Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 13, No. 3, pp. 527–545, July 2017

GENERALIZED EIGENVALUE COMPLEMENTARITY PROBLEM FOR TENSORS

Zhongming Chen*, Qingzhi Yang† and Lu Ye‡

Abstract: In this paper, the generalized eigenvalue complementarity problem for tensors (GEiCP-T) is addressed, which arises from the stability analysis of finite dimensional mechanical systems and finds application in differential dynamical systems. The general properties of GEiCP-T have been studied. We establish its relationship with the generalized tensor eigenvalue problem. It follows that if exist, the number of λ-solutions can be bounded. We also give some sufficient conditions for the existence of the solution. In particular, there exists a unique solution of eigenvalue complementarity problem for irreducible nonnegative tensors. For the symmetric case, we derive a sufficient and necessary condition for the solvability of GEiCP-T by reformulating it as a nonlinear program. It has also been proved that deciding the solvability of eigenvalue complementarity problem for tensors is NP-hard in general. Moreover, a shifted projected power method is proposed to solve the symmetric GEiCP-T. The monotonic convergence is also established. The numerical experiments demonstrate convergence behavior of our method and show that the algorithm presented is promising.

Key words: *generalized eigenvalue complementarity, generalized tensor eigenpair, solution existence, nonlinear program, NP-hard, shifted projected power method*

Mathematics Subject Classification: *15A18, 15A69, 90C33*

1 Introduction

Denote $[n] = \{1, 2, \ldots, n\}$. A real mth order n-dimensional tensor $\mathcal{A} = (a_{i_1\cdots i_m})$ is a multidimensional array with each entry $a_{i_1\cdots i_m} \in \mathbb{R}$ for any $i_1, \ldots, i_m \in [n]$. Denote the set of all real mth order n-dimensional tensors by $T_{m,n}$. For any vector $\mathbf{x} \in \mathbb{R}^n$, let $\mathcal{A}\mathbf{x}^{m-1}$ be a vector in $\mathbb{R}^n$ whose ith component is defined by

$$(\mathcal{A}\mathbf{x}^{m-1})_i = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m}.$$

And let $\mathcal{A}\mathbf{x}^m$ be the scale denoted by $\mathcal{A}\mathbf{x}^m = \mathbf{x}^\top \mathcal{A}\mathbf{x}^{m-1}$. We say a tensor $\mathcal{A}$ is *symmetric* if its entries are invariant under permutation. Denote the set of all real symmetric mth order n-dimensional tensors by $S_{m,n}$. Clearly, $S_{m,n}$ is a linear subspace of $T_{m,n}$. A tensor

*Zhongming Chen's work was supported by the National Natural Science Foundation of China, Grant No. 11701132.

†Qingzhi Yang's work was supported by the National Natural Science Foundation of China, Grant No. 11671217 and Grant No. 11271206.

‡Corresponding author.

$\mathcal{A} \in S_{m,n}$ is called *positive definite* if $\mathcal{A}\mathbf{x}^m > 0$ for all $\mathbf{x} \neq 0$. Clearly, when m is odd, there is no positive definite tensors.

In 2005, the definition of eigenvalue for tensors was introduced by Qi [19] and Lim [14], independently. Later, these definitions were unified by Chang, Person and Zhang [3] as follows. Let $\mathcal{A} \in T_{m,n}$. Assume that m is even and $\mathcal{B} \in S_{m,n}$ is positive definite. We say $(\lambda, \mathbf{x}) \in \mathbb{R} \times (\mathbb{R}^n \setminus \{0\})$ is a *generalized eigenpair* of $(\mathcal{A}, \mathcal{B})$ if

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathcal{B}\mathbf{x}^{m-1}. \tag{1.1}$$

The advantage of this definition is that the definitions of H-eigenvalue [19], Z-eigenvalue [19] and D-eigenvalue [21] can be unified by replacing $\mathcal{B}$ with some special forms, respectively [3]. After that, the study of tensors and the spectra of tensors with their various applications has attracted extensive attention and interest [4, 8, 13].

However, in many practical problems, the eigenpair $(\lambda, \mathbf{x})$ in equations (1.1) should be solved to satisfy actual requirement, or can always be constrained according to prior information. For example, in the stability analysis of finite dimensional mechanical systems with frictional contact [6], a necessary and sufficient condition for the occurrence of divergence instability along a constant admissible direction, is to find $\lambda \geq 0$ and a nonzero vector $\mathbf{x} \in \mathbb{R}^n$ such that

$$\begin{cases} \mathbf{w} = (\lambda M + K)\mathbf{x} \\ \mathbf{w}_f = 0 \\ 0 \leq \mathbf{x}_c \perp \mathbf{w}_c \geq 0, \end{cases} \qquad \mathbf{x} = \begin{bmatrix} \mathbf{x}_f \\ \mathbf{x}_c \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} \mathbf{w}_f \\ \mathbf{w}_c \end{bmatrix},$$

where the matrices $M \in \mathbb{R}^{n\times n}$ and $K \in \mathbb{R}^{n\times n}$ are linear pencils of matrices in the coefficient of friction at the particles in impending slip. For the matrix case, this problem was well-studied and found application in different areas of science and engineering, for instance, see [5, 22].

More general, we consider a homogeneous differential dynamical system constrained by linear complementarity conditions. To be more specific, given a homogeneous mapping $F : \mathbb{R}^n \to \mathbb{R}^n$, we consider an equilibrium system of the form:

$$\begin{cases} \mathbf{u}(t) \geq 0 \\ \mathbf{u}'(t) - F(\mathbf{u}(t)) \geq 0 \\ \langle \mathbf{u}(t), \mathbf{u}'(t) - F(\mathbf{u}(t)) \rangle = 0. \end{cases} \tag{1.2}$$

When the mapping F is linear, i.e., $F(\mathbf{x}) = A\mathbf{x}$ with a given matrix $A \in \mathbb{R}^{n\times n}$, the solution of (1.2) is called the *linear complementarity process* in [23]. Here, we consider a mapping defined by a nonnegative tensor $\mathcal{A} \in T_{m,n}$, i.e.,

$$F_{\mathcal{A}}(\mathbf{x}) = (\mathcal{A}\mathbf{x}^{m-1})^{[\frac{1}{m-1}]}, \quad \forall\, \mathbf{x} \in \mathbb{R}^n_+,$$

where the ith component of the vector $\mathbf{y}^{[r]}$ is given by y_i^r. Clearly, the definition of $F_{\mathcal{A}}(\mathbf{x})$ is well-defined since $\mathcal{A}$ and $\mathbf{x}$ are nonnegative. And the mapping $F_{\mathcal{A}} : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is positively homogeneous, i.e., $F_{\mathcal{A}}(t\mathbf{x}) = tF_{\mathcal{A}}(\mathbf{x})$ for any $t > 0$. Like the linear case, if we require a solution which has the form $\mathbf{u}(t) = e^{\lambda t}\mathbf{x}$ with $\lambda > 0$, then the equilibrium system (1.2) can be expressed as

$$\begin{cases} \mathbf{x} \geq 0 \\ \lambda\mathbf{x} - F_{\mathcal{A}}(\mathbf{x}) \geq 0 \\ \langle \mathbf{x}, \lambda\mathbf{x} - F_{\mathcal{A}}(\mathbf{x}) \rangle = 0. \end{cases} \tag{1.3}$$

This transformation makes use of the positive homogeneity of $F_{\mathcal{A}}$. Note that the second inequality in (1.3) is equivalent to $\lambda^{m-1}\mathbf{x}^{[m-1]} - \mathcal{A}\mathbf{x}^{m-1} \geq 0$ since $\lambda > 0$ with $\mathbf{x} \geq 0$. It follows that the complementarity condition can also be equivalently written as

$$\langle \mathbf{x}, \lambda^{m-1}\mathbf{x}^{[m-1]} - \mathcal{A}\mathbf{x}^{m-1} \rangle = 0.$$

Denote the identity tensor by $\mathcal{I} = (\delta_{i_1 \cdots i_m}) \in S_{m,n}$, where $\delta_{i_1 \cdots i_m}$ is the generalized Kronecker symbol defined as

$$\delta_{i_1 \cdots i_m} = \begin{cases} 1 & \text{if } i_1 = \ldots = i_m, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the equilibrium system (1.3) is to find $\lambda > 0$ and $\mathbf{x} \in \mathbb{R}^n$ such that

$$\begin{cases} \mathbf{x} \geq 0 \\ (\lambda^{m-1}\mathcal{I} - \mathcal{A})\mathbf{x}^{m-1} \geq 0 \\ \langle \mathbf{x}, (\lambda^{m-1}\mathcal{I} - \mathcal{A})\mathbf{x}^{m-1} \rangle = 0. \end{cases} \tag{1.4}$$

This is exactly an eigenvalue complementarity problem for tensors. We also mention that if $(\lambda, \mathbf{x}) \in \mathbb{R} \times (\mathbb{R}^n \setminus \{\mathbf{0}\})$ is a solution (1.4), λ is called Pareto H-eigenvalue of $\mathcal{A}$, corresponding to the Pareto H-eigenvector $\mathbf{x}$ [24]. More generally, the cone eigenvalue complementarity problem for high-order tensors was also discussed in [15, 16].

