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A NEW PARAMETRIC KERNEL FUNCTION YIELDING THE BEST KNOWN ITERATION BOUNDS OF INTERIOR-POINT METHODS FOR THE CARTESIAN $P_*(\kappa)$-SCLCP*

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Abstract: In this paper, we introduce a new parametric kernel function with trigonometric barrier term, which yields a class of large- and small-update interior-point methods for the Cartesian $P_*(\kappa)$-LCP over symmetric cones. By using Euclidean Jordan algebras, together with the feature of the new parametric kernel function, we establish the currently best known iteration bounds for large- and small-update methods. This result reduces the gap between the practical behavior of the algorithms and their theoretical performance result.

Key words: *interior-point methods, linear complementarity problem, Cartesian $P_*(\kappa)$-property, Euclidean Jordan algebras, large-update method, small-update method, polynomial complexity*

Mathematics Subject Classification: *90C33, 90C51*

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1 Introduction

Let $\mathcal{V} = \mathcal{V}_1 \times \cdots \times \mathcal{V}_N$ be a Cartesian product of a finite number of simple Euclidean Jordan algebras (EJAs) $(\mathcal{V}_j, \circ)$ with dimensions n_j and ranks r_j for $j = 1, \ldots, N$, and $\mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_N$ be a Cartesian product of a finite number of $\mathcal{K}_j$ (i.e., the corresponding cones of squares of $\mathcal{V}_j$) for $j = 1, \ldots, N$. The dimension and the rank of $\mathcal{V}$ are $n = \sum_{j=1}^N n_j$ and $r = \sum_{j=1}^N r_j$, respectively.

The linear complementarity problem over symmetric cones (SCLCP), is to find $x, s \in \mathcal{V}$ such that

$$x \in \mathcal{K}, \quad s = \mathcal{A}(x) + q \in \mathcal{K}, \quad \text{and } x \diamond s = 0,$$

where $\mathcal{A} : \mathcal{V} \to \mathcal{V}$ is a linear transformation and $q \in \mathcal{V}$. The SCLCP is a wide class of problems that contains linear complementarity problem, second-order cone linear complementarity problem and semidefinite linear complementarity problem as special cases. Moreover, the Karush-Kuhn-Tucker (KKT) condition of symmetric optimization (SO) can be written in the form of SCLCP. By means of EJAs, Faybusovich [9–11] made the first attempt to study the interior-point methods (IPMs) for the monotone SCLCP and proved

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the existence and uniqueness of the central path. For a brief survey on the recent developments related to SO and symmetric cone complementarity problems (SCCP), we refer to [1, 4, 12, 13, 15, 18–22, 30–32, 34].

Let $I_+(x) = \{\nu : \langle x^{(\nu)}, [\mathcal{A}(x)]^{(\nu)}\rangle > 0\}$ and $I_-(x) = \{\nu : \langle x^{(\nu)}, [\mathcal{A}(x)]^{(\nu)}\rangle < 0\}$ be two index sets. We call SCLCP the Cartesian $P_*(\kappa)$-SCLCP if $\mathcal{A}$ has the Cartesian $P_*(\kappa)$-property, i.e.,

$$(1+4\kappa)\sum_{\nu\in I_+(x)}\langle x^{(\nu)}, [\mathcal{A}(x)]^{(\nu)}\rangle + \sum_{\nu\in I_-(x)}\langle x^{(\nu)}, [\mathcal{A}(x)]^{(\nu)}\rangle \geq 0,\ \kappa \geq 0.$$

This problem has been recently considered in [22] as the generalization of the more commonly known and more widely used monotone SCLCPs (see, e.g., [12, 15, 18, 21, 29, 33, 34]).

The recent development of primal-dual IPMs is based on the barrier functions that are defined by a large class of univariate functions. The univariate functions called eligible kernel functions [2] which have been successfully used to design new IPMs for various optimization problems. It is well known that the use of certain eligible kernel functions lead to significant reduction of the complexity gap between large- and small-update methods comparing to the logarithmic kernel function. This was one of the main motivations of considering eligible kernel functions as an alternative to classical logarithmic kernel function. For some other related kernel-function based IPMs we refer to the monograph [6] and the references [3, 5, 7, 14, 16, 17, 24, 25, 28, 35].

In this paper, we introduce a new kind of parametric kernel function with trigonometric barrier term as follows:

$$\psi(t) = \frac{t^2-1}{2} - \log t - \int_1^t \frac{u^2}{2p(x+2u)^2}\tan^{2p}(h(x))\mathrm{d}x, \quad t>0, \quad p\in\mathbf{N}, \quad p>1 \tag{1.1}$$

where

$$h(x) = \frac{\pi u(1-x)}{x+2u}, \tag{1.2}$$

and $0 < u \leq u^*$, ($u^* \approx 0.4275$), u^* is the unique solution of the following equation

$$g(u) := \tan\left(\frac{(1-2u)\pi}{4}\right) - \frac{2}{3\pi(1+2u)} = 0. \tag{1.3}$$

It should be noted that if $u = 0$, then $\psi(t) = \frac{t^2-1}{2} - \log t$, which is the kernel function of the classic barrier function. Some properties of the parametric kernel function, as well as the corresponding barrier function, are studied. Based on this new parametric kernel function, we proposed a class of primal-dual IPMs for the Cartesian $P_*(\kappa)$-SCLCP. The obtained complexity results match the currently best known iteration bounds for large-update methods, $O((1+2\kappa)\sqrt{r}\log r\log\frac{r}{\epsilon})$, and small-update methods, $O((1+2\kappa)\sqrt{r}\log\frac{r}{\epsilon})$, respectively. Thus, the iteration bounds are as good as they can be in the current state-of-the-art.

The paper is organized as follows. In Section 2, some well known results on EJAs that are needed in this paper are studied. In Section 3, we introduce the new parametric kernel function with a trigonometric barrier term and develop some useful properties of the new kernel function, as well as the corresponding barrier function. In Section 4, we present the framework of kernel function-based IPMs for the Cartesian $P_*(\kappa)$-SCLCP. The analysis and complexity of the algorithms for large- and small-update methods are presented in Section 5. Finally, some conclusions and remarks are made in Section 6.

2 Preliminaries

In this section, we briefly recall some well known results on EJAs that are used in this paper. A comprehensive treatment of EJAs can be found in the monograph [8] and the references [23, 27, 31, 32].

The bilinear form on $\mathcal{V}$ is defined as

$$x \diamond s := \left(x^{(1)} \circ s^{(1)}, \ldots, x^{(N)} \circ s^{(N)}\right)^T,$$

where $x = \left(x^{(1)}, \ldots, x^{(N)}\right)^T$ and $s = \left(s^{(1)}, \ldots, s^{(N)}\right)^T$ in $\mathcal{V}$ with $x^{(j)}, s^{(j)} \in \mathcal{V}_j,\ j = 1, \ldots, N$. Similarly, the identity element in $\mathcal{V}$ is defined as

$$e = \left(e^{(1)}, \ldots, e^{(N)}\right)^T,$$

where $e^{(j)} \in \mathcal{V}_j$ is the identity element in $\mathcal{V}_j$,

The spectral decomposition of $x = \left(x^{(1)}, \ldots, x^{(N)}\right)^T$ in $\mathcal{V}$ is given by

$$x = \left(\sum_{i=1}^{r_1} \lambda_i(x^{(1)}) c_i^{(1)}, \ldots, \sum_{i=1}^{r_N} \lambda_i(x^{(N)}) c_i^{(N)}\right)^T,$$

where

$$x^{(j)} = \sum_{i=1}^{r_j} \lambda_i(x^{(j)}) c_i^{(j)}$$

is the spectral decomposition of $x^{(j)} \in \mathcal{V}_j$ with respect to the Jordan frame $\{c_1^{(j)}, \ldots, c_{r_j}^{(j)}\}$ for $j = 1, \ldots, N$. Corresponding, the vector-valued function $\psi(x)$ can be defined as

$$\psi(x) = (\psi(x^{(1)}), \ldots, \psi(x^{(N)}))^T, \tag{2.1}$$

where

$$\psi(x^{(j)}) = \psi(\lambda_1(x^{(j)}))\, c_1^{(j)} + \cdots + \psi(\lambda_{r_j}(x^{(j)}))\, c_{r_j}^{(j)}, \quad j = 1, \ldots, N.$$

