $\theta > 0$. Otherwise, $\theta = 0$, then (2.4) reduces to

$$(\alpha p^* + \beta r^*)^T[\nabla^2 f(x^*)] + \beta^T[\nabla^2 y^{*T} g(x^*)] = 0. \tag{2.10}$$

Multiplying the above equation by p^*, we have

$$\alpha p^{*T}\nabla^2 f(x^*)p^* + r^*\beta^T\nabla^2 f(x^*)p^* + \beta^T\nabla^2 y^* g(x^*)p^* = 0. \tag{2.11}$$

Multiplying (2.2) by $y_j^*, j = 1, \ldots, m$, summering over j and using (2.5) and (2.7), we get

$$\beta^T[\nabla y^{*T} g(x^*) + \nabla^2 y^{*T} g(x^*)p^*] = 0. \tag{2.12}$$

Multiplying (2.3) by r^* and from (2.6), we have

$$r^*\beta^T\nabla f(x^*) + r^*\beta^T\nabla^2 f(x^*)p^* = 0. \tag{2.13}$$

Using(2.12) and (2.13) in (2.11), we have

$$\alpha p^{*T}\nabla^2 f(x^*)p^* = \beta^T[r^*\nabla f(x^*) + \nabla y^* g(x^*)].$$

If assumption (i) holds, the above equation gives

$$\alpha p^{*T}\nabla^2 f(x^*)p^* = 0,$$

and using assumption (ii), we obtain $\alpha p^* = 0$. We claim that $\alpha > 0$. Suppose to the contrary that $\alpha = 0$, then (2.1) reduces to

$$[r^*(\nabla^2 f(x^*) + \nabla(\nabla^2 f(x^*)p^*)) + \nabla^2(y^{*T} g(x^*)) + \nabla(\nabla^2(y^* g(x^{*T})p^*))]\beta = 0.$$

If assumption (iii) holds, the above equation gives $\beta = 0$. From (2.2) and (2.3), we have $\xi = 0$ and $\eta = 0$. So $(\alpha, \beta, \theta, \xi, \eta) = 0$, and which contradicts with (2.9). Therefore $\alpha > 0$. Since $\alpha p^* = 0$, $p^* = 0$. Hence (2.1) and (2.4) reduce to

$$[r^*\nabla^2 f(x^*) + \nabla^2 y^{*T} g(x^*)]\beta = \alpha\nabla f(x^*),$$
$$[r^*\nabla^2 f(x^*) + \nabla^2 y^{*T} g(x^*)]\beta = 0.$$

By using assumption (iv) and $\alpha > 0$, we have $\alpha\nabla f(x^*) \neq 0$. And we get a contradiction. Therefore, $\theta \neq 0$. That is $\theta > 0$.

Since $\theta > 0$, it follows from (2.5), we have

$$y^{*T} g(x^*) = \frac{1}{2}p^{*T}\nabla^2 y^{*T} g(x^*)p^*.$$

If assumption (v) holds, the above equation gives $p^* = 0$. Therefore, (2.4) reduces to

$$\beta^T[r^*\nabla^2 f(x^*) + \nabla^2 y^* g(x^*)] = 0.$$

Which combining with assumption (iii), we get $\beta = 0$. Therefore, (2.2) reduces to

$$\theta g_j(x^*) = -\eta_j, \quad j = 1, \ldots, m.$$

Since $\theta > 0$ and $\eta \geqq 0$, we have $g(x^*) \leqq 0$. This implies that x^* is a feasible solution of problem (ND1).

On the other hand, since $p^* = 0$, the two objectives are equal. Also, by weak duality theorem in [3], x^* is an optimal solution of problem (NP1).

Remark 2.2. If $p^* = 0$, condition (i) is the first constraint condition of problem (NP1), and for first order converse duality theorem, this condition can be removed. If $p^* = 0$, condition (ii) is the condition $\nabla f(x^*) + \nabla^2 f(x^*)p^* \neq 0$ in [2], condition (iii) reduces to $r^*\nabla^2 f(x^*) + \nabla^2(y^{*T}g(x^*)$ is nonsingular. If $\nabla^2(y^{*T}g(x^*)$ is negative define or positive define, condition (ii) implies $r^*\nabla^2 f(x^*) + \nabla^2(y^{*T}g(x^*)$ is nonsingular.