In this paper, we consider the Generalized Eigenvalue Complementarity Problem for Tensors $(\text{GEiCP-T})_J$ which has the form

$$(\text{GEiCP-T})_J : \text{Find } \lambda > 0, \mathbf{x} \neq 0 \text{ such that } \begin{cases} \mathbf{w} = (\lambda\mathcal{B} - \mathcal{A})\mathbf{x}^{m-1} \\ \mathbf{w}_{\bar{J}} = 0 \\ \mathbf{w}_J \geq 0 \\ \mathbf{x}_J \geq 0 \\ \mathbf{w}_J^\top \mathbf{x}_J = 0, \end{cases} \tag{1.5}$$

where $\mathcal{A} \in T_{m,n}$ and $\mathcal{B} \in S_{m,n}$ is positive definite. Here, $J \subseteq [n]$ is given and $\bar{J} = [n] \setminus J$. For a vector $\mathbf{x} \in \mathbb{R}^n$, we denote by $\mathbf{x}_J$ the vector in $\mathbb{R}^{|J|}$ such that $\mathbf{x}_J = (x_j) \in \mathbb{R}^{|J|}$ for all $j \in J$, where $|J|$ denotes the cardinality of J. The Eigenvalue Complementarity Problem for Tensors (EiCP-T) is a special case of $(\text{GEiCP-T})_J$ with $J = [n]$. If the index set J is clear in the content, we simply write GEiCP-T, abbreviated to $(\text{GEiCP-T})_J$. The rest of this paper is organized as follows. In Section 2, the general properties of GEiCP-T are studied. We establish its relationship with the generalized tensor eigenvalue problem. We also give some sufficient conditions for the existence of the solution. In Section 3, we consider the symmetric GEiCP-T, i.e., $\mathcal{A} \in S_{m,n}$. By reformulating it as a nonlinear program, we derive a sufficient and necessary condition for the solvability of the symmetric GEiCP-T. In Section 4, we proposed a shifted projected power method to solve the symmetric GEiCP-T. The monotonic convergence is also established. Some numerical experiments are reported in Section 5.

Throughout this paper, we assume that m is even and $\mathcal{B} \in S_{m,n}$ is positive definite. We use small letters $x, y, \ldots$, for scalars, small bold letters $\mathbf{x}, \mathbf{y}, \ldots$, for vectors, capital letters $A, B, \ldots$, for matrices, calligraphic letters $\mathcal{A}, \mathcal{B}, \ldots$, for tensors. All the tensors discussed in this paper are real.

$\boxed{2}$ The Generalized Eigenvalue Complementarity Problem for Tensors

In this section, we concentrate on the properties of generalized eigenvalue complementarity problem for tensors, including the existence of solution. As stated in Introduction, EiCP-T is a special case of GEiCP-T, which can be expressed as

$$\text{(EiCP-T): Find } \lambda > 0, \mathbf{x} \neq 0 \text{ such that } \begin{cases} \mathbf{w} = (\lambda \mathcal{B} - \mathcal{A})\mathbf{x}^{m-1} \\ \mathbf{w} \geq 0 \\ \mathbf{x} \geq 0 \\ \mathbf{w}^\top \mathbf{x} = 0, \end{cases} \tag{2.1}$$

where $\mathcal{A} \in T_{m,n}$ and $\mathcal{B} \in S_{m,n}$ is positive definite. Note that any solution with $\mathbf{w} = 0$ is a generalized tensor eigenpair of $(\mathcal{A}, \mathcal{B})$.

Given an index set $J \subseteq [n]$, if $(\lambda, \mathbf{x})$ is a solution of (1.5), the value $\lambda > 0$ is called a *complementary eigenvalue* of $(\mathcal{A}, \mathcal{B})$ and the vector $\mathbf{x}$ is called a corresponding *complementary eigenvector*. Clearly, for any $\alpha > 0$, $\alpha\mathbf{x}$ is also a complementary eigenvector of $(\mathcal{A}, \mathcal{B})$ associated to λ. Without loss of generality, we may restrict $\|\mathbf{x}\| = 1$ to replace the constraint $\mathbf{x} \neq 0$. In the case of EiCP-T, we use the linear constraint $\|\mathbf{x}\|_1 = \mathbf{e}^\top \mathbf{x} = 1$ since $\mathbf{x} \geq 0$. Here, $\mathbf{e} \in \mathbb{R}^n$ denotes all-ones vector.

The relationship between GEiCP-T and the generalized eigenvalue problem is stated as follows. Recall that given a set $J \subseteq [n]$, the principal sub-tensor of a tensor $\mathcal{A} = (a_{i_1 \cdots i_m}) \in T_{m,n}$, denoted by $\mathcal{A}_J$, is the tensor in $T_{m,|J|}$ such that

$$\mathcal{A}_J = (a_{i_1 \cdots i_m}) \quad \text{for all } i_1, \ldots, i_m \in J.$$

Proposition 2.1. *Suppose that $(\lambda, \mathbf{x})$ is a solution of (GEiCP-T)$_J$ with a given set $J \subseteq [n]$. Then, there exists a set I satisfying $\bar{J} \subseteq I \subseteq [n]$, such that λ is a positive generalized eigenvalue of $(\mathcal{A}_I, \mathcal{B}_I)$ and $\mathbf{x}_I$ is a corresponding eigenvector with $\mathbf{x}_{J \cap I} \geq 0$.*

Proof. Let $I = \{i \in [n] : w_i = 0\}$. Obviously, $\bar{J} \subseteq I \subseteq [n]$ and $\mathbf{w}_{\bar{I}} > 0$. By complementarity, we have $\mathbf{x}_{\bar{I}} = 0$. It follows that $\mathbf{x}_I \neq 0$ and $\mathbf{x}_{J \cap I} \geq 0$. On the other hand, by definition, for any $i \in [n]$,

$$w_i = \sum_{i_2, \ldots, i_m \in [n]} (\lambda \mathcal{B} - \mathcal{A})_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} = \sum_{i_2, \ldots, i_m \in I} (\lambda \mathcal{B} - \mathcal{A})_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}.$$

Hence, $0 = \mathbf{w}_I = (\lambda \mathcal{B}_I - \mathcal{A}_I)\mathbf{x}_I^{m-1}$, i.e., $\mathcal{A}_I \mathbf{x}_I^{m-1} = \lambda \mathcal{B}_I \mathbf{x}_I^{m-1}$. □

It follows that if exist, the number of λ-solutions, i.e., complementarity eigenvalues of GEiCP-T can be bounded. Before that, two important equalities are presented.

Lemma 2.2. *Given a positive integer n, the following equalities hold:*

(a) $\sum_{k=0}^{n} \binom{n}{k} t^k = (1+t)^n, \quad \forall t \neq 0;$

(b) $\sum_{k=0}^{n} k \binom{n}{k} t^k = nt(1+t)^{n-1}, \quad \forall t \neq 0.$

Proof. Denote $\phi(t) = (1+t)^n$. The first conclusion

$$\phi(t) = \sum_{k=0}^{n} \binom{n}{k} t^k$$

can be obtained by the binomial theorem, which implies that $\phi'(t) = n(1+t)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k} t^{k-1}$. By multiplying by t, we obtain the second conclusion. □

Theorem 2.3. *Given an index set $J \subseteq [n]$, the (GEiCP-T)$_J$ has at most*

$$(mn - |J|)m^{|J|-1}(m-1)^{n-1-|J|}$$

distinct λ-solutions. In particular, EiCP-T has at most nm^{n-1} distinct λ-solutions.

Proof. Given a set J, there are $2^{|J|}$ possible subsets I such that $\bar{J} \subseteq I \subseteq [n]$. For each possible I, it has been shown that there are at most $|I|(m-1)^{|I|-1}$ generalized eigenvalues of $(\mathcal{A}_I, \mathcal{B}_I)$ [3]. By Proposition 2.1, the total number of λ-solutions is at most

$$\begin{aligned}
&\sum_{k=0}^{|J|} \binom{|J|}{k} (|\bar{J}| + k)(m-1)^{|\bar{J}|+k-1} \\
= \ & |\bar{J}|(m-1)^{|\bar{J}|-1} \sum_{k=0}^{|J|} \binom{|J|}{k} (m-1)^k + (m-1)^{|\bar{J}|-1} \sum_{k=0}^{|J|} k \binom{|J|}{k} (m-1)^k \\
= \ & |\bar{J}|(m-1)^{|\bar{J}|-1} m^{|J|} + (m-1)^{|\bar{J}|-1} |J|(m-1)m^{|J|-1} \\
= \ & (m|\bar{J}| + m|J| - |J|)m^{|J|-1}(m-1)^{|\bar{J}|-1} \\
= \ & (mn - |J|)m^{|J|-1}(m-1)^{n-1-|J|}.
\end{aligned}$$

The second equality holds by substituting $t = m-1$ in Lemma 2.2.

Since EiCP is a special case of (GEiCP-T)$_J$ by taking $J = [n]$, it follows that EiCP-T has at most nm^{n-1} distinct λ-solutions. □

In the following, we give some sufficient conditions to check the existence of solution for (GEiCP-T)$_J$ with a given set $J \subseteq [n]$.

Proposition 2.4. *Suppose $J \subseteq [n]$. If $\mathcal{A} \in T_{m,n}$ is negative semi-definite (i.e., $\mathcal{A}\mathbf{x}^m \le 0$ for all $\mathbf{x} \in \mathbb{R}^n$), then the corresponding (GEiCP-T)$_J$ is unsolvable.*

Proof. Suppose that $(\lambda, \mathbf{x})$ is a solution of (GEiCP-T)$_J$. By Proposition 2.1, there exists a set I satisfying $\bar{J} \subseteq I \subseteq [n]$ such that $\mathcal{A}_I \mathbf{x}_I^{m-1} = \lambda \mathcal{B} \mathbf{x}_I^{m-1}$. Taking the dot product with $\mathbf{x}_I$, it follows that

$$\lambda = \frac{\mathcal{A}_I \mathbf{x}_I^m}{\mathcal{B}_I \mathbf{x}_I^m}.$$

On the other hand, let $\mathbf{y}$ be the vector whose entries are the same with $\mathbf{x}_I$ in the index set I and 0 otherwise. Since $\mathcal{A}$ is negative semi-definite, we have $\mathcal{A}\mathbf{y}^m = \mathcal{A}_I \mathbf{x}_I^m \le 0$. Similarly, we have $\mathcal{B}\mathbf{y}^m = \mathcal{B}_I \mathbf{x}_I^m > 0$. Thus, $\lambda \le 0$ which contradicts the constraint $\lambda > 0$. □

Now we consider the case $\mathcal{B} = \mathcal{I}$. For any $i \in [n]$, we denote by $\mathbf{e}_i$ the ith unit vector in $\mathbb{R}^n$. Then we have the following proposition which can be easily derived.