Furthermore, if $\psi(t)$ is differentiable, the derivative $\psi'(t)$ exists, and we also have the vector-valued function $\psi'(x)$, namely

$$\psi'(x) = (\psi'(x^{(1)}), \ldots, \psi'(x^{(N)}))^T, \tag{2.2}$$

where

$$\psi'(x^{(j)}) = \psi'(\lambda_1(x^{(j)}))\, c_1^{(j)} + \cdots + \psi'(\lambda_{r_j}(x^{(j)}))\, c_{r_j}^{(j)}, \quad j = 1, \ldots, N. \tag{2.3}$$

The Peirce decomposition of $x^{(j)} \in \mathcal{V}_j$ with respect to the Jordan frame $\{c_1^{(j)}, \ldots, c_{r_j}^{(j)}\}$ is given by

$$x^{(j)} = \sum_{i=1}^{r_j} x_i^{(j)} c_i^{(j)} + \sum_{i<m_j} x_{i,m_j}^{(j)}, \quad j = 1, \ldots, N,$$

with $x_i^{(j)} \in \mathbf{R}$, $i = 1, \ldots, r_j$ and $x_{i,m_j}^{(j)} \in \mathcal{V}_{i,m_j}^{(j)}$, $1 \le i < m_j \le r_j$. The $\mathcal{V}_{im_j}^{(j)}$ for $1 \le i < m_j \le r_j$ are the Peirce subspaces of $\mathcal{V}_j$ induced by the Jordan frame $\{c_1^{(j)}, \ldots, c_{r_j}^{(j)}\}$. The Peirce decomposition of $x \in \mathcal{V}$ can be defined straightforwardly by using the Peirce decomposition of components $x^{(j)} \in \mathcal{V}_j$ as follows

$$x = \left(\sum_{i=1}^{r_1} x_i^{(1)} c_i^{(1)} + \sum_{i<m_1} x_{im_1}^{(1)}, \ldots, \sum_{i=1}^{r_N} x_i^{(N)} c_i^{(N)} + \sum_{i<m_N} x_{i,m_N}^{(N)} \right)^T.$$

The trace and the determinant of x in $\mathcal{V}$ are given by

$$\mathrm{tr}(x) = \sum_{j=1}^{N} \sum_{i=1}^{r_j} \lambda_i(x^{(j)}) \text{ and } \det(x) = \prod_{j=1}^{N} \prod_{i=1}^{r_j} \lambda_i(x^{(j)}).$$

Furthermore, we define the canonical inner product and the Frobenius norm as follows

$$\langle x, s \rangle = \sum_{j=1}^{N} \langle x^{(j)}, s^{(j)} \rangle = \sum_{j=1}^{N} \mathrm{tr}\left(x^{(j)} \circ s^{(j)} \right)$$

and

$$\|x\|_F = \sqrt{\sum_{j=1}^{N} \sum_{i=1}^{r_j} \lambda_i^2(x^{(j)})}.$$

Let

$$\lambda_{max}(x) = \max\{\lambda_i(x^{(j)}) : \ 1 \le i \le r_j, j = 1, \ldots, N\}$$

and

$$\lambda_{min}(x) = \min\{\lambda_i(x^{(j)}) : \ 1 \le i \le r_j, j = 1, \ldots, N\}.$$

Then

$$|\lambda_{max}(x)| \le \|x\|_F \text{ and } |\lambda_{min}(x)| \le \|x\|_F.$$

Let $x^{(j)} = \sum_{i=1}^{r_j} \lambda_i(x^{(j)}) c_i^{(j)}$ be the spectral decomposition of $x^{(j)} \in \mathcal{V}_j$ with respect to the Jordan frame $\{c_1^{(j)}, \ldots, c_{r_j}^{(j)}\}$ for $j = 1, \cdots, N$, f and g are continuously differentiable functions in a suitable domain that contains all the eigenvalues of $x^{(j)}$. Then we define $F : \mathcal{V}_j \to \mathbf{R}$ and $G : \mathcal{V}_j \to \mathcal{V}_j$ by

$$F(x^{(j)}) := \sum_{i=1}^{r_j} f(\lambda_i(x^{(j)})) \text{ and } G(x^{(j)}) := \sum_{i=1}^{r_j} g(\lambda_i(x^{(j)})) c_i^{(j)}. \tag{2.4}$$

The first derivatives $D_x F(x^{(j)})$ and $D_x G(x^{(j)})$ of the function $F(x^{(j)})$ and $G(x^{(j)})$ are given by

$$\nabla F(x^{(j)}) = D_x F(x^{(j)}) = F'(x^{(j)}) = \sum_{i=1}^{r_j} f'(\lambda_i(x^{(j)})) c_i^{(j)}, \tag{2.5}$$

and

$$D_x G(x^{(j)}) = G'(x^{(j)}) \quad = \quad \sum_{i=1}^{r_j} g'(\lambda_i(x^{(j)})) P_{ii}^{(j)} + \sum_{\substack{l<k \\ \lambda_l(x^{(j)}) = \lambda_k(x^{(j)})}} g'(\lambda_l(x^{(j)})) P_{lk}^{(j)}$$

$$+ \sum_{\substack{l<k \\ \lambda_l(x^{(j)}) \neq \lambda_k(x^{(j)})}} \frac{g(\lambda_l(x^{(j)})) - g(\lambda_k(x^{(j)}))}{\lambda_l(x^{(j)}) - \lambda_k(x^{(j)})} P_{lk}^{(j)} \tag{2.6}$$

respectively. Here $P_{lk}^{(j)}$, $1 \leq l \leq k \leq r_j$ are orthogonal projection operators that appear in the Peirce decomposition of $\mathcal{V}_j$ with respect to the Jordan frame $\{c_1^{(j)}, \dots, c_{r_j}^{(j)}\}$.

The separable spectral functions $F(x)$ and $G(x)$ are defined as

$$F(x) = \sum_{j=1}^{m} F(x^{(j)}) \text{ and } G(x) = \left(G(x^{(1)}), \dots, G(x^{(m)})\right)^T,$$

where the components are defined by (2.4). Then we have

$$F'(x) = \left(F'(x^{(1)}), \dots, F'(x^{(m)})\right)^T, \tag{2.7}$$

and

$$G'(x) = \left(G'(x^{(1)}), \dots, G'(x^{(m)})\right)^T, \tag{2.8}$$

where the derivatives of the components are given by (2.5) and (2.6), respectively.

3 New Parametric Kernel Function and its Properties

For ease of reference, we give the first three derivatives of $\psi(t)$ given by (1.1) with respect to t as follows

$$\psi'(t) = t - \frac{1}{t} - \frac{u^2}{2p(t+2u)^2} \tan^{2p}(h(t)), \tag{3.1}$$

$$\begin{aligned} \psi''(t) &= 1 + \frac{1}{t^2} + \frac{u^2}{p(t+2u)^3} \tan^{2p}(h(t)) \\ &+ \frac{\pi u^3(1+2u)}{(t+2u)^4} \tan^{2p-1}(h(t)) \sec^2(h(t)), \end{aligned} \tag{3.2}$$

$$\begin{aligned} \psi'''(t) &= -\frac{2}{t^3} - \frac{3u^2}{p(t+2u)^4} \tan^{2p}(h(t)) - \frac{6\pi u^3(1+2u)}{(t+2u)^5} \tan^{2p-1}(h(t)) \sec^2(h(t)) \\ &- \frac{\pi^2 u^4 (1+2u)^2 (2p-1)}{(t+2u)^6} \tan^{2p-2}(h(t)) \sec^4(h(t)) \\ &- \frac{2\pi u^4 (1+2u)^2}{(t+2u)^6} \tan^{2p}(h(t)) \sec^2(h(t)). \end{aligned} \tag{3.3}$$

One can conclude that

$$\psi(1) = \psi'(1) = 0, \quad \text{and } \lim_{t \to 0^+} \psi(t) = \lim_{t \to +\infty} \psi(t) = +\infty.$$

This implies that the proposed kernel function $\psi(t)$ is completely defined by its second derivative, namely,

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta) d\zeta d\xi.$$

In what follows, we develop some technical lemmas that are needed in this paper.