Hence, we give the proof of the converse duality theorem by replacing condition (A3) in Theorem 1.2 by two reasonable conditions (ii) and (iii).

3 Converse Duality for Multiobjective Programming Problems

In [2], Gulati et al studied the following multiobjecitve nonlinear programming problem.

$$\text{(VP)} \qquad \begin{aligned} &\text{Minimize } \ f(x) \\ &\text{subject to } g(x) \leqq 0. \end{aligned}$$

where $f : R^n \to R^k, g : R^n \to R^m$ are twice differentiable functions.

And they formulated second order dual problem (VD) for (VP) as follows:

$$\text{(VD)} \qquad \begin{aligned} &\text{Maxmize } \ (f_1(u) - \frac{1}{2}p^T\nabla^2 f_1(u)p, \ldots, f_k(u) - \frac{1}{2}p^T\nabla^2 f_k(u)p) \\ &\text{subject to } \nabla(\lambda^T f(u) + \nabla y^T g(u)) + \nabla^2(\lambda^T f(u) + \nabla^2 y^T g(u))p = 0, \\ &y^T g(u) - \frac{1}{2}p^T\nabla(y^T g(u))p \geqq 0, \\ &\lambda \geq 0, \\ &y \geqq 0, \\ &\sum_{i=1}^{k} \lambda_i = 1, \end{aligned}$$

where $r \in R, y \in R^m$ and $p, u \in R^n$.

Observing that in Huard type converse duality Theorem 5 in [2], the assumption that the vectors $\{[\nabla^2 f_i(x^*)]_j, [\nabla^2 y^{*T}g(x^*)]_j, i = 1, \ldots, k, j = 1, \ldots, n\}$ are linearly independent, where $[\nabla^2 f_i(x^*)]_j$ and $[\nabla^2 y^{*T}g(x^*)]_j$ are jth rows of $\nabla^2 f_i(x^*)$ and $\nabla^2 y^{*T}g(x^*)$, respectively, is erroneous. Since $\{[\nabla^2 f(x^*)]_j, [\nabla^2 y^* g(x^*)]_j, j = 1, \ldots, n\}$ are $(k+1)n$ vectors in R^n. They are linearly dependent.

Remark 3.1. In problem (VD), since $\lambda \geq 0$, that is $\lambda_i \geqq 0, \imath = 1, \ldots, k$, and $\lambda \neq 0$, the constraint condition $\sum_{i=1}^k \lambda_i = 1$ can be eliminated. In fact, if $\sum_{i=1}^k \lambda_i \neq 1$, we can take $\lambda = \frac{\lambda}{\sum_{i=1}^k \lambda_i}$ and $y = \frac{y}{\sum_{i=1}^k \lambda_i}$ in the constrains of problem (VD). Therefore, in the proof of following converse duality theorem, we do not consider this condition.

Now we give a modified proof of converse duality theorems for (VP) and (VD).

Theorem 3.2 (Converse Duality for (VD))**.** *Let $(x^*, \lambda^*, y^*, p^*)$ be a weak efficient solution of* (VD)*). Assume that*

(i) $\sum_{i=1}^k \lambda_i^* \nabla f_i(x^*) + \nabla y^{*T} g(x^*) = 0.$

(ii) $\nabla^2 f_i(x^*), i = 1, \ldots, k$ *are negative define or positive define.*

(iii) $\nabla^2 \lambda^{*T} f(x^*) + \nabla(\nabla^2 \lambda^{*T} f(x^*)p^*) + \nabla^2(y^{*T}g(x^*)) + \nabla(\nabla^2(y^{*T}g(x^*)p^*))$ *is nonsingular.*

(iv) $\{\nabla f_1(x^*), \ldots, \nabla f_k(x^*)\}, k \leq n$, *are linearly independent.*

(v) $\nabla^2 y^{*T} g(x^*)$ *is positive define and* $y^{*T} g(x^*) \leqq 0$ *or* $\nabla^2 y^{*T} g(x^*)$ *is negative define and* $y^{*T} g(x^*) \geqq 0$.

Then $\bar{p} = 0$, x^* *is a feasible solution of* (VP) *and the two objective functions values are equal. Furthermore, if the generalized convexity hypotheses of Theorem 5 in [2] are satisfied, then* x^* *is a weak efficient solution of* (VP).