Proposition 2.5. *Suppose that $\mathcal{A} = (a_{i_1 \cdots i_m}) \in T_{m,n}$ and $\mathcal{B} = \mathcal{I}$. If there is an index $j \in [n]$ such that $a_{jj\cdots j} > 0$ and $a_{ij\cdots j} \le 0$ for all $i \ne j$, then $(\lambda, \mathbf{x})$ is a solution of EiCP-T with $\lambda = a_{jj\cdots j}$ and $\mathbf{x} = \mathbf{e}_j$.*

In particular, this property holds for the well-known class of nonsingular M-tensors [7] (also called strong M-tensors in [27]).

Recall that a tensor $\mathcal{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$ is called *reducible* if there exists a nonempty proper index subset $I \subseteq [n]$ such that

$$a_{ii_2\cdots i_m} = 0, \qquad \forall i \in I, \quad i_2, \ldots, i_m \in \bar{I}.$$

If $\mathcal{A}$ is not reducible, we call $\mathcal{A}$ *irreducible*. The Perron-Frobenius theorem is a well-known result that describes the spectral radius of a nonnegative matrix. Recently, Perron-Frobenius theorem has been extended to nonnegative tensors, see [2, 25, 26]. It can been stated as follows.

Lemma 2.6. *Suppose $\mathcal{B} = \mathcal{I}$. If $\mathcal{A} \in T_{m,n}$ is irreducible nonnegative, then there exists a pair $(\lambda^*, \mathbf{x}^*)$ satisfying (1.1) such that:*

(a) $\lambda^* > 0$ *is an eigenvalue.*

(b) $\mathbf{x}^* > 0$*, i.e., all components of* $\mathbf{x}^*$ *are positive.*

(c) *If λ is an eigenvalue with nonnegative eigenvector, then $\lambda = \lambda^*$. Moreover, the nonnegative eigenvector is unique up to a multiplicative constant.*

(d) *If λ is an eigenvalue of $\mathcal{A}$, then $|\lambda| \leq \lambda^*$.*

Based on this lemma, we can derive the existence and uniqueness of the solution of EiCP-T for irreducible nonnegative tensors.

Theorem 2.7. *Suppose $\mathcal{B} = \mathcal{I}$. If $\mathcal{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$ is irreducible nonnegative, then EiCP-T has a solution $(\lambda^*, \mathbf{x}^*)$ with $\lambda^* > 0$ and $\mathbf{x}^* > 0$. Moreover, the solution λ^* is unique and $\mathbf{x}^*$ is unique up to a multiplicative constant.*

Proof. Since $\mathcal{A}$ is irreducible nonnegative, by Lemma 2.6, there exists $\lambda^* > 0$ amd $\mathbf{x}^* > 0$ satisfying (1.1), i.e., $\mathcal{A}(\mathbf{x}^*)^{m-1} = \lambda^* \mathcal{I}(\mathbf{x}^*)^{m-1}$. It follows that $(\lambda^*, \mathbf{x}^*)$ is a solution of EiCP-T. Now we prove the uniqueness of the solution of EiCP-T. Suppose that $(\lambda, \mathbf{x})$ is a solution of EiCP-T. Clearly, $\mathbf{x} \neq 0$ and $\mathbf{x} \geq 0$. Let I be the index set defined by

$$I = \{i \in [n] : x_i = 0\}.$$

Then $I \subsetneq [n]$ and $x_i > 0$ for all $i \in \bar{I}$. Moreover, we claim that $I = \emptyset$. Otherwise, I is a nonempty proper index subset of $[n]$. By definition, we have that for any $i \in I$,

$$\begin{aligned} w_i &= \lambda x_i^{m-1} - \sum_{i_2,\ldots,i_m \in [n]} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m} \\ &= -\sum_{i_2,\ldots,i_m \in \bar{I}} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m} \\ &\geq 0. \end{aligned}$$

On the other hand, since $\mathbf{x} \geq 0$ and $\mathcal{A}$ is nonnegative, we have $\sum_{i_2,\ldots,i_m \in \bar{I}} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m} = 0$ for any $i \in I$. Note that $x_i > 0$ for all $i \in \bar{I}$. It follows that

$$a_{ii_2\cdots i_m} = 0, \qquad \forall i \in I, \quad i_2, \ldots, i_m \in \bar{I}.$$

It means that $\mathcal{A}$ is reducible, which is a contradiction. Hence, $I = \emptyset$, i.e., $\mathbf{x} > 0$. By complementary condition of EiCP-T, λ is an eigenvalue with a positive eigenvector $\mathbf{x}$. Again, by Lemma 2.6, the conclusion follows immediately. $\square$

In fact, for any $J \subseteq [n]$, the $(\text{GEiCP-T})_J$ still has a solution $(\lambda^*, \mathbf{x}^*)$ with $\lambda^* > 0$ and $\mathbf{x}_J^* > 0$ for irreducible nonnegative tensors. Before that, we give a useful lemma, which describes the relationship of solutions between different generalized eigenvalue complementarity problems.

Lemma 2.8. *Suppose that $J \subseteq [n]$ and $(\lambda, \mathbf{x})$ is a solution of the $(\text{GEiCP-T})_J$. Let J_0 and J_1 be the subsets of $[n]$ given by*

$$J_0 = \{i \in [n] : \mathbf{w}_i > 0\} \quad \textit{and} \quad J_1 = \{i \in [n] : \mathbf{x}_i \geq 0\},$$

where $\mathbf{w}$ is defined in (1.5). Then $(\lambda, \mathbf{x})$ is a solution of the $(\text{GEiCP-T})_{\hat{J}}$ for any $J_0 \subseteq \hat{J} \subseteq J_1$.

Proof. By definition, $J_0 \subseteq J \subseteq J_1$. And for any index set $\hat{J}$ satisfying $J_0 \subseteq \hat{J} \subseteq J_1$, we have $\mathbf{w}_{\hat{J}} \geq 0$, $\mathbf{x}_{\hat{J}} \geq 0$ and $\mathbf{w}_{\hat{J}}^\top \mathbf{x}_{\hat{J}} = \mathbf{w}_J^\top \mathbf{x}_J = 0$. It follows that $(\lambda, \mathbf{x})$ is a solution of the $(\text{GEiCP-T})_{\hat{J}}$. □

Theorem 2.9. *Suppose $\mathcal{B} = \mathcal{I}$. If $\mathcal{A} = (a_{i_1 \cdots i_m}) \in T_{m,n}$ is irreducible nonnegative, then for any $J \subseteq [n]$, the $(\text{GEiCP-T})_J$ has a solution $(\lambda^*, \mathbf{x}^*)$ with $\lambda^* > 0$ and $\mathbf{x}_J^* > 0$.*

Proof. By Theorem 2.7, EiCP-T has a unique solution $(\lambda^*, \mathbf{x}^*)$ with $\lambda^* > 0$ and $\mathbf{x}^* > 0$. It follows that $\mathbf{w}^* = 0$. By the definition of J_0 and J_1 in Lemma 2.8, we have $J_0 = \emptyset$ and $J_1 = [n]$. According to Lemma 2.8, the conclusion follows immediately. □

Note that for irreducible nonnegative tensors, the uniqueness of the solution of GEiCP-T does not hold in general. For example, let $B \in S_{2,2}$ be the identity matrix and let $A \in T_{2,2}$ be the irreducible nonnegative matrix given by

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

For the index set $J = \{1\}$, it is easy to verify that $\lambda_1 = 3$, $\mathbf{x}_1 = (1, 1)^\top$ and $\lambda_2 = 1$, $\mathbf{x}_2 = (1, -1)^\top$ are the solutions of the $(\text{GEiCP-T})_J$.

3 The symmetric Generalized Eigenvalue Complementarity Problem for Tensors

In this section, we focus on the symmetric GEiCP-T, i.e., $\mathcal{A} \in S_{m,n}$. Its relationship with an optimization problem is established. And we give a sufficient and necessary condition for the solution existence of symmetric GEiCP-T.

Suppose that $(\lambda, \mathbf{x})$ is a solution of the $(\text{GEiCP-T})_J$ with a given $J \subseteq [n]$. Then we have $0 = \mathbf{w}_J^\top \mathbf{x}_J = \mathbf{w}^\top \mathbf{x} = \lambda \mathcal{B}\mathbf{x}^m - \mathcal{A}\mathbf{x}^m$. It follows that

$$\lambda(\mathbf{x}) = \frac{\mathcal{A}\mathbf{x}^m}{\mathcal{B}\mathbf{x}^m}, \tag{3.1}$$

which can be seen as the generalized Rayleigh quotient for tensors.

Theorem 3.1. *Suppose $J \subseteq [n]$. The symmetric $(\text{GEiCP-T})_J$ is equivalent to the following optimization problem*

$$(P)_J \qquad \left\{ \begin{array}{ll} \max & \lambda(\mathbf{x}) \\ s.t. & \mathbf{x}^\top \mathbf{x} = 1 \\ & \mathbf{x}_J \geq 0 \end{array} \right.$$

in the sense that any equilibrium solution $\mathbf{x}$ of $(P)_J$ with $\lambda(\mathbf{x}) > 0$ is a solution of the symmetric $(\text{GEiCP-T})_J$.