Lemma 3.1. *For the function $g(u)$, defined in (1.3), one has*

$$g(u) > 0, \quad 0 < u < u^*.$$

Proof. Consider the function $g(u) = \tan\left(\frac{(1-2u)\pi}{4}\right) - \frac{2}{3\pi(1+2u)}$ as defined in (1.3), and using that $\cos(x) = \sin(\frac{\pi}{2} - x) < \frac{\pi}{2} - x$ for $0 \leq x < \frac{\pi}{2}$. Hence, we have

$$\begin{aligned} g'(u) &= -\frac{\pi}{2}\sec^2\left(\frac{\pi(1-2u)}{4}\right) + \frac{4}{3\pi(1+2u)^2} \\ &= \sec^2\left(\frac{\pi(1-2u)}{4}\right)\left(-\frac{\pi}{2} + \frac{4}{3\pi(1+2u)^2}\cos^2\left(\frac{\pi(1-2u)}{4}\right)\right) \\ &\leq \sec^2\left(\frac{\pi(1-2u)}{4}\right)\left(-\frac{\pi}{2} + \frac{4}{3\pi(1+2u)^2}\frac{\pi^2(1+2u)^2}{16}\right) \\ &= -\frac{5\pi}{12}\sec^2\left(\frac{\pi(1-2u)}{4}\right) < 0. \end{aligned}$$

This implies that $g(u)$ is decreasing in $(0, u^*)$. Due to the fact that $g(u^*) = 0$, we can conclude that $g(u) > 0$ for $0 < u < u^*$. This completes the proof. □

Lemma 3.2. *Let $h(t)$ be given by (1.2). Then*

$$f(t,u) := \tan(h(t)) - \frac{4u}{3\pi(1+2u)t} > 0, \quad 0 < t \leq 2u\ ,\ 0 < u < u^*.$$

Proof. For $0 < t \leq 1$, one has $0 \leq h(t) < \frac{\pi}{2}$, therefore $\cos(h(t)) \leq \frac{\pi}{2} - h(t)$. Differentiating the function $f(t,u)$ with respect to t, we have

$$\begin{aligned} \frac{\partial f(t,u)}{\partial t} &= \frac{1}{\cos^2 h(t)}h'(t) + \frac{4u}{3(1+2u)\pi t^2} \\ &= \frac{1}{3\pi t^2 \cos^2 h(t)}\left(3\pi t^2 h'(t) + \frac{4u}{1+2u}\cos^2 h(t)\right) \\ &\leq \frac{1}{3\pi t^2 \cos^2 h(t)}\left(3\pi t^2 h'(t) + \frac{4u}{1+2u}\left(\frac{\pi}{2} - h(t)\right)^2\right) \\ &= \frac{1}{3\pi t^2 \cos^2 h(t)}\left(-3\pi t^2\frac{\pi u(1+2u)}{(t+2u)^2} + \frac{4u}{1+2u}\frac{\pi^2(1+2u)^2 t^2}{4(t+2u)^2}\right) \\ &= -\frac{2\pi u(1+2u)}{3(t+2u)^2\cos^2 h(t)} < 0. \end{aligned}$$

This implies that $f(t,u)$ is strictly monotonically decreasing with respect to $t \in (0, 2u]$. It follows from Lemma 3.1 that

$$f(2u,u) = \tan\left(\frac{(1-2u)\pi}{4}\right) - \frac{2}{3\pi(1+2u)} = g(u) > 0, \quad 0 < u < u^*.$$

Then, we can conclude that $f(t,u) > 0$ for $t \in (0, 2u]$. This completes the proof. □

Lemma 3.3 (Lemma 2 in [3]). *Let a be a constant, and*

$$w(t,\lambda) = L_n(\lambda)t^n + L_{n-1}(\lambda)t^{n-1} + \cdots + L_1(\lambda)t + L_0(\lambda), \quad t \in \mathbf{R}.$$

Here $L_i(\lambda)$ are functions of parameter $\lambda \in \mathbf{R}$ for $i = 0, 1, \ldots, n$. If $L_n(\lambda) > 0$, $w(a,\lambda) > 0$ and $\frac{\partial^i w(t,\lambda)}{\partial t^i}|_{t=a} > 0$ for $i = 1, \ldots, n-1$, then we have $w(t,\lambda) > 0$ for all $t > a$.

The next lemma serves to prove that the proposed parametric kernel function has some good properties.

Lemma 3.4. *Let $\psi(t)$ be as defined in (1.1). Then*

$$\begin{aligned} \psi''(t) &> 1, \quad \forall t > 0; && \text{(4-a)} \\ t\psi''(t) + \psi'(t) &> 0, \quad \forall t > 0; && \text{(4-b)} \\ t\psi''(t) - \psi'(t) &> 0, \quad \forall t > 1; && \text{(4-c)} \\ \psi'''(t) &< 0, \quad \forall t > 0. && \text{(4-d)} \end{aligned}$$

Proof. We first prove (4-a). The second derivative of $\psi(t)$ is given in (3.2). Using that $\tan(h(t)) > 0$ for all $0 < t < 1$, thus $\psi''(t) > 1$ for $0 < t < 1$.

Now let $t \geq 1$. Define the function

$$\xi(t) := \frac{1}{t^2} + \frac{u^2}{p(t+2u)^3}\tan^{2p}(h(t)) + \frac{\pi u^3(1+2u)}{(t+2u)^4}\tan^{2p-1}(h(t))\sec^2(h(t)),$$

we need to prove that when $0 < u < u^*$ and $t \geq 1$, $\xi(t) > 0$ holds. To do this we consider the following two cases:

Case 4-a.1: For $0 < u \leq \frac{1}{4}$. Then we have $-\frac{\pi}{4} \leq -\pi u < h(t) < 0$. This implies that $-1 < -\tan(\pi u) < \tan(h(t)) \leq 0$ for $t \geq 1$. We have

$$\begin{aligned} \xi(t) &\geq \frac{1}{t^2} + \frac{u^2}{p(t+2u)^3}\tan^{2p}(h(t)) - \frac{2\pi u^3(1+2u)}{(t+2u)^4} \\ &= \frac{u^2}{p(t+2u)^3}\tan^{2p}(h(t)) + \frac{\eta_1(t)}{t^2(t+2u)^4}, \end{aligned}$$

where

$$\eta_1(t) := (t+2u)^4 - 2\pi u^3(1+2u)t^2, \quad t \geq 1.$$

One can easily verify that

$$\eta_1(1) = (1+2u)^4 - 2\pi u^3(1+2u) = (1+2u)((1+2u)^3 - 2\pi u^3) > 0.$$

Similarly, we can prove that $\eta_1'(1) > 0$, $\eta_1''(1) > 0$ and $\eta_1'''(t) = 24(t+2u) > 0$. From Lemma 3.3, we have $\eta_1(t) > 0$ for $t \geq 1$. This shows that $\psi''(t) > 1$ holds when $0 < u \leq \frac{1}{4}$ and $t > 0$.

Case 4-a.2: Let $\frac{1}{4} < u \leq u^*$. We consider two situations to prove that $\xi(t) > 0$ holds for $t \geq 1$.

Situation 4-a.2.1: Let $1 \leq t < \frac{6u}{4u-1}$. Then $-\frac{\pi}{4} < h(t) \leq 0$, which implies that $-1 < \tan(h(t)) \leq 0$. Similar to the proof in the Case 4-a.1, we can easily verify that (4-a) holds.