Proof. Since $(x^*, \lambda^*, y^*, p^*)$ is a weak efficient solution of (VD), by Fritz John type necessary condition, there exist $\alpha \in R, \beta \in R^n, \theta \in R, \xi \in R$ and $\eta \in R^m$ such that

$$-\sum_{i=1}^{k} \alpha_i [\nabla f_i(x^*) - \frac{1}{2}\nabla(p^{*T}\nabla^2 f_i(x^*)p^*)] - \theta[\nabla(y^{*T}g(x^*)) - \frac{1}{2}\nabla(p^{*T}\nabla^2(y^{*T}g(x^*))p^*)]$$
$$+[\nabla^2\lambda^{*T}f(x^*) + \nabla(\nabla^2\lambda^{*T}f(x^*)p^*) + \nabla^2(y^{*T}g(x^*)) + \nabla(\nabla^2(y^{*T}g(x^*)p^*))]\beta = 0, \quad (3.1)$$

$$\beta^T[\nabla g_j(x^*) + \nabla^2 g_j(x^*)p^*] - \theta[g_j(x^*) - \frac{1}{2}p^{*T}\nabla^2 g_j(x^*)p^*] - \eta_j = 0, \quad j = 1, \ldots, m, \quad (3.2)$$

$$\beta^T[\nabla f_i(x^*) + \nabla^2 f_i(x^*)p^*] - \xi_i = 0, \quad i = 1, \ldots, k, \quad (3.3)$$

$$\sum_{i=1}^{k}(\alpha_i p^* + \beta\lambda_i^*)^T[\nabla^2 f_i(x^*)] + (\theta p^* + \beta)^T[\nabla^2 y^{*T} g(x^*)] = 0, \quad (3.4)$$

$$\theta(y^{*T}g(x^*) - \frac{1}{2}p^{*T}\nabla(y^{*T}g(x^*))p^*] = 0, \quad (3.5)$$

$$\xi^T\lambda^* = 0, \quad (3.6)$$

$$\eta^T y^* = 0, \quad (3.7)$$

$$(\alpha, \theta, \xi, \eta) \geqq 0, \quad (3.8)$$

$$(\alpha, \beta, \theta, \xi, \eta) \neq 0. \quad (3.9)$$

First, we claim that $\theta > 0$. Indeed, if $\theta = 0$, then (3.4) reduces to

$$\sum_{i=1}^{k}(\alpha_i p^* + \beta\lambda_i^*)^T[\nabla^2 f_i(x^*)] + \beta^T\nabla^2 y^{*T} g(x^*) = 0. \quad (3.10)$$

Multiplying the above equation by p^*, we have

$$\sum_{i=1}^{k}[\alpha_i p^{*T}\nabla^2 f_i(x^*)p^* + \lambda_i^*\beta^T\nabla^2 f_i(x^*)p^*] + \beta^T\nabla^2 y^{*T}g(x^*)p^* = 0. \quad (3.11)$$

Multiplying (3.2) by $y_j^*, j = 1, \ldots, m$, summering over j and using (3.5) and (3.7), we get

$$\beta^T[\nabla y^{*T}g(x^*) + \nabla^2 y^{*T}g(x^*)p^*] = 0. \quad (3.12)$$

Multiplying (3.3) by $\lambda_i^*, i = 1, \ldots, k$, summering over i and using (3.6), we get

$$\sum_{i=1}^{k}\beta^T[\lambda_i^*\nabla f_i(x^*) + \lambda_i^*\nabla^2 f_i(x^*)p^*] = 0. \quad (3.13)$$

Using (3.12) and (3.13) in (3.11), we have

$$\sum_{i=1}^{k} \alpha_i p^{*T} \nabla^2 f(x^*) p^* = \beta^T [\sum_{i=1}^{k} \lambda_i^* \nabla f_i(x^*) + \nabla y^{*T} g(x^*)].$$

If assumption (i) holds, the above equation gives

$$\sum_{i=1}^{k} \alpha_i p^{*T} \nabla^2 f(x^*) p^* = 0.$$

And using assumption (ii), we get $\alpha_i p^* = 0, i = 1, \ldots, k$. That is, $\alpha = 0$ or $p^* = 0$.