Proof. The Lagrangian associated with the problem $(P)_J$ is defined as

$$L(\mathbf{x}, \mu, \mathbf{v}) = \lambda(\mathbf{x}) + \mu(\mathbf{x}^\top \mathbf{x} - 1) + \mathbf{v}^\top \mathbf{x}_J,$$

where $\mu \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^{|J|}$ are the Lagrange multipliers. Without loss of generality, we may assume that J denotes the first $|J|$ indexes in [n], i.e., $J = [|J|]$. Any equilibrium solution of $(P)_J$ satisfies the KKT conditions

$$\begin{cases} \nabla\lambda(\mathbf{x}) + 2\mu\mathbf{x} + \begin{pmatrix} \mathbf{v} \\ 0 \end{pmatrix} = 0, \\ \mathbf{v} \geq 0, \\ \mathbf{v}^\top \mathbf{x}_J = 0, \\ \mathbf{x}_J \geq 0, \\ \mathbf{x}^\top \mathbf{x} = 1, \end{cases} \tag{3.2}$$

where $\nabla\lambda(\mathbf{x})$ is the gradient of $\lambda(\mathbf{x})$. Since $\mathcal{A}, \mathcal{B} \in S_{m,n}$, by simple computation, we have

$$\nabla\lambda(\mathbf{x}) = \frac{m}{\mathcal{B}\mathbf{x}^m}(\mathcal{A}\mathbf{x}^{m-1} - \lambda(\mathbf{x})\mathcal{B}\mathbf{x}^{m-1}), \quad \forall \mathbf{x} \neq 0.$$

According to the definition of $\lambda(\mathbf{x})$, $\mathbf{x}^\top \nabla\lambda(\mathbf{x}) = 0$. By taking the dot product with $\mathbf{x}$ in the first equation, we have $\mu = 0$. By taking $\mathbf{w} = \frac{\mathcal{B}\mathbf{x}^m}{m}\begin{pmatrix} \mathbf{v} \\ 0 \end{pmatrix}$, we have $\mathbf{w} = \lambda(\mathbf{x})\mathcal{B}\mathbf{x}^{m-1} - \mathcal{A}\mathbf{x}^{m-1}$ which satisfies $\mathbf{w}_J \geq 0$ and $\mathbf{w}_{\bar{J}} = 0$. Since $\lambda(\mathbf{x}) > 0$, it follows that $(\lambda(\mathbf{x}), \mathbf{x})$ is a solution of $(\text{GEiCP-T})_J$. The result follows immediately. □

Here, we use $\|\mathbf{x}\|_2^2 = 1$ to normalize the nonzero vector $\mathbf{x}$. In fact, from the proof above, the conclusion still holds if we normalize the nonzero vector $\mathbf{x}$ by $\|\mathbf{x}\|_k^k = 1$ for any $k \geq 1$. For EiCP, i.e., $J = [n]$, it is simple to normalize $\mathbf{x}$ by $\|\mathbf{x}\|_1 = 1$. Notice that $\mathbf{x} \geq 0$. The constraint can be written as $\mathbf{e}^\top \mathbf{x} = 1$. Then, we have the following corollary.

Corollary 3.2. *The symmetric EiCP-T is equivalent to the following optimization problem*

$$(P) \quad \begin{cases} \max & \lambda(\mathbf{x}) \\ s.t. & \mathbf{e}^\top \mathbf{x} = 1 \\ & \mathbf{x} \geq 0 \end{cases}$$

in the sense that any equilibrium solution $\mathbf{x}$ *of (P) with* $\lambda(\mathbf{x}) > 0$ *is a solution of the symmetric EiCP-T.*

From the reformulation of the symmetric GEiCP-T as a nonlinear program, several important conclusions are derived. The following result gives a sufficient and necessary condition for the solvability of the symmetric GEiCP-T.

Theorem 3.3. *Suppose* $J \subseteq [n]$. *The symmetric* $(GEiCP\text{-}T)_J$ *is solvable if and only if there exists a vector* $\mathbf{x} \in \mathbb{R}^n$ *such that* $\mathbf{x}_J \geq 0$ *and* $\mathcal{A}\mathbf{x}^m > 0$.

Proof. Suppose that $(\lambda, \mathbf{x})$ is a solution of the $(\text{GEiCP-T})_J$. Then, $\mathbf{x}_J \geq 0$ and $\mathcal{A}\mathbf{x}^m = \lambda\mathcal{B}\mathbf{x}^m$. Since $\lambda > 0$ and $\mathcal{B}$ is positive definite, we have $\mathcal{A}\mathbf{x}^m > 0$.

On the other hand, suppose that there is a vector $\bar{\mathbf{x}} \in \mathbb{R}^n$ such that $\bar{\mathbf{x}}_J \geq 0$ and $\mathcal{A}\bar{\mathbf{x}}^m > 0$. Without loss of generality, we assume that J denotes the first $|J|$ indexes in $[n]$. Clearly, $\bar{\mathbf{x}} \neq 0$ and $\lambda(\frac{\bar{\mathbf{x}}}{\|\bar{\mathbf{x}}\|_2}) = \lambda(\bar{\mathbf{x}}) > 0$. Denote

$$\Omega = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}_J \geq 0, \ \mathbf{x}^\top \mathbf{x} = 1\}.$$

Since the set Ω is compact and $\lambda(\mathbf{x})$ is continuous on Ω, then there exists a vector $\mathbf{x}^* \in \Omega$ satisfying $\lambda(\mathbf{x}^*) \geq \lambda(\mathbf{x})$ for any $\mathbf{x} \in \Omega$. In particular, $\lambda(\mathbf{x}^*) \geq \lambda(\frac{\bar{\mathbf{x}}}{\|\bar{\mathbf{x}}\|_2}) = \lambda(\bar{\mathbf{x}}) > 0$. Moreover, we claim that $\mathbf{x}^*$ is an equilibrium solution of $(P)_J$, i.e., there exist $\mu \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^{|J|}$ such that the KKT conditions (3.2) hold. For any $j \in [n]$, let's consider the one-variable function

$$g_j(t) = \lambda(\mathbf{x}^* + t\mathbf{e}_j),$$

where $\mathbf{e}_j \in \mathbb{R}^n$ denotes the jth unit vector. Then we have

$$g_j(0) = \begin{cases} \max_{t \geq -x_j^*} g_j(t) & \text{if } j \in J, \\ \max_{t \in \mathbb{R}} g_j(t) & \text{if } j \in [n] \setminus J. \end{cases}$$

Otherwise, there exists $t \geq -x_j^*$ if $j \in J$, or $t \in \mathbb{R}$ if $j \in [n] \setminus J$ such that $g_j(t) > g_j(0)$. Let $\bar{\mathbf{y}} = \frac{\mathbf{y}}{\|\mathbf{y}\|_2}$ where $\mathbf{y} = \mathbf{x}^* + t\mathbf{e}_j$. It follows that $\bar{\mathbf{y}} \in \Omega$ and $\lambda(\bar{\mathbf{y}}) = \lambda(\mathbf{y}) = g_j(t) > g_j(0) = \lambda(\mathbf{x}^*)$, which is a contradiction. As a result, for any $j \in [n]$, a necessary condition is

$$g_j'(0) = (\nabla \lambda(\mathbf{x}^*))_j \begin{cases} \leq 0 & \text{if } j \in J \text{ and } x_j^* = 0, \\ = 0 & \text{otherwise .} \end{cases}$$

Let $\mu = 0$ and $\mathbf{v} = -(\nabla \lambda(\mathbf{x}^*))_J$. It is easy to verify that the KKT conditions (3.2) hold. Hence, by Theorem 3.1, we can see that $(\lambda(\mathbf{x}^*), \mathbf{x}^*)$ is a solution of the symmetric (GEiCP-T)$_J$. $\square$

Based on the theorem above, we have the following two corollaries.

Corollary 3.4. *The symmetric EiCP-T is solvable if and only if there exists a vector* $\mathbf{x} \geq 0$ *such that* $\mathcal{A}\mathbf{x}^m > 0$.

Corollary 3.5. *If the symmetric EiCP-T is solvable, then the symmetric (GEiCP-T)*$_J$ *is also solvable for any given* $J \subseteq [n]$.

It has been shown [11] that most tensor problems are NP-hard. Here, we also show that in general, deciding the solvability of EiCP-T is NP-hard.

Theorem 3.6. *The solvability of EiCP-T is an NP-hard decision problem.*

Proof. Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Note that m is even. Suppose $m = 2k$. Consider the tensor $\mathcal{T} = (t_{i_1 \cdots i_m}) \in T_{m,n}$ defined by

$$t_{i_1 \cdots i_m} = \begin{cases} a_{i_1 i_{k+1}} & \text{if } i_1 = \ldots = i_k \text{ and } i_{k+1} = \ldots = i_m, \\ 0 & \text{otherwise.} \end{cases}$$

In general, $\mathcal{T}$ is not symmetric. Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in S_{m,n}$ be the symmetrization of $\mathcal{T}$, i.e.,

$$a_{i_1 i_2 \cdots i_m} = \frac{1}{n!} \sum_{\sigma \in S_n} t_{i_{\sigma(1)} i_{\sigma(2)} \cdots i_{\sigma(m)}},$$

where S_n denotes the permutation group on $[n]$. Clearly, for any $\mathbf{x} \in \mathbb{R}^n$,

$$\mathcal{A}\mathbf{x}^m = \mathcal{T}\mathbf{x}^m = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i^k x_j^k.$$

Since $\mathcal{A}$ is symmetric, by Corollary 3.4, EiCP-T corresponding to $\mathcal{A}$ is solvable if and only if there exists a vector $\mathbf{x} \geq 0$ such that $\mathcal{A}\mathbf{x}^m > 0$. Hence, deciding the solvability of EiCP-T is at least as difficult as finding a vector $\mathbf{y} \geq 0$ such that $\mathbf{y}^\top A\mathbf{y} > 0$ ($y_i = x_i^k$). It has been proved that the latter problem is NP-hard (see Theorem 2.18 of [17]). The conclusion follows immediately. □

It follows that solving EiCP-T is NP-hard in general. Despite the fact, the solvability of EiCP-T can be answered easily for some structured tensors.