Situation 4-a.2.2: Let $t \geq \frac{6u}{4u-1}$. Then $-\pi u^* < h(t) \leq -\frac{\pi}{4}$, which implies that $-\tan(\pi u^*) < \tan(h(t)) \leq -1$. We have

$$\begin{aligned}\xi(t) &= \frac{1}{\tan^{2p}(h(t))}\left(\frac{\tan^{2p}(h(t))}{t^2} + \frac{u^2}{p(t+2u)^3} + \frac{\pi u^3(1+2u)}{(t+2u)^4}\tan^{-1}(h(t))\sec^2(h(t))\right)\\ &\geq \frac{1}{\tan^{2p}(h(t))}\left(\frac{1}{t^2} + \frac{u^2}{p(t+2u)^3} - \frac{\pi u^3(1+2u)\sec^2(\pi u^*)}{(t+2u)^4}\right)\\ &= \frac{1}{\tan^{2p}(h(t))}\left(\frac{u^2}{p(t+2u)^3} + \frac{\eta_2(t)}{t^2(t+2u)^4}\right),\end{aligned}$$

where

$$\eta_2(t) := (t+2u)^4 - \pi u^3(1+2u)\sec^2(\pi u^*)t^2,\ t \geq \frac{6u}{4u-1}.$$

Since $u^* \approx 0.4275 < 0.428$, so $\sec^2(\pi u^*) < \sec^2(0.428\pi) < 20$, and use the fact that $\pi < 3.2$, thus for $\frac{1}{4} < u < u^* < \frac{1}{2}$, we have

$$\begin{aligned}\eta_2\left(\frac{6u}{4u-1}\right) &= \frac{4u^4(1+2u)}{(4u-1)^4}(64(1+2u)^3 - 9\pi u(4u-1)^2)\sec^2(\pi u^*)\\ &> \frac{4u^4(1+2u)}{(4u-1)^4}(64(1+2u)^3 - 576u(4u-1)^2)\\ &= \frac{256u^4(1+2u)}{(4u-1)^4}(1+3u(4u-1)+4u^2+68u^2(1-2u))\\ &> 0.\end{aligned}$$

Similarly, we can verify that $\eta_2'(\frac{6u}{4u-1}) > 0$, $\eta_2''(\frac{6u}{4u-1}) > 0$ and $\eta_2'''(t) = 24(t+2u) > 0$. From Lemma 3.3, we have $\eta_2(t) > 0$ when $t \geq \frac{6u}{4u-1}$. This means that $\psi''(t) > 1$ when $\frac{1}{4} < u < u^*$ and $t \geq \frac{6u}{4u-1}$.

From the two cases above we can conclude that (4-a) holds.

By using (3.1) and (3.2), we have

$$t\psi''(t) + \psi'(t) = 2t + \frac{u^2(t-2u)}{2p(t+2u)^3}\tan^{2p}(h(t)) + \frac{\pi u^3(1+2u)t}{(t+2u)^4}\tan^{2p-1}(h(t))(1+\tan^2(h(t))).$$

We will consider three cases to prove (4-b).

Case 4-b.1: Let $0 < t \leq 2u$. After some elementary reductions, we have, by Lemma 3.2,

$$t\psi''(t) + \psi'(t) \quad > \quad 2t + \frac{\pi u^3(1+2u)t}{(t+2u)^4}\tan^{2p-1}(h(t)) + \frac{3u^2t^2 + (8p-12)u^4}{6p(t+2u)^4}\tan^{2p}(h(t)),$$

which indicates that $t\psi'(t) + \psi'(t) > 0$ holds in this case.

Case 4-b.2: Let $2u < t \leq 1$. Then $t - 2u > 0$ and $\tan(h(t)) \geq 0$. It follows that $t\psi''(t) + \psi'(t) > 0$ holds in this case.

Case 4-b.3: Let $t > 1$. It follows from (4-a) that $\psi'(t)$ is an increasing function for $t > 0$. Due to the fact that $\psi'(1) = 0$, we can conclude that $\psi'(t) > 0$ holds for $t > 1$, so $t\psi''(t) + \psi'(t) > 0$.

From the three cases above we can conclude that (4-b) holds.

To prove (4-c), we consider two cases:

Case 4-c.1: Let $0 < u \leq \frac{1}{4}$. Then $-\frac{\pi}{4} \leq -\pi u < h(t) \leq 0$ for $t \geq 1$, which implies that $-1 < \tan(h(t)) \leq 0$ for $t \geq 1$. We have

$$\begin{aligned} t\psi''(t) - \psi'(t) &\geq \frac{2}{t} + \frac{3u^2t + 2u^3}{2p(t+2u)^3}\tan^{2p}(h(t)) - \frac{2\pi u^3(1+2u)t}{(t+2u)^4} \\ &= \frac{3u^2t + 2u^3}{2p(t+2u)^3}\tan^{2p}(h(t)) + \frac{\eta_3(t)}{t(t+2u)^4}, \end{aligned}$$

where

$$\eta_3(t) := 2(t+2u)^4 - 2\pi u^3(1+2u)t^2 = (t+2u)^4 + \eta_1(t) > 0, \quad t > 1.$$

Thus $t\psi''(t) - \psi'(t) > 0$ holds for $t > 1$.

Case 4-c.2: Let $\frac{1}{4} < u < u^*$. We consider two situations to prove (4-c) holds in this case.

Situation 4-c.2.1: Let $1 \leq t < \frac{6u}{4u-1}$. Then $-\frac{\pi}{4} < h(t) \leq 0$, which implies that $-1 < \tan(h(t)) \leq 0$. Similar to the proof in Situation 4-a.2.1, we can easily verify that (4-c) holds.

Situation 4-c.2.2: Let $t \geq \frac{6u}{4u-1}$. Then $-\pi u^* < h(t) \leq -\frac{\pi}{4}$, which implies that $-\tan(\pi u^*) < \tan(h(t)) \leq -1$. We have

$$\begin{aligned} t\psi''(t) - \psi'(t) &\geq \frac{1}{\tan^{2p}(h(t))}\left(\frac{2}{t} + \frac{3u^2t + 2u^3}{2p(t+2u)^3} - \frac{\pi u^3(1+2u)\sec^2(\pi u^*)t}{(t+2u)^4}\right) \\ &= \frac{1}{\tan^{2p}(h(t))}\left(\frac{3u^2t + 2u^3}{2p(t+2u)^3} + \frac{\eta_4(t)}{t(t+2u)^4}\right), \end{aligned}$$

where

$$\eta_4(t) := 2(t+2u)^4 - \pi u^3(1+2u)(1+\tan^2(\pi u^*))t^2 = (t+2u)^4 + \eta_2(t) > 0, \quad t \geq \frac{6u}{4u-1}.$$

This implies that $t\psi''(t) - \psi'(t) > 0$ holds when $\frac{1}{4} < u < u^*$ and $t \geq \frac{6u}{4u-1}$.

From the two cases above we can conclude that (4-c) holds.

Finally we need to prove that (4-d) holds.

Using (3.3) and since $\tan(h(t)) > 0$ for $0 < t < 1$, therefore $\psi'''(t) < 0$.

Now let $t \geq 1$. To prove $\psi'''(t) < 0$ we consider two cases.

Case 4-d.1: Let $0 < u \leq \frac{1}{4}$. Then $-\frac{\pi}{4} \leq -\pi u < h(t) \leq 0$ for $t \geq 1$, which implies that $-1 < \tan(h(t)) \leq 0$ for $t \geq 1$. We have

$$\begin{aligned} \psi'''(t) &\leq -\frac{2}{t^3} - \frac{6\pi u^3(1+2u)}{(t+2u)^5}\tan^{2p-1}(h(t))\sec^2(h(t)) \\ &\leq -\frac{2}{t^3} + \frac{12\pi u^3(1+2u)}{(t+2u)^5} = -\frac{2\eta_5(t)}{t^3(t+2u)^5}, \end{aligned}$$

where

$$\eta_5(t) := (t+2u)^5 - 6\pi u^3(1+2u)t^3, \quad t \geq 1.$$

Let $0 < u \leq \frac{1}{4}$. Then

$$\eta_5(1) = (1+2u)^5 - 6\pi u^3(1+2u) \geq (1+2u)\left(1 - \frac{3\pi}{32}\right) > 0.$$

Similarly, we can verify that ${\eta_5}'(1) > 0$, ${\eta_5}''(1) > 0$, ${\eta_5}'''(1) > 0$ and ${\eta_5}^{(4)}(t) = 120(t+2u) > 0$. From Lemma 3.3, we have $\eta_5(t) > 0$ for $t \geq 1$. This shows that $\psi'''(t) < 0$ when $0 < u \leq \frac{1}{4}$ and $t \geq 1$.

Case 4-d.2: Let $\frac{1}{4} < u < u^*$. We consider two situations to prove (4-d) holds in this case.

Situation 4-d.2.1: Let $1 \leq t < \frac{6u}{4u-1}$. Then $-\frac{\pi}{4} < h(t) \leq 0$. This implies that $-1 < \tan(h(t)) \leq 0$. Similar to the proof in Case 4-a.2.1, we can easily verify that (4-d) holds.