We claim that $\alpha \neq 0$. Suppose to the contrary that $\alpha = 0$, then (3.1) reduces to

$$[\nabla^2 \lambda^{*T} f(x^*) + \nabla(\nabla^2 \lambda^{*T} f(x^*) p^*) + \nabla^2 (y^{*T} g(x^*)) + \nabla(\nabla^2 (y^{*T} g(x^*) p^*))]\beta = 0.$$

If assumption (iii) holds, the above equation gives $\beta = 0$. From (3.2) and (3.3), we have $\xi = 0$ and $\eta = 0$. So $(\alpha, \beta, \theta, \xi, \eta) = 0$, and which contradicts with (3.9). Therefore $\alpha \neq 0$. Since $\alpha_i p^* = 0, i = 1, \ldots, k$ and $p^* = 0$. Hence (3.1) and (3.4) reduce to

$$[\nabla^2 \lambda^{*T} f(x^*) + \nabla^2 y^{*T} g(x^*)]\beta = \sum_{i=1}^{k} \alpha_i \nabla f_i(x^*),$$
$$[\nabla^2 \lambda^{*T} f(x^*) + \nabla^2 y^{*T} g(x^*)]\beta = 0.$$

By using assumption (iv) and $\alpha \neq 0$, we have $\sum_{i=1}^{k} \alpha_i \nabla f_i(x^*) \neq 0$. And we get a contradiction. Therefore, $\theta \neq 0$. That is $\theta > 0$.

Since $\theta > 0$, it follows from (3.5), we have

$$y^{*T} g(x^*) = \frac{1}{2} p^{*T} \nabla^2 y^{*T} g(x^*) p^*$$

If assumption (v) holds, the above equation gives $p^* = 0$. Hence, (3.4) reduces to

$$\beta^T [\sum_{i=1}^{k} \lambda_i^* \nabla^2 f_i(x^*) + \nabla^2 y^{*T} g(x^*)] = 0.$$

Which combining with assumption (iii), we get $\beta = 0$. Hence, (3.2) reduces to

$$\theta g_j(x^*) = -\eta_j, \quad j = 1, \ldots, m.$$

Since $\theta > 0$ and $\eta \geqq 0$, we have $g(x^*) \leqq 0$. This implies x^* is a feasible solution of problem (VP).

On the other hand, since $p^* = 0$, the two objectives are equal, and x^* is weak efficient solution of problem (VP).

Remark 3.3. (1) If $k = 1$, conditions (i)-(v) reduce to conditions (i)-(v) in Theorem 2.1. And we give the proof of the converse duality theorem by replacing condition (C3) of Theorem 5 in [2] by two reasonable conditions (ii) and (iii).

(2) If $p^* = 0$, the dual problem (VD) reduces to first order dual model considered in [11]. Condition (i) is the first constraint condition of problem (VD), this condition can be removed. Condition (iii) reduces to $\nabla^2[\lambda^{*T} f(x^*) + y^{*T} g(x^*)]$ is nonsingular. And conditions (ii) and (v) can imply this condition. Hence, this condition can be removed.

Therefore, we can have the first order converse duality theorem under the following conditions.

$(C1)$ $\nabla^2 f_i(x^*), i = 1, \ldots, k$, $\nabla^2 y^{*T} g(x^*)$ are negative define and $y^{*T} g(x^*) \geqq 0$ or $\nabla^2 f_i(x^*)$, $i = 1, \ldots, k$, $\nabla^2 y^{*T} g(x^*)$ are positive define and $y^{*T} g(x^*) \leqq 0$,

$(C2)$ $\{\nabla f_1(x^*), \ldots, \nabla f_k(x^*)\}, k \leq n$, are linearly independent, Yang et al . [11] consider the first order converse duality for multiobjective optimization problems (VP) and (VD) under the following assumptions.

$(A1)$ $\nabla^2[\lambda^{*T} f(x^*) + y^{*T} g(x^*)]$ is negative define,

$(A2)$ $\nabla y^{*T} g(x^*) \neq 0$,

$(A3)$ $\nabla y^{*T} g(x^*) \notin span\{\nabla f_1(x^*), \ldots, \nabla f_k(x^*)\}$.

We can see that our assumptions can not reduce to the above conditions $(A1) - (A3)$. In fact, conditions $(A1) - (A3)$ are more weaker than the conditions $(C1) - (C2)$. This imply that there are some differences in the methods of proof for first order converse duality theorem and second order converse duality theorem.