Proposition 3.7. *Suppose $\mathcal{A} = (a_{i_1\cdots i_m}) \in S_{m,n}$. If $\mathcal{A}$ satisfies one of the conditions*

(a) $\exists\, i \in [n]$ *such that* $a_{ii\cdots i} > 0$;

(b) $\mathcal{A} \geq 0$ *and* $\mathcal{A} \neq 0$;

(c) $\mathcal{A}$ *is a nonsingular* M*-tensor;*

(d) $\mathcal{A}$ *is strictly copositive [20], i.e.,* $\mathcal{A}\mathbf{x}^m > 0$ *for all* $\mathbf{x} \in \mathbb{R}^n_+ \setminus \{0\}$,

then there exists $\bar{\mathbf{x}} \geq 0$ *such that* $\mathcal{A}\bar{\mathbf{x}}^m > 0$ *and the corresponding EiCP-T is solvable.*

Proof. (a) Let $\bar{\mathbf{x}} = \mathbf{e}_i$; (b) Let $\bar{\mathbf{x}} = \mathbf{e}$; (c) Since $\mathcal{A}$ is a nonsingular M-tensor, by Theorem 2 of [7], there exists $\bar{\mathbf{x}} \geq 0$ such that $\mathcal{A}\bar{\mathbf{x}}^{m-1} > 0$. It follows that $\mathcal{A}\bar{\mathbf{x}}^m > 0$. (d) Trivial. □

Recall that a tensor $\mathcal{A} \in T_{m,n}$ is called *weakly symmetric* [3] if

$$\nabla(\mathcal{A}\mathbf{x}^m) = m\mathcal{A}\mathbf{x}^{m-1}, \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

And it has also been shown [3] that a symmetric tensor is weakly symmetric, but the converse is not true in general. It is worth mentioning that most of the results in this section can be extended to weakly symmetric tensors since only the derivative information is used here.

4 Shifted Projected Power Method

According to Proposition 2.1, the relationship between GEiCP-T and the generalized eigenvalue problem is established. Based on this, the solution set GEiCP-T can be obtained via complete enumeration.

Algorithm 1. A procedure to compute all the complementary eigenpairs of $(\text{GEiCP-T})_J$
Input: Given a subset $J \subseteq [n]$, $\mathcal{A} \in T_{m,n}$ and a positive definite tensor $\mathcal{B} \in S_{m,n}$.
Output: The solution set S.
Step 1. Find all the subsets I such that $\bar{J} \subseteq I \subseteq [n]$.
Step 2. Compute all the generalized eigenpairs $(\lambda, \mathbf{x}_I)$ of $(\mathcal{A}_I, \mathcal{B}_I)$ for each I, where $\mathbf{x}_I$ is normalized by $\|\mathbf{x}_I\| = 1$. Denote the set by $\sigma(I)$.
Step 3. For any $(\lambda, \mathbf{x}_I) \in \sigma(I)$, let $\mathbf{x} = (\mathbf{x}_I, 0)^\top \in \mathbb{R}^n$. Check

$$\begin{cases} \lambda > 0, \\ \mathbf{x}_{I\cap J} \geq 0, \\ \lambda\mathcal{B}\mathbf{x}^{m-1} - \mathcal{A}\mathbf{x}^{m-1} \geq 0. \end{cases}$$

If all the conditions hold, then $(\lambda, \mathbf{x}) \in S$.

Here, we may assume that the solution set $\sigma(I)$ is finite. For example, when $\mathcal{A}$ is symmetric, it has been shown [4] that $\sigma(I)$ is finite due to the fact that the corresponding polynomial optimization problem has finitely many critical values. On the the other hand, we can see that the Step 2 in Algorithm 1 is exactly a generalized eigenvalue problem. Recently, several methods have been proposed to solve this problem (or partially). For example, Ng et al. [18] proposed an iterative method for finding the largest eigenvalue of nonnegative tensors. Kolda et al. [12] presented a shifted power method for computing Z-eigenvalues, and they also generalized this method to compute generalized eigenpairs by choosing the shift adaptively [13]. Hao et al. [10] presented a sequential subspace projection method for computing extreme Z-eigenvalues. Han [9] introduced an unconstrained optimization method for even order symmetric tensors. By using the Jacobian SDP relaxations in polynomial optimization, Cui et al. [4] proposed an method for computing all the real eigenvalues of symmetric tensors.

However, Algorithm 1 may do not work in practice. The computational cost can be very high, especially when the scale of the problem becomes large. And it is not necessary to compute all the solutions of the $(\text{GEiCP-T})_J$. In fact, it provides a way to decide the solvability of the $(\text{GEiCP-T})_J$, i.e., the $(\text{GEiCP-T})_J$ is unsolvable if the solution set S is empty. As stated in Theorem 3.6, it is an NP-hard problem in general.

Motivated by the work of Kolda and Mayo [12,13], we propose a shifted projected power method for finding a solution of the symmetric GEiCP-T. Here, we assume that the GEiCP-T given is solvable. Instead of using the gradient directly, we do a projection of the gradient to make the new iterative point feasible. And it will be shown that the monotonic convergence is also guaranteed.

Given a set $J \subseteq [n]$, denote $\Omega = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}_J \geq 0,\ \mathbf{x}^\top \mathbf{x} = 1\}$. Clearly, Ω is compact. Consider the following nonlinear program

$$\max f(\mathbf{x}) \qquad \text{subject to } \mathbf{x} \in \Omega,$$

where $f(\mathbf{x})$ is a given nonlinear function.

Theorem 4.1. *Suppose that $\mathbf{w} \in \Omega$ and $f(\mathbf{x})$ is a given function. Let $\Sigma(\mathbf{w})$ be the open neighborhood of $\mathbf{w}$ such that $f(\mathbf{x})$ is convex and continuously differentiable on it. Let $d(\mathbf{w}) \in \mathbb{R}^n$ be given by*

$$d(\mathbf{w})_i = \begin{cases} 0 & \text{if } i \in J \text{ and } \nabla f(\mathbf{w})_i < 0, \\ \nabla f(\mathbf{w})_i & \text{otherwise.} \end{cases} \tag{4.1}$$

Assume $\|d(\mathbf{w})\| \neq 0$. Define $\mathbf{v} = \frac{d(\mathbf{w})}{\|d(\mathbf{w})\|}$. If $\mathbf{v} \in \Sigma(\mathbf{w})$ and $\mathbf{v} \neq \mathbf{w}$, then $f(\mathbf{v}) - f(\mathbf{w}) > 0$.

Proof. Clearly, $\mathbf{v} \in \Omega$. By definition, we have $\nabla f(\mathbf{w})^\top d(\mathbf{w}) = \|d(\mathbf{w})\|^2$. Note that $\mathbf{w}_J \geq 0$. It follows that $\nabla f(\mathbf{w})^\top \mathbf{w} \leq d(\mathbf{w})^\top \mathbf{w}$. Then, by Cauchy-Schwarz inequality, we get $\nabla f(\mathbf{w})^\top (\mathbf{v} - \mathbf{w}) \geq \|d(\mathbf{w})\| - d(\mathbf{w})^\top \mathbf{w} > 0$ since $\mathbf{w} \in \Omega$ and $\mathbf{v} \neq \mathbf{w}$. On the other hand, since $f(\mathbf{x})$ is convex on $\Sigma(\mathbf{w})$, we have

$$f(\mathbf{v}) - f(\mathbf{w}) \geq \nabla f(\mathbf{w})^\top (\mathbf{v} - \mathbf{w}).$$

Consequently, $f(\mathbf{v}) - f(\mathbf{w}) > 0$. □

We can see that $d(\mathbf{w})$ is a ascent direction since $\nabla f(\mathbf{w})^\top d(\mathbf{w}) = \|d(\mathbf{w})\|^2 \geq 0$. In fact, by the definition of $d(\mathbf{w})$, it can be seen as the projection of the gradient $\nabla f(\mathbf{w})$ on the set $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}_J \geq 0\}$. According to Theorem 4.1, the simple iterative algorithm, i.e.,

$$\mathbf{x}^+ = d(\mathbf{x}) / \|d(\mathbf{x})\|,$$

will monotonically ascent if $f(\mathbf{x})$ is local convex. Unfortunately, the objective function $f(\mathbf{x})$ in $(P)_J$ is $\lambda(\mathbf{x})$, defined as (3.1). It is not convex in general.