Situation 4-d.2.2: Let $t \geq \frac{6u}{4u-1}$. Then $-\tan(\pi u^*) < \tan(h(t)) \leq -1$. We have

$$\begin{aligned}
\psi'''(t) &\leq -\frac{2}{t^3} - \frac{6\pi u^3(1+2u)}{(t+2u)^5}\tan^{2p-1}(h(t))\sec^2(h(t)) \\
&= -\frac{2}{\tan^{2p}(h(t))}\left(\frac{\tan^{2p}(h(t))}{t^3} + \frac{3\pi u^3(1+2u)}{(t+2u)^5}\tan^{-1}(h(t))\sec^2(h(t))\right) \\
&\leq -\frac{2}{\tan^{2p}(h(t))}\left(\frac{1}{t^3} - \frac{3\pi u^3(1+2u)}{(t+2u)^5}\sec^2(\pi u^*)\right) \\
&= -\frac{2\eta_6(t)}{t^3(t+2u)^5\tan^{2p}(h(t))},
\end{aligned}$$

where

$$\eta_6(t) := (t+2u)^5 - 3\pi u^3(1+2u)\sec^2(\pi u^*)t^3, \quad t \geq \frac{6u}{4u-1}.$$

Since $\sec^2(\pi u^*) < 20$ and $\pi < 3.2$, thus for $\frac{1}{4} < u < u^* < \frac{1}{2}$, we have

$$\begin{aligned}
\eta_6\left(\frac{6u}{4u-1}\right) &= \left(\frac{4u(1+2u)}{4u-1}\right)^5 - 3\pi(1+\tan^2(\pi u^*))u^3(1+2u)\left(\frac{6u}{4u-1}\right)^3 \\
&> \frac{512u^5(1+2u)}{(4u-1)^5}\left(2(1+2u)^4 - 81u(4u-1)^2\right) > 0.
\end{aligned}$$

Similarly, we can verify that ${\eta_6}'(\frac{6u}{4u-1}) > 0$, ${\eta_6}''(\frac{6u}{4u-1}) > 0$, ${\eta_6}'''(\frac{6u}{4u-1}) > 0$ and ${\eta_6}^{(4)}(t) = 120(t+2u) > 0$. From Lemma 3.3, we have $\eta_6(t) > 0$ when $t \geq \frac{6u}{4u-1}$. This means that $\psi'''(t) < 0$ for $\frac{1}{4} < u < u^*$ and $t \geq \frac{6u}{4u-1}$.

From the above all we complete the proof of the lemma. □

The following lemma means that the proposed parametric kernel function $\psi(t)$ is exponential convex, which is equivalent to the second property (4-b) in Lemma 3.4, (see, e.g., Lemma 1 in [25]).

Lemma 3.5. *Let $t_1, t_2 \geq 0$. Then*

$$\psi(\sqrt{t_1 t_2}) \leq \frac{1}{2}(\psi(t_1) + \psi(t_2)).$$

We consider the barrier function of the form

$$\Psi(x, s; \mu) := \Psi(v) := \operatorname{tr}(\psi(v)) = \sum_{j=1}^{N} \operatorname{tr}(\psi(v^{(j)})) = \sum_{j=1}^{N} \sum_{i=1}^{r_j} \psi(\lambda_i(v^{(j)}))$$

defined by the parametric kernel function $\psi(t)$. It follows immediately from (2.7) that

$$D_v \Psi(v) = \psi'(v) = \left(\sum_{i=1}^{r_1} \psi'(\lambda_i(v^{(1)}))c_i^{(1)}, \ldots, \sum_{i=1}^{r_N} \psi'(\lambda_i(v^{(N)}))c_i^{(N)} \right)^T.$$

Furthermore, we can conclude that $\Psi(v)$ is nonnegative and strictly convex with respect to $v \succ_{\mathcal{K}} 0$ and vanishes at its global minimal point $v = e$, i.e.,

$$\Psi(v) = 0 \Leftrightarrow \psi(v) = 0 \Leftrightarrow \psi'(v) = 0 \Leftrightarrow v = e.$$

As a result of Lemma 3.5, we have the following theorem, which is crucial for the analysis of kernel function-based IPMs for the Cartesian $P_*(\kappa)$-SCLCP, (see, e.g., Theorem 4.1 in [31]).

Theorem 3.6. *$x, s \in \mathcal{K}_+$. Then*

$$\Psi\left((P(x)^{1/2}s)^{1/2} \right) \leq \frac{1}{2}(\Psi(x) + \Psi(s)),$$

where $\mathcal{K}_+$ is the interior of $\mathcal{K}$ and the map $P(x)$ is the quadratic representation of $\mathcal{K}$.

The norm-based proximity measure $\delta(v)$ is given by

$$\delta(v) := \frac{1}{2}\|\psi'(v)\|_F = \frac{1}{2}\sqrt{\sum_{j=1}^{N}\sum_{i=1}^{r_j} \psi'(\lambda_i(v^{(j)}))^2}. \tag{3.5}$$

This implies that $\delta(v) \geq 0$, and $\delta(v) = 0$ if and only if $\Psi(v) = 0$.

The proposed parametric kernel function $\psi(t)$ is strongly convex due to the fact that (4-a) of Lemma 3.4, i.e., $\psi''(t) > 1$. As a consequence of this property, the following lemma can be directly obtained from the corresponding results in [2].

Lemma 3.7. *Let $t > 0$. Then*

$$\frac{1}{2}(t-1)^2 \leq \psi(t) \leq \frac{1}{2}\psi'(t)^2.$$

Lemma 3.8. *Let $\varrho : [0, \infty) \to [1, \infty)$ be the inverse function of the parametric kernel function $\psi(t)$ for $t \geq 1$. Then*

$$\varrho(s) \leq 1 + \sqrt{2s}.$$

Proof. It follows immediately from the first inequality of Lemma 3.7 that the result is obvious. □

As the consequences of Lemma 3.7, we have the following two corollaries, which provide a lower bound on $\delta(v)$ and an upper bound on $\|v\|_F$ in terms of $\Psi(v)$, respectively.

Corollary 3.9. *Let $\Psi(v) \geq 1$. Then*

$$\delta(v) \geq \sqrt{\frac{\Psi(v)}{2}}.$$

Corollary 3.10. *Let* $\Psi(v) \geq 1$. *Then*

$$\|v\|_F \leq \sqrt{r} + \sqrt{2\Psi(v)}.$$

In the analysis of the algorithms, we need to consider the derivatives of the function $\Psi(x(t))$ with respect to t, where $x(t) = \left(x^{(1)}(t), \ldots, x^{(N)}(t)\right)^T$ such that $x^{(j)}(t) := x_0^{(j)} + tu^{(j)}$ with $t \in \mathbf{R}$ and $u^{(j)} \in \mathcal{V}_j$ and assume that $x^{(j)}(t) \in (\mathcal{K}_j)_+$. For more details, we refer to [23].

Let $x^{(j)}(t) = \sum_{i=1}^{r_j} \lambda_i(x^{(j)}(t))c_i^{(j)}$ be the spectral decomposition of $x^{(j)}(t)$ and $u^{(j)} = \sum_{i=1}^{r_j} u_i^{(j)} c_i^{(j)} + \sum_{l<k} u_{lk}^{(j)}$ be the Peirce decomposition of $u^{(j)}$ with respect to the Jordan frame $\{c_1^{(j)}, \ldots, c_{r_j}^{(j)}\}$ for $j = 1, \cdots, N$. From (2.7) with (2.5), we have

$$D_t\Psi(x^{(j)}(t)) = \mathrm{tr}\left(\sum_{i=1}^{r_j} \psi'(\lambda_i(x^{(j)}(t)))c_i^{(j)} \circ u^{(j)}\right). \tag{3.6}$$