4 Converse Duality for Nondifferentiable Nonlinear Programming

In [1], Ahmad et al studied the following nondifferentiable fractional programming problem.

$$\text{(NP2)} \qquad \text{Minimize } \frac{f(x) + (x^T Bx)^{\frac{1}{2}}}{g(x) - (x^T Cx)^{\frac{1}{2}}}$$
$$\text{subject to } h(x) \leqq 0.$$

where $f, g : R^n \to R, h : R^n \to R^m$ are twice differentiable functions. B and C are $n \times n$ positive semi-definite symmetric matrices.

And they formulated second order dual problem (ND2) for (NP2) as follows.

$$\text{(ND2)} \qquad \text{Maxmize } K(y, u, v, w, z, p) = v$$
subject to
$$[\nabla f(y) + \nabla^2 f(y)p + Bw] - v[\nabla g(y) + \nabla^2 g(y)p - Cz]$$
$$- \nabla u^T h(y) - \nabla^2 u^T h(y)p = 0, \tag{4.1}$$
$$[f(y) - \frac{1}{2} p^T \nabla^2 f(y)p + y^T Bw] - v[g(y) - \frac{1}{2} p^T \nabla^2 g(y)p - y^T CZ] \leq 0,$$
$$u^T h(y) - \frac{1}{2} p^T \nabla^2 u^T h(y)p \geqq 0,$$
$$w^T Bw \leqq 1,$$
$$z^T Cz \leqq 1,$$
$$u \geqq 0, v \geqq 0,$$

where $u, w, p \in R^n, y \in R^m$, $I_\alpha \subset M = \{1, \ldots, m\}, \alpha = 0, 1, \ldots, r$ with $\bigcup_{\alpha=0}^{r} I_\alpha = M$.

In converse duality Theorem 3.4 in [1], the assumption that the vectors $\{[\nabla^2 f(\bar{y}) - \bar{v}\nabla^2\nabla^2 g(\bar{y})]_j, [\nabla^2 \bar{u}^T h(\bar{y}]_j, j = 1, \ldots, n\}$ are linearly independent, is erroneous. In the following theorem, we give appropriate rectifications for Theorem 3.4 in [1].

Theorem 4.1 (Converse Duality for (ND2))**.** *Let $(\bar{y}, \bar{u}, \bar{v}, \bar{w}, \bar{z}, \bar{p})$ be a optimal solution of* (ND3)*). Assume that*

(i) $\nabla f(\bar{y}) + B\bar{w} - \bar{v}\nabla g(\bar{y}) + \bar{v}C\bar{z} - \nabla \bar{u}^T h(\bar{y}) = 0$.

(ii) $\nabla^2 f(\bar{y}) - \bar{v}\nabla^2 g(\bar{y})$ *is negative define or positive define.*

(iii) $\nabla^2 f(\bar{y}) - \bar{v}\nabla^2 g(\bar{y}) + \nabla(\nabla^2 f(\bar{y})\bar{p} - \bar{v}\nabla^2 g(\bar{y})\bar{p}) - \nabla^2 \bar{u}^T h(\bar{y}) - \nabla(\nabla^2 \bar{u}^T h(\bar{y})\bar{p})$ *is nonsingular.*

(iv) $\nabla f(\bar{y}) + B\bar{w} - \bar{v}\nabla g(\bar{y}) + \bar{v}C\bar{z} \neq 0$.

(v) $\nabla^2 \bar{u}^T h(\bar{y})$ *is positive define and* $\bar{u}^T h(\bar{y}) \leqq 0$ *or* $\nabla^2 \bar{u}^T h(\bar{y})$ *is negative define and* $\bar{u}^T h(\bar{y}) \geqq 0$.

Then $\bar{p} = 0$, $\bar{y}$ *is a feasible solution of* (NP2) *and the two objective functions values are equal. Furthermore, if the generalized convexity hypotheses of Theorem 3.4 in [1] are satisfied, then* $\bar{y}$ *is an optimal solution of* (NP2).