To fix this, we consider an equivalent optimization problem of $(P)_J$, i.e.,

$$\max \hat{f}(\mathbf{x}) = \lambda(\mathbf{x}) + \alpha\|\mathbf{x}\|^m \qquad \text{subject to } \mathbf{x} \in \Omega,$$

where α is a shifted factor which makes the objective function $\hat{f}(\mathbf{x})$ locally convex. Denote by $H(\mathbf{x})$ the Hessian of $\lambda(\mathbf{x})$. By simple computation, we have

$$\nabla\lambda(\mathbf{x}) = \frac{m}{\mathcal{B}\mathbf{x}^m}[\mathcal{A}\mathbf{x}^{m-1} - \lambda(\mathbf{x})\mathcal{B}\mathbf{x}^{m-1}] \tag{4.2}$$

and

$$\begin{aligned} H(\mathbf{x}) = \frac{m(m-1)}{\mathcal{B}\mathbf{x}^m}\mathcal{A}\mathbf{x}^{m-2} &- \frac{m^2}{(\mathcal{B}\mathbf{x}^m)^2}\mathcal{A}\mathbf{x}^{m-1}(\mathcal{B}\mathbf{x}^{m-1})^\top - \frac{m(m-1)}{(\mathcal{B}\mathbf{x}^m)^2}\mathcal{A}\mathbf{x}^m\mathcal{B}\mathbf{x}^{m-2} \\ &- \frac{m^2}{(\mathcal{B}\mathbf{x}^m)^3}\mathcal{B}\mathbf{x}^{m-1}(\mathcal{B}\mathbf{x}^m\mathcal{A}\mathbf{x}^{m-1} - 2\mathcal{A}\mathbf{x}^m\mathcal{B}\mathbf{x}^{m-1})^\top. \end{aligned} \tag{4.3}$$

Here, for a tensor $\mathcal{A} \in T_{m,n}$ and a vector $\mathbf{x} \in \mathbb{R}^n$, $\mathcal{A}\mathbf{x}^{m-2}$ is a matrix in $\mathbb{R}^{n\times n}$ whose (i,j)-th component is defined by

$$(\mathcal{A}\mathbf{x}^{m-2})_{ij} = \sum_{i_3,\ldots,i_m=1}^{n} a_{iji_3\cdots i_m}x_{i_3}\cdots x_{i_m}.$$

It follows that for $\mathbf{x} \in \Omega$,

$$\nabla\hat{f}(\mathbf{x}) = \nabla\lambda(\mathbf{x}) + \alpha m\mathbf{x} \tag{4.4}$$

and the corresponding Hessian of $\hat{f}$ is given by

$$\hat{H}(\mathbf{x}) = H(\mathbf{x}) + \alpha mI + \alpha m(m-2)\mathbf{x}\mathbf{x}^\top. \tag{4.5}$$

Our goal is choose the shifted factor α to make $\hat{f}(\mathbf{x})$ locally convex, i.e., $\hat{H}(\mathbf{x})$ is positive semi-definite. The following result gives a way to choose the shifted factor adaptively. Denote by $\lambda_{\min}(H)$ the minimal eigenvalue of the matrix H.

Lemma 4.2 (Corollary 4.4 of [13])**.** *Assume* $\mathbf{x} \in \Omega$. *Let* $\tau > 0$. *If*

$$\alpha = \max\{0, (\tau - \lambda_{\min}(H))/m\},$$

the $\lambda_{\min}(\hat{H}) > \tau$.

Then, given $\mathbf{x} \in \Omega$, a shifted factor can be chosen to make $\hat{f}(\mathbf{x})$ locally convex according to Lemma 4.2. It follows that

$$\mathbf{x}^+ = \frac{\hat{d}(\mathbf{x})}{\|\hat{d}(\mathbf{x})\|} \in \Sigma(\mathbf{x}) \;\Rightarrow\; \hat{f}(\mathbf{x}^+) - \hat{f}(\mathbf{x}) > 0 \;\Rightarrow\; \lambda(\mathbf{x}^+) - \lambda(\mathbf{x}) > 0,$$

where $\hat{d}(\mathbf{x})$ is defined as (4.1) for $\nabla\hat{f}(\mathbf{x})$. Though the shifted factor changes adaptively, we can see that the original function increases after each iteration. In particular, the shifted projected power method can be described as follows.

Algorithm 2. Shifted projected power method for solving the symmetric $(\text{GEiCP-T})_J$
Step 0. Given $J \subseteq [n]$, $\tau > 0$ and let $\mathbf{x}^0 \in \Omega$ be the vector so that $\mathcal{A}(\mathbf{x}^0)^m > 0$. Set $k = 0$.

Step 1. Compute the gradient $\nabla\lambda(\mathbf{x}^k)$ and the Hessian $H(\mathbf{x}^k)$ by (4.2) and (4.3), respectively. Let $\alpha_k = \max\{0, (\tau - \lambda_{\min}(H(\mathbf{x}^k)))/m\}$, $\nabla\hat{f}(\mathbf{x}^k) = \nabla\lambda(\mathbf{x}^k) + \alpha_k m\mathbf{x}^k$ and $\hat{d}(\mathbf{x}^k) \in \mathbb{R}^n$ be given by

$$\hat{d}(\mathbf{x}^k)_i = \begin{cases} 0 & \text{if } i \in J \text{ and } \nabla\hat{f}(\mathbf{x}^k)_i < 0, \\ \nabla\hat{f}(\mathbf{x}^k)_i & \text{otherwise.} \end{cases} \tag{4.6}$$

Step 2. If $\|\hat{d}(\mathbf{x}^k)\| = 0$, stop. Otherwise,

$$\mathbf{x}^{k+1} = \frac{\hat{d}(\mathbf{x}^k)}{\|\hat{d}(\mathbf{x}^k)\|}.$$

Step 3. Set $k = k + 1$ and go back to Step 1.

The convergence of this method is also established as follows.

Theorem 4.3. *If there is a vector $\mathbf{x} \in \Omega$ such that $\hat{d}(\mathbf{x}) = 0$, then $(\lambda(\mathbf{x}), \mathbf{x})$ is a solution of the symmetric (GEiCP-T)$_J$ with a given index set $J \subseteq [n]$.*

Proof. According to (4.6), we have $\nabla\hat{f}(\mathbf{x})_{\bar{J}} = 0$ and $\nabla\hat{f}(\mathbf{x})_J \leq 0$ since $\hat{d}(\mathbf{x}) = 0$. Clearly, $\mathbf{x}_J \geq 0$ since $\mathbf{x} \in \Omega$. It follows that $\mathbf{x}^\top\nabla\hat{f}(\mathbf{x}) = \mathbf{x}_J^\top\nabla\hat{f}(\mathbf{x})_J \leq 0$. On the other hand, $\nabla\hat{f}(\mathbf{x}) = \nabla\lambda(\mathbf{x}) + \alpha m\mathbf{x}$. Note that $\|\mathbf{x}\| = 1$ and $\mathbf{x}^\top\nabla\lambda(\mathbf{x}) = \frac{m}{\mathcal{B}\mathbf{x}^m}\mathbf{x}^\top[\mathcal{A}\mathbf{x}^{m-1} - \lambda(\mathbf{x})\mathcal{B}\mathbf{x}^{m-1}] = 0$. Thus, $\mathbf{x}^\top\nabla\hat{f}(\mathbf{x}) = \alpha m \geq 0$ since $\alpha \geq 0$. As a result, $\mathbf{x}^\top\nabla\hat{f}(\mathbf{x}) = \alpha m = 0$. It implies that $\alpha = 0$. Recall that $\mathbf{w} = \lambda(\mathbf{x})\mathcal{B}\mathbf{x}^{m-1} - \mathcal{A}\mathbf{x}^{m-1}$ for the symmetric (GEiCP-T)$_J$. Then, we have $\mathbf{x}_J \geq 0$, $\mathbf{w}_{\bar{J}} = 0$ and $\mathbf{w}_J \geq 0$. Moreover, $\mathbf{w}_J^\top\mathbf{x}_J = \mathbf{w}^\top\mathbf{x} = 0$. This exactly means that $(\lambda(\mathbf{x}), \mathbf{x})$ is a solution of the symmetric (GEiCP-T)$_J$. The proof is completed. □

Theorem 4.4. *Suppose that $\hat{d}(\mathbf{x}^k) \neq 0$ for all $k \geq 0$ in Algorithm 2. Let $\{\mathbf{x}^k\}_{k=0}^\infty$ be the corresponding generated sequence. Then, $\mathbf{x}^k \in \Omega$ for any $k \geq 0$. Furthermore, if $\mathbf{x}^{k+1} \in \Sigma(\mathbf{x}^k)$ for all $k \geq 0$, then the sequence $\{\lambda(\mathbf{x}^k)\}_{k=0}^\infty$ increases monotonically, and converges to a λ-solution of the symmetric (GEiCP-T)$_J$ with a given index set $J \subseteq [n]$.*

Proof. Based on the analysis above, it is obvious that $\mathbf{x}^k \in \Omega$ for any $k \geq 0$, and $\lambda(\mathbf{x}^{k+1}) \geq \lambda(\mathbf{x}^k)$ if $\mathbf{x}^{k+1} \in \Sigma(\mathbf{x}^k)$. Note that Ω is compact and $\lambda(\mathbf{x})$ is continuous. Then $\lambda(\mathbf{x})$ is bounded on Ω. Hence, the sequence $\{\lambda(\mathbf{x}^k)\}$ converges. Let $\lambda^* = \lim_{k\to\infty}\lambda(\mathbf{x}^k)$. It follows that λ^* is a local maximal value of the optimization problem $(P)_J$. By Theorem 3.1, λ^* is a λ-solution of the symmetric (GEiCP-T)$_J$. □

Note that $\mathbf{x}^k$ need not converge to optimal point, although any accumulation point of the sequence $\{\mathbf{x}^k\}$ is a local maximal point of the optimization problem $(P)_J$. Here, we assume that $\mathbf{x}^+ \in \Sigma(\mathbf{x})$ for each iteration. It means that $\mathbf{x}^+$ should be located in the open neighborhood of $\mathbf{x}$, on which $\hat{f}(\mathbf{x})$ is convex. If the condition does not hold, we may see decrease in the origin function. As explained in [13], one strategy is to choose a sufficiently large α until the condition is satisfied. However, from the numerical experiments in the next section, it is not necessary to worry about the unexpected case. Thus, we do not include this technique in the implementation of the algorithm.