Furthermore, we have, by (2.8) with (2.6),

$$\begin{aligned} D_t^2\Psi(x^{(j)}(t)) &= \sum_{i=1}^{r_j} \psi''(\lambda_i^{(j)})(u_i^{(j)})^2 + \sum_{\substack{l<k \\ \lambda_l^{(j)} = \lambda_k^{(j)}}} \psi''(\lambda_l^{(j)})\mathrm{tr}\left((u_{lk}^{(j)})^2\right) \\ &\quad + \sum_{\substack{l<k \\ \lambda_l^{(j)} \neq \lambda_k^{(j)}}} \frac{\psi'(\lambda_l^{(j)}) - \psi'(\lambda_k^{(j)})}{\lambda_l^{(j)} - \lambda_k^{(j)}}\mathrm{tr}\left((u_{lk}^{(j)})^2\right), \end{aligned}$$

where $\lambda_l^{(j)}$ and $\lambda_k^{(j)}$ represent $\lambda_l(x^{(j)}(t))$ and $\lambda_k(x^{(j)}(t))$, respectively. Now the derivatives on the Cartesian product $\mathcal{V}$ are given by

$$D_t\Psi(x(t)) = \sum_{j=1}^{N} D_t\Psi(x^{(j)}(t)) = \mathrm{tr}\left(\Psi'(x(t)) \circ x'(t)\right)$$

and

$$D_t^2\Psi(x(t)) = \sum_{j=1}^{N} D_t^2\Psi(x^{(j)}(t)).$$

Let $\lambda_l(x(t)) \geq \lambda_k(x(t))$ under the assumption that $l < k$. Then the following inequality gives an upper bound of the second-order derivative of $\Psi(x(t))$ with respect to t, i.e.,

$$D_t^2\Psi(x(t)) \leq \sum_{j=1}^{N}\left(\sum_{i=1}^{r_j} \psi''(\lambda_i^{(j)})(u_i^{(j)})^2 + \sum_{l<k} \psi''(\lambda_l^{(j)})\mathrm{tr}\left((u_{lk}^{(j)})^2\right)\right). \tag{3.7}$$

The detailed can be founded in [23].

4 Kernel Function-based IPMs for the Cartesian $P_*(\kappa)$-SCLCP

Throughout the paper we assume that the Cartesian $P_*(\kappa)$-SCLCP satisfies the interior-point condition (IPC), i.e., there exists $(x^0 \succ_\kappa 0, s^0 \succ_\kappa 0)$ such that $s^0 = \mathcal{A}(x^0) + q$. For this and other properties of the Cartesian $P_*(\kappa)$-SCLCP, we refer to [22]. Under the IPC holds, by relaxing the complementarity slackness $x \diamond s = 0$ with $x \diamond s = \mu e$ for $\mu > 0$. We have

$$\begin{pmatrix} \mathcal{A}(x) - s \\ x \diamond s \end{pmatrix} = \begin{pmatrix} -q \\ \mu e \end{pmatrix}, \quad x, s \succeq_{\mathcal{K}} 0 \tag{4.1}$$

The parameterized system (4.1) has a unique solution, for each $\mu > 0$. This solution is denoted as $(x(\mu), s(\mu))$ and we call $(x(\mu), s(\mu))$ the μ-center of the Cartesian $P_*(\kappa)$-SCLCP. The set of μ-centers (with μ running through all positive real numbers) gives a homotopy path, which is called the central path of the Cartesian $P_*(\kappa)$-SCLCP. If $\mu \to 0$, then the limit of the central path exists and since the limit points satisfy the complementarity condition $x \diamond s = 0$, the limit yields a solution for the Cartesian $P_*(\kappa)$-SCLCP.

The basic idea of kernel function-based IPMs is to follow the central path and approach the optimal set of the Cartesian $P_*(\kappa)$-SCLCP by letting μ go to zero. Applying Newton's method, we have the following system.

$$\begin{pmatrix} \mathcal{A}(\Delta x) - \Delta s \\ s \diamond \Delta x + x \diamond \Delta s \end{pmatrix} = \begin{pmatrix} 0 \\ -x \diamond s + \mu e \end{pmatrix}.$$

Since x and s do not operator commute in general, that is, $L(x)L(s) \neq L(s)L(x)$, then the above system unfortunately does not have a unique solution. We can use the following scaling scheme to overcome this difficulty.

Lemma 4.1 (Lemma 28 in [27])**.** *Let $u \in \mathcal{K}_+$. Then*

$$x \circ s = \mu e \quad \Leftrightarrow \quad P(u)x \circ P(u)^{-1}s = \mu e.$$

Replacing $x \diamond s = \mu e$ with $P(u)x \diamond P(u)^{-1}s = \mu e$ and applying Newton's method again, we have

$$\begin{pmatrix} \mathcal{A}(\Delta x) - \Delta s \\ P(u^{-1})s \circ P(u)\Delta x + P(u)x \circ P(u^{-1})\Delta s \end{pmatrix} = \begin{pmatrix} 0 \\ -P(u)x \circ P(u^{-1})s + \mu e \end{pmatrix}. \tag{4.2}$$

The appropriate choices of u that lead to obtaining unique search directions from the system (4.2) can be generalized from the semidefinite optimization (SDO) case.

In this paper, we consider the classical NT-scaling scheme to find the unique NT-direction. Let $u = w^{-\frac{1}{2}}$, where

$$w = P(x^{\frac{1}{2}}) \left(P(x^{\frac{1}{2}})s\right)^{-\frac{1}{2}} = P(s^{-\frac{1}{2}}) \left(P(s^{\frac{1}{2}})x\right)^{\frac{1}{2}}. \tag{4.3}$$

Furthermore, we define

$$v := \frac{P(w)^{-\frac{1}{2}}x}{\sqrt{\mu}} \left[= \frac{P(w)^{\frac{1}{2}}s}{\sqrt{\mu}}\right], \tag{4.4}$$

and

$$\overline{\mathcal{A}} := P(w)^{\frac{1}{2}}\mathcal{A}P(w)^{\frac{1}{2}},\ d_x := \frac{P(w)^{-\frac{1}{2}}\Delta x}{\sqrt{\mu}},\ d_s := \frac{P(w)^{\frac{1}{2}}\Delta s}{\sqrt{\mu}}. \tag{4.5}$$

It follows from Proposition 3.4 in [22] that the transformation $\overline{\mathcal{A}}$ has the Cartesian $P_*(\kappa)$-property if the linear transformation $\mathcal{A}$ has the Cartesian $P_*(\kappa)$-property. From (4.4) and (4.5), we have

$$\begin{pmatrix} \overline{\mathcal{A}}(d_x) - d_s \\ d_x + d_s \end{pmatrix} = \begin{pmatrix} 0 \\ v^{-1} - v \end{pmatrix}. \tag{4.6}$$

We can conclude that the system (4.6) has a unique solution (cf. Theorem 3.6 in [22]).

The classical logarithmic barrier function is given by

$$\Psi_c(v) := \mathrm{tr}(\psi_c(v)) = \sum_{j=1}^{N} \mathrm{tr}(\psi_c(v^{(j)})) = \sum_{j=1}^{N}\sum_{i=1}^{r_j}\left(\frac{\lambda_i(v^{(j)})^2-1}{2} - \log \lambda_i(v^{(j)})\right).$$

We have $v^{-1} - v = -\psi_c'(v)$. This means that the system (4.6) can be rewritten as

$$\begin{pmatrix} \overline{\mathcal{A}}(d_x) - d_s \\ d_x + d_s \end{pmatrix} = \begin{pmatrix} 0 \\ -\psi_c'(v) \end{pmatrix}. \tag{4.7}$$

Given the parametric kernel function $\psi(t)$ and the associated vector-valued function $\psi'(v)$ defined by (2.3), we replace the right-hand side of the second equation in (4.7) by $-\psi'(v)$, i.e., $-D_v\Psi(v)$. Thus we have

$$\begin{pmatrix} \overline{\mathcal{A}}(d_x) - d_s \\ d_x + d_s \end{pmatrix} = \begin{pmatrix} 0 \\ -\psi'(v) \end{pmatrix}. \tag{4.8}$$

The new search directions d_x and d_s are obtained by solving (4.8) and then Δx and Δs can be computed via (4.5). If $(x, s) \neq (x(\mu), s(\mu))$ then $(\Delta x, \Delta s)$ is nonzero. By taking a default step size α along the search directions, we get the new iteration point (x_+, s_+) according to

$$x_+ := x + \alpha\triangle x \text{ and } s_+ := s + \alpha\triangle s. \tag{4.9}$$

The generic IPM for the Cartesian $P_*(\kappa)$-SCLCP presented in the Figure 1.