Proof. Since $(\bar{y}, \bar{u}, \bar{v}, \bar{w}, \bar{z}, \bar{p})$ be a optimal solution of (ND2)), by Fritz John type necessary condition, there exist $\alpha \in R, \beta \in R^n, \gamma \in R, \xi \in R$, $\mu \in R$, $\delta \in R$ and $v \in R^m$ such that

$$\beta^T[\nabla^2 f(\bar{y}) - \bar{v}\nabla^2 g(\bar{y}) + \nabla(\nabla^2 f(\bar{y})\bar{p} - \bar{v}\nabla^2 g(\bar{y})\bar{p}) - \nabla^2 \bar{u}^T h(\bar{y}) - \nabla(\nabla^2 \bar{u}^T h(\bar{y})\bar{p})]$$
$$+\gamma[(\nabla f(\bar{y}) + B\bar{w} - \frac{1}{2}\bar{p}^T\nabla(\nabla^2 f(\bar{y})\bar{p})) - \bar{v}(\nabla g(\bar{y}) - C\bar{z} - \frac{1}{2}\bar{p}^T\nabla(\nabla^2 g(\bar{y})\bar{p}))]$$
$$+\xi[\nabla \bar{u}^T h(\bar{y}) - \frac{1}{2}\bar{p}^T\nabla(\nabla^2 \bar{u}^T h(\bar{y})\bar{p})] = 0, \quad (4.2)$$

$$\alpha + [\nabla g(\bar{y} + \nabla^2 g(\bar{y})\bar{p} - C\bar{z}]\beta + \gamma[g(\bar{y}) - \frac{1}{2}\bar{p}^T\nabla^2 g(\bar{y})\bar{p} - \bar{y}^T C\bar{z}] = 0, \quad (4.3)$$

$$[\nabla h(\bar{y}) + \nabla^2 h(\bar{y})\bar{p}]\beta - \xi[h(\bar{y}) - \frac{1}{2}\bar{p}^T\nabla^2 h(\bar{y})\bar{p}] - v = 0, \quad (4.4)$$

$$\beta^T B + \gamma B\bar{y} - 2\mu B\bar{w} = 0, \quad (4.5)$$

$$\bar{v}\beta^T C + \bar{v}\gamma C\bar{y} - 2\delta C\bar{z} = 0, \quad (4.6)$$

$$\beta^T[\nabla^2 f(\bar{y}) - \bar{v}\nabla^2 g(\bar{y}) - \nabla^2 \bar{u}^T h(\bar{y})] - \gamma[\nabla^2 f(\bar{y})\bar{p} - \bar{v}\nabla^2 g(\bar{y})\bar{p}] - \xi\nabla^2 \bar{u}^T h(\bar{y})\bar{p} = 0, \quad (4.7)$$

$$\gamma[(f(\bar{y}) + \bar{y}^T B\bar{w} - \frac{1}{2}\bar{p}^T\nabla^2 f(\bar{y})\bar{p}) - \bar{v}(g(\bar{y}) - \bar{y}^T C\bar{z} - \frac{1}{2}\bar{p}^T\nabla^2 g(\bar{y})\bar{p})] = 0, \quad (4.8)$$

$$\xi[\bar{u}^T h(\bar{y}) - \frac{1}{2}\bar{p}^T\nabla^2 \bar{u}^T h(\bar{y})\bar{p}] = 0, \quad (4.9)$$

$$\mu(\bar{w}^T B\bar{w} - 1) = 0, \quad (4.10)$$

$$\delta(\bar{z}^T C\bar{z} - 1) = 0, \quad (4.11)$$

$$v^T\bar{u} = 0, \quad (4.12)$$

$$(\alpha, \gamma, \xi, \mu, \delta, v) \geqq 0, \quad (4.13)$$

$$(\alpha, \beta, \gamma, \xi, \mu, \delta, v) \neq 0. \quad (4.14)$$

First, we claim that $\xi > 0$. Otherwise, $\xi = 0$, then (4.7) reduces to

$$(\beta - \gamma\bar{p})^T[\nabla^2 f(\bar{y}) - \bar{v}\nabla^2 g(\bar{y})] - \beta^T\nabla^2 \bar{u}^T h(\bar{y}) = 0.$$

Multiplying the above equation by $\bar{p}$, we have

$$\beta^T[\nabla^2 f(\bar{y}) - \bar{v}\nabla^2 g(\bar{y}) - \nabla^2 \bar{u}^T h(\bar{y})]\bar{p} = \gamma\bar{p}^T[\nabla^2 f(\bar{y}) - \bar{v}\nabla^2 g(\bar{y})]\bar{p}. \quad (4.15)$$