5 Numerical Experiments

In this section, the numerical performance of the shifted projected power method is presented. All codes were written by using Matlab Version R2012b and the Tensor Toolbox

Version 2.6 [1]. And the numerical experiments were done on a laptop with an Intel Core i5-2430M CPU (2.4GHz) and RAM of 5.58GB.

In the implementation of Algorithm 2, we set the parameter $\tau = 10^{-6}$, where τ is the tolerance on being positive definite. We consider the iterates to be converged once $|\lambda(\mathbf{x}^{k+1}) - \lambda(\mathbf{x}^k)| \leq 10^{-10}$. The maximum iterations is 1000. For simplicity, the positive definite tensor $\mathcal{B} \in S_{m,n}$ is chosen such that $\mathcal{B}\mathbf{x}^{m-1} = \mathbf{x}$ for all $\mathbf{x}^\top \mathbf{x} = 1$, where m is even [13]. To solve the minimal eigenvalue of the matrix in Step 1, we use the **eig** function built in Matlab to compute all the eigenvalues and select the minimal one. In some ways, the speed of our method can be accelerated if a better algorithm is used to find the minimal eigenvalue of a given matrix.

First, we use a randomly generated example to show the performance of Algorithm 2. In the example, the tensor $\mathcal{A} \in S_{6,4}$ is generated randomly as follows: we select random entries from $[-1, 1]$, symmetrize the result, and round to four decimal places. To make the $(\text{GEiCP-T})_J$ solvable, we reset its first entry by $a_{111111} = 0.5$. It follows that $\mathcal{A}\mathbf{e}_1^6 = a_{111111} > 0$. By Theorem 3.3, the corresponding $(\text{GEiCP-T})_J$ is solvable if $1 \in J$. Hence, we execute our experiment under all possible cases, i.e.,

$$J = \{1\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{1,2,3,4\}.$$

For all cases, the initial point is chosen by $\mathbf{x}^0 = \mathbf{e}_1$. The tensor $\mathcal{A}$ is specified in Table 1. The numerical results are reported in Table 2. To show the monotonic convergence of the shifted projective power method, the courses of iteration are presented in Figure 1.

In Table 2, *Ite.* denotes the iteration number of Algorithm 2 and $(\lambda^*, \mathbf{x}^*)$ denote the solution derived at the final iteration. And $\mathbf{w}^*$ is computed by $\mathbf{w}^* = \mathcal{B}(\mathbf{x}^*)^{m-1} - \lambda^* \mathcal{A}(\mathbf{x}^*)^{m-1}$. We can see that for all cases, the shifted projected power method can find a solution of the corresponding $(\text{GEiCP-T})_J$ successfully, i.e.,

$$\lambda^* > 0, \quad \mathbf{x}_J^* \geq 0, \quad \mathbf{w}_J^* \geq 0, \quad \mathbf{w}_{\bar{J}}^* = 0 \quad \text{and} \quad (\mathbf{w}_J^*)^\top \mathbf{x}_J^* = 0.$$

Notice that there are four typical solutions for the cases $J = \{1\}, \{1,2\}, \{1,3,4\}, \{1,2,3,4\}$. Any other solution is the same with one of these typical solutions. This phenomenon can be well explained by Lemma 2.8. For instance, if $J = \{1,3,4\}$, $(\lambda^*, \mathbf{x}^*)$ is a solution of the $(\text{GEiCP-T})_J$ with $\lambda^* = 1.2894$ and $\mathbf{x}^* = (0.8801, -0.2669, 0.3927, 0)^\top$. It follows that $\mathbf{w}^* = (0, 0, 0, 0.4676)^\top$. By definition, we have $J_0 = \{4\}$ and $J_1 = \{1,3,4\}$. According to Lemma 2.8, since $J_0 \subseteq \{1,4\} \subseteq J_1$, $(\lambda^*, \mathbf{x}^*)$ is still a solution of the $(\text{GEiCP-T})_{\hat{J}}$ with $\hat{J} = \{1,4\}$.

For these typical cases, the courses of iteration are shown in Figure 1. We can see that the generated sequence $\{\lambda(\mathbf{x}^k)\}$ is monotonically increasing, and converges to a local maximal value quickly. It is also shown that the new iteration point $\mathbf{x}^+$ is always located in $\Sigma(\mathbf{x})$, on which $\hat{f}(\mathbf{x})$ is convex.

Second, we test our method by randomly generated tensors. Given order m and dimension n, the tensor $\mathcal{A} \in S_{m,n}$ is randomly generated as before, i.e., we select random entries from $[-1, 1]$ and symmetrize the result. To make the $(\text{GEiCP-T})_J$ solvable, we reset its first entry by $a_{111111} = 0.5$. The initial point is chosen by $\mathbf{x}^0 = \mathbf{e}_1$. And the (EiCP-T) is considered i.e., $J = [n]$. For each case, ten symmetric tensors are generated and the average time, the average λ-solution and the average number of iteration are recorded, respectively. The results are reported in Table 3.

From Table 3, we can see that our method is very efficient for randomly generated tensors. For each case, one can find a λ-solution within 500 iterations. Note that the tensor may be out of the memory of our laptop when its scale becomes large.

Table 1: A randomly generated tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_6}) \in S_{6,4}$

$a_{111111} = 0.5000$,	$a_{111112} = -0.2369$,	$a_{111113} = 0.1953$,	$a_{111114} = -0.2691$,
$a_{111122} = 0.0835$,	$a_{111123} = -0.2016$,	$a_{111124} = -0.0441$,	$a_{111133} = 0.0567$,
$a_{111134} = -0.2784$,	$a_{111144} = 0.2321$,	$a_{111222} = -0.1250$,	$a_{111223} = 0.0333$,
$a_{111224} = 0.0235$,	$a_{111233} = 0.0093$,	$a_{111234} = -0.0304$,	$a_{111244} = -0.0167$,
$a_{111333} = 0.1028$,	$a_{111334} = -0.0385$,	$a_{111344} = 0.0068$,	$a_{111444} = 0.1627$,
$a_{112222} = -0.1002$,	$a_{112223} = 0.0733$,	$a_{112224} = 0.0607$,	$a_{112233} = -0.1125$,
$a_{112234} = 0.0096$,	$a_{112244} = -0.0810$,	$a_{112333} = -0.0299$,	$a_{112334} = 0.0153$,
$a_{112344} = 0.0572$,	$a_{112444} = 0.0251$,	$a_{113333} = 0.1927$,	$a_{113334} = -0.1024$,
$a_{113344} = -0.0885$,	$a_{113444} = 0.0289$,	$a_{114444} = -0.0668$,	$a_{122222} = -0.2707$,
$a_{122223} = -0.1066$,	$a_{122224} = -0.1592$,	$a_{122233} = 0.0805$,	$a_{122234} = -0.0540$,
$a_{122244} = -0.0434$,	$a_{122333} = -0.0048$,	$a_{122334} = -0.0118$,	$a_{122344} = 0.0196$,
$a_{122444} = -0.0585$,	$a_{123333} = -0.0442$,	$a_{123334} = -0.0618$,	$a_{123344} = 0.0318$,
$a_{123444} = 0.0332$,	$a_{124444} = -0.2490$,	$a_{133333} = 0.1291$,	$a_{133334} = 0.0704$,
$a_{133344} = -0.0032$,	$a_{133444} = 0.0270$,	$a_{134444} = 0.0232$,	$a_{144444} = -0.3403$,
$a_{222222} = -0.6637$,	$a_{222223} = 0.2191$,	$a_{222224} = 0.3280$,	$a_{222233} = 0.1834$,
$a_{222234} = 0.0627$,	$a_{222244} = 0.0860$,	$a_{222333} = 0.1590$,	$a_{222334} = -0.0217$,
$a_{222344} = 0.1198$,	$a_{222444} = -0.1674$,	$a_{223333} = 0.0549$,	$a_{223334} = -0.0868$,
$a_{223344} = 0.0043$,	$a_{223444} = 0.0101$,	$a_{224444} = -0.0307$,	$a_{233333} = -0.3553$,
$a_{233334} = 0.0207$,	$a_{233344} = 0.1544$,	$a_{233444} = -0.1707$,	$a_{234444} = -0.3557$,
$a_{244444} = -0.1706$,	$a_{333333} = 0.7354$,	$a_{333334} = -0.3628$,	$a_{333344} = -0.2650$,
$a_{333444} = -0.0479$,	$a_{334444} = -0.0084$,	$a_{344444} = -0.0559$,	$a_{444444} = 0.6136$.

Table 2: Numerical results for the tensor $\mathcal{A}$ given in Table 1

J	Ite.	Time(s)	λ^*	$\mathbf{x}^*$	$\mathbf{w}^*$
$\{1\}$	38	0.5636	1.7230	$(0.8646, -0.1272, 0.4080, -0.2642)^\top$	$(0, 0, 0, 0)^\top$
$\{1, 2\}$	41	0.6048	1.6381	$(0.8513, 0, 0.4315, -0.2985)^\top$	$(0, 0.2180, 0, 0)^\top$
$\{1, 3\}$	38	0.5662	1.7230	$(0.8646, -0.1272, 0.4080, -0.2642)^\top$	$(0, 0, 0, 0)^\top$
$\{1, 4\}$	101	1.4926	1.2894	$(0.8801, -0.2669, 0.3927, 0)^\top$	$(0, 0, 0, 0.4676)^\top$
$\{1, 2, 3\}$	41	0.6066	1.6381	$(0.8513, 0, 0.4315, -0.2985)^\top$	$(0, 0.2180, 0, 0)^\top$
$\{1, 2, 4\}$	52	0.7673	1.1666	$(0.5781, 0, 0.8160, 0)^\top$	$(0, 0.3347, 0, 0.4207)^\top$
$\{1, 3, 4\}$	101	1.4949	1.2894	$(0.8801, -0.2669, 0.3927, 0)^\top$	$(0, 0, 0, 0.4676)^\top$
$\{1, 2, 3, 4\}$	52	0.7684	1.1666	$(0.5781, 0, 0.8160, 0)^\top$	$(0, 0.3347, 0, 0.4207)^\top$

6 Conclusion

In this paper, the generalized eigenvalue complementarity problem for tensors (GEiCP-T) is considered, which arises from the stability analysis of finite dimensional mechanical systems. In theory, we mainly discuss the existence of the solution of GEiCP-T. It is shown that if exist, the number of λ-solutions is finite. And there always exists a unique solution of EiCP-T (i.e. $J = [n]$) for irreducible nonnegative tensors. For the symmetric case, GEiCP-T is solvable if and only if there exists a feasible point of the corresponding nonlinear program such that the objective function value is positive. It has also been proved that deciding the solvability of EiCP-T is NP-hard in general.