Generic IPM for the Cartesian $P_*(\kappa)$-SCLCP

Input:
A threshold parameter $\tau \geq 1$;
an accuracy parameter $\varepsilon > 0$;
a fixed barrier update parameter $\theta, 0 < \theta < 1$;
a strictly feasible (x^0, s^0) and $\mu^0 = \langle x^0, s^0\rangle / r$ such that $\Psi(x^0, s^0; \mu^0) \leq \tau$.
begin
  $x := x^0$; $s := s^0$; $\mu := \mu^0$;
  while $r\mu \geq \varepsilon$ **do** (outer iteration)
  begin
    $\mu := (1-\theta)\mu$;
    while $\Psi(x, s; \mu) > \tau$ **do** (inner iteration)
    begin
      calculate the search direction $(\Delta x, \Delta s)$;
      choose a suitable step size α;
      update $x = x + \alpha\Delta x,\ s = s + \alpha\Delta s$;
    end
  end
end

Figure 1: Generic IPM for the Cartesian $P_*(\kappa)$-SCLCP

5 Analysis and Complexity of the Algorithms

5.1 Growth behavior of the barrier function during an outer iteration

During the course of the algorithms, the largest values of $\Psi(v)$ occur just after the update of μ. In what follows, we need to consider the effect of a μ-update on the value of $\Psi(v)$.

Lemma 5.1. *Let $\beta \geq 1$. Then*

$$\psi(\beta t) \leq \psi(t) + \frac{1}{2}(\beta^2 - 1)t^2.$$

Proof. Let

$$w(t) := -\log t - \int_1^t \frac{u^2}{2p(x+2u)^2} \tan^{2p}(h(x))dx\,, \quad 0 < u \leq u^*.$$

We have

$$\psi(t) = \frac{1}{2}(t^2 - 1) + w(t)$$

and

$$\psi(\beta t) - \psi(t) = \frac{1}{2}(\beta^2 - 1)t^2 + w(\beta t) - w(t).$$

As $\beta \geq 1$, to prove the lemma, it is sufficient to show that the function $w(t)$ is a decreasing function. This can be seen from the following inequality:

$$w'(t) = -\frac{1}{t} - \frac{u^2}{2p(t+2u)^2} \tan^{2p}(h(t)) < 0.$$

This completes the proof. □

Theorem 5.2. *Let $0 < \theta < 1$ and $v_+ = \frac{v}{\sqrt{1-\theta}}$. Then*

$$\Psi(v_+) \leq \Psi(v) + \frac{\theta}{2(1-\theta)} \left(2\Psi(v) + 2\sqrt{2r\Psi(v)} + r\right).$$

Proof. It follows from Lemma 5.1 with $\beta = \frac{1}{\sqrt{1-\theta}}$ that

$$\Psi(\beta v) \leq \Psi(v) + \frac{1}{2}\sum_{j=1}^{N}\sum_{i=1}^{r_j}(\beta^2 - 1)\lambda_i^2(v^{(j)}) = \Psi(v) + \frac{\theta\|v\|_F^2}{2(1-\theta)}.$$

Thus, we have, by Corollary 3.10,

$$\Psi(v_+) \leq \Psi(v) + \frac{\theta}{2(1-\theta)} \left(2\Psi(v) + 2\sqrt{2r\Psi(v)} + r\right).$$

This completes the proof. □

5.2 Choice of the default step size

By using (4.9) and (4.5), we have

$$x_+ = \sqrt{\mu} P(w)^{\frac{1}{2}}(v + \alpha d_x) \text{ and } s_+ = \sqrt{\mu} P(w)^{-\frac{1}{2}}(v + \alpha d_s).$$

Then

$$v_+ = P(w_+)^{-\frac{1}{2}} P(w)^{\frac{1}{2}}(v + \alpha d_x) = P(w_+)^{\frac{1}{2}} P(w)^{-\frac{1}{2}}(v + \alpha d_s),$$

where, w_+ is given as it is defined in (4.3),

$$w_+ = P\left(x_+^{\frac{1}{2}}\right)\left(P\left(x_+^{\frac{1}{2}}\right) s_+\right)^{-\frac{1}{2}}.$$

To calculate the decrease of the barrier function $\Psi(v)$ during an inner iteration it is standard to consider the decrease as a function of α defined by

$$f(\alpha) := \Psi(v_+) - \Psi(v).$$

Our aim is to find an upper bound for $f(\alpha)$ by using the exponential convexity of $\psi(t)$, and according to Lemma 3.5. However, working with $f(\alpha)$ may not be easy because in general $f(\alpha)$ is not convex. Thus, we are searching for the convex function $f_1(\alpha)$ that is an upper bound of $f(\alpha)$ and whose derivatives are easier to calculate than those of $f(\alpha)$.

It follows from Proposition 5.9.3 in [23] that

$$v_+ \sim \left(P(v + \alpha d_x)^{\frac{1}{2}}(v + \alpha d_s)\right)^{\frac{1}{2}}.$$

Then

$$\Psi(v_+) = \Psi\left(P(v + \alpha d_x)^{\frac{1}{2}}(v + \alpha d_s)\right)^{\frac{1}{2}}.$$

We have, by Theorem 3.6,

$$\Psi(v_+) \leq \frac{1}{2}(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)).$$

Let

$$f_1(\alpha) := \frac{1}{2}\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s) - \Psi(v).$$

Then, we have $f(\alpha) \leq f_1(\alpha)$ and $f(0) = f_1(0) = 0$. This means that $f_1(\alpha)$ gives an upper bound for the decrease of the barrier function $\Psi(v)$.

It follows from (3.6) that

$$\begin{aligned} f_1'(\alpha) &= \frac{1}{2}\sum_{j=1}^{N}\left(\operatorname{tr}(\psi'(v^{(j)} + \alpha d_x^{(j)}) \circ d_x^{(j)}) + \operatorname{tr}(\psi'(v^{(j)} + \alpha d_s^{(j)}) \circ d_s^{(j)})\right) \\ &= \frac{1}{2}(\operatorname{tr}(\psi'(v + \alpha d_x) \diamond d_x) + \operatorname{tr}(\psi'(v + \alpha d_s) \diamond d_s)). \end{aligned}$$

By using (4.7) and (3.5), we have

$$f_1'(0) = \frac{1}{2}\operatorname{tr}(\psi'(v) \diamond (d_x + d_s)) = -\frac{1}{2}\operatorname{tr}(\psi'(v) diamond \psi'(v)) = -\frac{1}{2}\|\psi'(v)\|_F^2 = -2\delta(v)^2 < 0.$$

Let

$$d_x^{(j)} = \sum_{i=1}^{r_j}(d_x)_i^{(j)} c_i^{(j)} + \sum_{i<m_j}(d_x)_{i,m_j}^{(j)},\ j = 1, \ldots, N$$

be the Peirce decomposition of $d_x^{(j)}$ with respect to the Jordan frame $\{c_1^{(j)}, \ldots, c_{r_j}^{(j)}\}$, and

$$d_s^{(j)} = \sum_{i=1}^{r_j} (d_s)_i^{(j)} c_i^{(j)} + \sum_{i<m_j} (d_s)_{i,m_j}^{(j)},\ j = 1, \ldots, N$$

be the Peirce decomposition of $d_s^{(j)}$ with respect to the Jordan frame $\{b_1^{(j)}, \ldots, b_{r_j}^{(j)}\}$. Furthermore, we can write

$$(v + \alpha d_x)^{(j)} = \sum_{i=1}^{r_j} \lambda_i (v + \alpha d_x)^{(j)} c_i^{(j)},\ (v + \alpha d_s)^{(j)} = \sum_{i=1}^{r_j} \lambda_i (v + \alpha d_s)^{(j)} b_i^{(j)}.$$

To simplify the notations we used (and will use below), $\lambda_i(\eta)^{(j)} = \lambda_i(v + \alpha d_x)^{(j)}$ and $\lambda_i(\eta)^{(j)} = \lambda_i(v + \alpha d_s)^{(j)}$ for $i = 1, \cdots, r_j$ and $j = 1, \cdots, N$.

From (3.7), we have

$$\begin{aligned} f_1''(\alpha) \ &\leq \ \frac{1}{2} \sum_{j=1}^{N} \left(\sum_{i=1}^{r_j} \psi''(\lambda_i(\eta^{(j)}))((d_x)_i^{(j)})^2 + \sum_{i<m_j} \psi''(\lambda_{m_j}(\eta^{(j)})) \mathrm{tr} \left(((d_x)_{i,m_j}^{(j)})^2 \right) \right) \\ &+ \ \frac{1}{2} \sum_{j=1}^{N} \left(\sum_{i=1}^{r_j} \psi''(\lambda_i(\gamma^{(j)}))((d_s)_i^{(j)})^2 + \sum_{i<m_j} \psi''(\lambda_{m_j}(\gamma^{(j)})) \mathrm{tr} \left(((d_s)_{i,m_j}^{(j)})^2 \right) \right). \end{aligned}$$

Below we use the shorthand notation: $\delta := \delta(v)$. The following lemma provides an upper bound of $f_1''(\alpha)$, which can be found in Lemma 3.3 in [31].