Multiplying (4.1) by β , we have

$$\beta^T[\nabla^2 f(\bar{y}) - v\nabla^2 g(y) - \nabla^2 \bar{u}^T h(\bar{y})]\bar{p} = -\beta^T[\nabla f(\bar{y}) + B\bar{w} - \bar{v}\nabla g(\bar{y}) + \bar{v}C\bar{z} - \nabla \bar{u}^T h(\bar{y})].$$

If assumption (i) holds, the above equation reduces to

$$\beta^T[\nabla^2 f(\bar{y}) - v\nabla^2 g(y) - \nabla^2 \bar{u}^T h(\bar{y})]\bar{p} = 0.$$

And (4.15) reduces to

$$\gamma\bar{p}^T[\nabla^2 f(\bar{y}) - \bar{v}\nabla^2 g(\bar{y})]\bar{p} = 0.$$

If assumption (ii) holds, the above equation gives $\gamma\bar{p} = 0$.

Now we claim that $\gamma \neq 0$. Indeed, if $\gamma = 0$, then (4.2) reduces to

$$\beta^T[\nabla^2 f(\bar{y}) - \bar{v}\nabla^2 g(\bar{y}) + \nabla(\nabla^2 f(\bar{y})\bar{p} - \bar{v}\nabla^2 g(\bar{y})\bar{p}) - \nabla^2 \bar{u}^T h(\bar{y}) - \nabla(\nabla^2 \bar{u}^T h(\bar{y})\bar{p})] = 0.$$

By using assumption (iii), we have $\beta = 0$. From (4.3)-(4.6), we have $(\alpha, \beta, \gamma, \xi, \mu, \delta, v) = 0$, and which contradicts with (4.14). Therefore $\gamma \neq 0$. Since $\gamma p^* = 0$, $p^* = 0$. Hence (4.2) and (4.5) reduce to

$$\beta^T[\nabla^2 f(\bar{y}) - \bar{v}\nabla^2 g(\bar{y}) - \nabla^2 \bar{u}^T h(\bar{y})] = -\gamma[\nabla f(\bar{y}) + B\bar{w} - \bar{v}\nabla g(\bar{y}) + \bar{v}C\bar{z}],$$
$$\beta^T[\nabla^2 f(\bar{y}) - \bar{v}\nabla^2 g(\bar{y}) - \beta^T\nabla^2 \bar{u}^T h(\bar{y})] = 0.$$

By using assumption (iv) and $\gamma \neq 0$, we get a contradiction.

Therefore, $\xi \neq 0$. That is $\xi > 0$.

Since $\xi > 0$, it follows from (4.9), we have

$$\bar{u}^T h(\bar{y}) - \frac{1}{2}\bar{p}^T\nabla^2 \bar{u}^T h(\bar{y})\bar{p} = 0.$$

If assumption (v) holds, the above equation gives $\bar{p} = 0$. Therefore, (4.7) reduces to

$$\beta^T[\nabla^2 f(\bar{y}) - v\nabla^2 g(y) - \nabla^2 \bar{u}^T h(\bar{y})] = 0.$$

Which combining with assumption (iii), we get $\beta = 0$. Therefore, (4.4) reduces to $\xi h(\bar{y}) = -v$. Since $\xi > 0$ and $v \geqq 0$, $h(\bar{y}) \leqq 0$. This implies that $\bar{y}$ is a feasible solution of problem (NP2).

Similar to the proof of Theorem 3.4 in [1], we have $\bar{y}$ is an optimal solution of problem (NP2).

Remark 4.2. (1) If $B = C = 0$ and $g(x) \equiv 1$, problem (NP2) reduces to problem (NP1), and conditions (i)-(v) reduce to conditions (i)-(v) in Theorem 2.1 of Section 2. And we give the proof of the converse duality theorem by replacing condition (ii) in Theorem 3.4 in [1] by reasonable conditions (ii) and (iii).

(2) If $\bar{p} = 0$ and $B = C = 0$, problems (NP2) and (ND2) reduce to first order dual problems considered in [4]. Lee et al. [4] gave first order converse duality theorem(see Theorem 2.1). But our assumptions can not reduce to the assumptions of Theorem 2.1 in [4].

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