In the aspect of algorithm, a shifted projected power method is proposed to solve the symmetric GEiCP-T. The monotonic convergence of the algorithm is established. And the

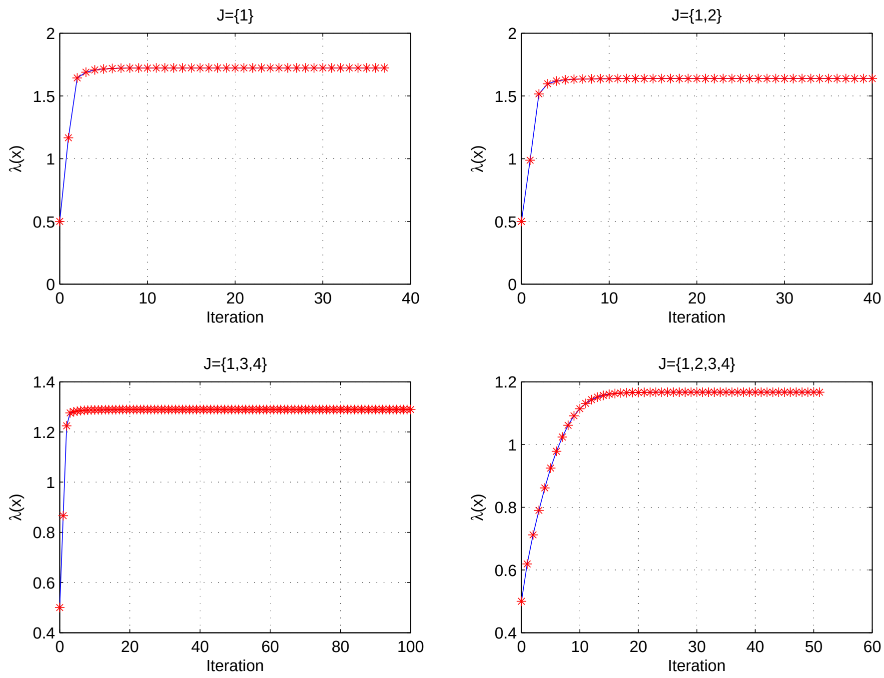

Figure 1: The course of iteration for the tensor $\mathcal{A}$ given in Table 1

Table 3: Numerical results for randomly generated tensors

m	n	Ite.	Time(s)	λ^*
4	5	50.0	0.3839	1.2143
4	10	65.7	0.5832	1.9260
4	15	116.4	1.1597	2.5069
4	20	162.8	2.3462	2.7829
4	25	187.7	4.3628	3.3754
4	30	195.1	9.9900	3.8642
4	35	225.6	21.3616	4.2010
4	40	278.7	44.2750	4.4314
4	45	432.7	91.8715	4.7232
6	4	23.8	0.3576	1.2637
6	5	38.6	0.6745	1.0581
6	6	40.5	0.8180	1.1849
6	7	51.6	1.0999	1.4305
6	8	82.3	2.2646	1.9459
6	9	95.7	3.8606	2.0624
8	4	29.3	0.8645	0.7863

numerical experiments show that the algorithm presented is efficient and promising.

References

[1] B.W. Bader, T.G. Kolda, et al., MATLAB Tensor Toolbox Version 2.6, February 2015, http://www.sandia.gov/ tgkolda/TensorToolbox/.

[2] K.C. Chang, K. Pearson, and T. Zhang, Perron-Frobenius theorem for nonnegative tensors, *Commun. Math. Sci.* 6 (2008) 507–520.

[3] K.C. Chang, K. Pearson, and T. Zhang, On eigenvalue problems of real symmetric tensors, *J. Math. Anal. Appl.* 350 (2009) 416–422.

[4] C. Cui, Y.-H. Dai, and J. Nie, All real eigenvalues of symmetric tensors, *SIAM J. Matrix Anal. Appl.* 35 (2014) 1582–1601.

[5] A. da Costa, I. Figueiredo, J. Júdice, and J. Martins, A complementarity eigenproblem in the stability analysis of finite dimensional elastic systems with frictional contact, in: *Complementarity: Applications, Algorithms and Extensions*, M. Ferris, J.S. Pang and O. Mangasarian (eds.), Kluwer, New York, 2001, pp. 67–83.

[6] A. da Costa, J. Martins, I. Figueiredo, and J. Júdice, The directional instability problem in systems with frictional contacts, *Comput. Methods Appl. Mech. Engrg.* 193 (2004) 357–384.

[7] W. Ding, L. Qi, and Y. Wei, M-tensors and nonsingular M-tensors, *Linear Algebra Appl.* 439 (2013) 3264–3278.

[8] W. Ding and Y. Wei, Generalized tensor eigenvalue problems, *SIAM J. Matrix Anal. Appl.* 36 (2015) 1073–1099.

[9] L. Han, An unconstrained optimization approach for finding real eigenvalues of even order symmetric tensors, *Numer. Algebra Control Optim.* 3 (2012) 583–599.

[10] C. Hao, C. Cui, and Y.-H. Dai. A sequential subspace projection method for extreme Z-eigenvalues of supersymmetric tensors, *Numer. Linear Algebra Appl.* 22 (2015) 283–298.

[11] C.J. Hillar and L.–H. Lim, Most tensor problems are NP–hard, *J. Amer. Math. Soc.* 60 (2013) 45.

[12] T.G. Kolda and J.R. Mayo, Shifted power method for computing tensor eigenpairs, *SIAM J. Matrix Anal. Appl.* 32 (2011) 1095–1124.

[13] T.G. Kolda and J.R. Mayo, An adaptive shifted power method for computing generalized tensor eigenpairs, *SIAM J. Matrix Anal. Appl.* 35 (2014) 1563–1581.

[14] L.-H. Lim, Singular values and eigenvalues of tensors: a variational approach, in: *Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Addaptive Processing*, CAMSAP05, IEEE Computer Society Press, Piscataway, 2005, pp. 129–132.

[15] C. Ling, H. He, and L. Qi, On the cone eigenvalue complementarity problem for higher-order tensors, *Comput. Optim. Appl.* 63 (2016) 143–168.

[16] C. Ling, H. He, and L. Qi, Higher-degree eigenvalue complementarity problems for tensors, *Comput. Optim. Appl.* 64 (2016) 149–176.

[17] K.G. Murty, *Linear complementarity, linear and nonlinear programming*, Heldermann Verlag, Berlin, 1988.

[18] M. Ng, L. Qi, and G. Zhou, Finding the largest eigenvalue of a nonnegative tensor, *SIAM J. Matrix Anal. Appl.* 31 (2009) 1090–1099.

[19] L. Qi, Eigenvalues of a real supersymmetric tensor, *J. Symbolic Comput.* 40 (2005) 1302–1324.

[20] L. Qi, Symmetric nonnegative tensors and copositive tensors, *Linear Algebra Appl.* 439 (2013) 228–238.

[21] L. Qi, Y. Wang, and E.X. Wu, D-eigenvalues of diffusion kurtosis tensors, *J. Comput. Appl. Math.* 221 (2008) 150–157.

[22] M. Queiroz, J.J. Júdice, and Jr.C. Humes, The symmetric eigenvalue complementarity problem, *Math. Comp.* 73 (2004) 1849–1863.

[23] A. Seeger, Eigenvalue analysis of equilibrium processes defined by linear complementarity conditions, *Linear Algebra Appl.* 292 (1999) 1–14.

[24] Y. Song and L. Qi, Eigenvalue analysis of constrained minimization problem for homogeneous polynomials, *J. Global Optim.* 64 (2016) 563–575.

[25] Q. Yang and Y. Yang, Further results for Perron-Frobenius theorem for nonnegative tensors II, *SIAM J. Matrix Anal. Appl.* 32 (2011) 1236–1250.

[26] Y. Yang and Q. Yang, Further results for Perron-Frobenius theorem for nonnegative tensors, *SIAM J. Matrix Anal. Appl.* 31 (2010) 2517–2530.

[27] L. Zhang, L. Qi, and G. Zhou, M-tensors and and some applications, *SIAM J. Matrix Anal. Appl.* 35 (2014) 437–452.

Manuscript received 15 September 2015
revised 7 December 2015
accepted for publication 12 December 2015

Zhongming Chen
Department of Mathematics
School of Science Hangzhou Dianzi University
Hangzhou 310018, P.R. China
E-mail address: czm183015@126.com

Qingzhi Yang
School of Mathematical Sciences and LPMC
Nankai University, Tianjin 300071, P.R. China
E-mail address: qz-yang@nankai.edu.cn

Lu Ye
College of Economics and Management and
Zhejiang Provincial Research Center for Ecological Civilization
Zhejiang Sci-Tech University, Hangzhou 310018, P.R. China
E-mail address: zjwzajyl@126.com