Lemma 5.3. *One has*

$$f_1''(\alpha) \leq 2(1 + 2\kappa)\delta^2 \psi''(\lambda_{min}(v) - 2\alpha\sqrt{1 + 2\kappa}\delta).$$

Following the strategy considered in [31], we briefly recall how to choose the default step size. Suppose that the step size α satisfies

$$-\psi'(v_{min} - 2\alpha\sqrt{1 + 2\kappa}\delta) + \psi'(v_{min}) \leq \frac{2\delta}{\sqrt{1 + 2\kappa}}. \tag{5.1}$$

Then $f_1(\alpha) \leq 0$. The largest possible value of the step size of α satisfying (5.1) is given by

$$\bar{\alpha} := \frac{1}{2\sqrt{1 + 2\kappa}\delta} \left(\rho(\delta) - \rho \left(\left(1 + \frac{1}{\sqrt{1 + 2\kappa}} \right) \delta \right) \right),$$

where $\rho(s) : [0, +\infty) \to (0, 1]$ is the inverse function of $-\frac{1}{2}\psi'(t)$ for $t \in (0, 1]$. Furthermore, we can conclude that

$$\bar{\alpha} \geq \frac{1}{(1 + 2\kappa)\psi'' \left(\rho \left(\left(1 + \frac{1}{\sqrt{1+2\kappa}} \right) \delta \right) \right)}.$$

Let $-\frac{1}{2}\psi'(t) = s$ for $t \in (0, 1]$. Then

$$-t + \frac{1}{t} + \frac{u^2}{2p(t + 2u)^2} \tan^{2p}(h(t)) = 2s.$$

For all $t \in (0,1]$, we have

$$\tan^{2p}(h(t)) = \frac{2p(t+2u)^2}{u^2}\left(2s+t-\frac{1}{t}\right) \le \frac{4(1+2u)^2 ps}{u^2}.$$

Hence, putting $t = \rho\left(\left(1+\frac{1}{\sqrt{1+2\kappa}}\right)\delta\right)$, we have $-\psi'(t) = 2\left(1+\frac{1}{\sqrt{1+2\kappa}}\right)\delta \le 4\delta$. Then

$$\tan^{2p}(h(t)) \le \frac{16(1+2u)^2 p\delta}{u^2} := \delta_u.$$

This implies that

$$\tan(h(t)) \le (\delta_u)^{\frac{1}{2p}}.$$

We have, by Lemma 3.2,

$$1+\tan(h(t)) > \frac{4u}{3\pi(1+2u)t}, \quad 0 < t \le 1.$$

This implies that

$$\frac{1}{t} < \frac{3\pi(1+2u)}{4u}\left(1+\tan(h(t))\right), \quad 0 < t \le 1.$$

Note that $\frac{1}{t+2u} < \frac{1}{2u}$ for $0 < t \le 1$, together with $p > 1$, then we have

$$\begin{aligned}(1+2\kappa)\tilde{\alpha} &= \psi''(t)^{-1} \\ &= \left(1+\frac{1}{t^2}+\frac{u^2}{p(t+2u)^3}\tan^{2p}(h(t))+\frac{\pi u^3(1+2u)}{(t+2u)^4}\tan^{2p-1}(h(t))\sec^2(h(t))\right)^{-1} \\ &\ge \left(1+\frac{9\pi^2(1+2u)^2}{16u^2}\left(1+(\delta_u)^{\frac{1}{2p}}\right)^2+\frac{1}{8u}\delta_u+\frac{\pi(1+2u)}{16u}(\delta_u)^{\frac{2p-1}{2p}}\left(1+(\delta_u)^{\frac{2}{2p}}\right)\right)^{-1}.\end{aligned}$$

Using Corollary 3.9 (i.e.,$\sqrt{2}\delta \ge \sqrt{\Psi(v)} \ge 1$) and note that $p \in N, p > 1$, we can conclude that $p\delta > 1$. After some elementary reductions, we have

$$\begin{aligned}(1+2\kappa)\tilde{\alpha} \ &\ge \ \left(1+\frac{9\pi^2(1+2u)^2}{16u^2}\left(1+\left(\frac{16(1+2u)}{u^2}\right)^{\frac{1}{2p}}\right)^2+\frac{(1+2u)^2}{u^3}\right. \\ &\quad \left.+\frac{\pi(1+2u)}{16u}\left(\frac{16(1+2u)}{u^2}\right)^{\frac{2p-1}{2p}}\left(1+\left(\frac{16(1+2u)}{u^2}\right)^{\frac{2}{2p}}\right)\right)^{-1}(p\delta)^{-\frac{2p+1}{2p}} \\ &\ge \ \left(1+\frac{9\pi^2(1+2u)^2}{16u^2}\left(1+\frac{4(1+2u)}{u}\right)^2+\frac{(1+2u)^2}{u^3}\right. \\ &\quad \left.+\frac{\pi(1+2u)}{16u}\frac{16(1+2u)^2}{u^2}\left(1+\frac{16(1+2u)}{u^2}\right)\right)^{-1}(p\delta)^{-\frac{2p+1}{2p}} \\ &= \ \frac{1}{C(u)(p\delta)^{\frac{2p+1}{2p}}},\end{aligned}$$

where

$$C(u) = 1+\frac{9\pi^2(1+2u)^2}{16u^2}\left(1+\frac{4(1+2u)}{u}\right)^2+\frac{(1+2u)^2}{u^3}+\frac{\pi(1+2u)^3}{2u^3}+\frac{4\pi(1+2u)^5}{u^5}.$$

In this paper, we use

$$\tilde{\alpha} := \frac{1}{(1+2\kappa)C(u)(p\delta)^{\frac{2p+1}{2p}}}, \tag{5.2}$$

as the default step size.

5.3 Decrease of the value of $\Psi(v)$ during an inner iteration

Lemma 5.4 (Lemma 12 in [25]). *Let $h(t)$ be a twice differentiable convex function with $h(0) = 0$, $h'(0) < 0$ and let $h(t)$ attain its (global) minimum at $t^* > 0$. If $h''(t)$ is increasing for $t \in [0, t^*]$, then*

$$h(t) \leq \frac{th'(0)}{2}, \quad 0 \leq t \leq t^*.$$

As a consequences of Lemma 5.4 and the fact that $f(\alpha) \leq f_1(\alpha)$, which is a twice differentiable convex function with $f_1(0) = 0$, and $f_1'(0) = -2\delta^2 < 0$, we can easily prove the following lemma.

Lemma 5.5. *Let the step size α is such that $\alpha \leq \tilde{\alpha}$. Then*

$$f(\alpha) \leq -\alpha\delta^2.$$

The following theorem shows that the default step size (5.2) yields the sufficient decrease of the barrier function value during each inner iteration.

Theorem 5.6. *One has*

$$f(\tilde{\alpha}) \leq -\frac{1}{\sqrt{2}(1+2\kappa)C(u)p^{\frac{2p+1}{2p}}}\Psi(v)^{\frac{2p-1}{4p}}.$$

Proof. Since $\tilde{\alpha} \leq \bar{\alpha}$, we have, by Corollary 3.9,

$$f(\tilde{\alpha}) \leq -\tilde{\alpha}\delta^2 \leq -\frac{\delta^2}{(1+2\kappa)C(u)(p\delta)^{\frac{2p+1}{2p}}} \leq -\frac{1}{\sqrt{2}(1+2\kappa)C(u)p^{\frac{2p+1}{2p}}}\Psi(v)^{\frac{2p-1}{4p}}.$$

This completes the proof. □

5.4 Iteration bounds for large-update methods

From Theorem 5.2, we have, by updating of the parameter μ,

$$\Psi(v_+) \leq \Psi(v) + \frac{\theta}{2(1-\theta)}(2\Psi(v) + 2\sqrt{2r\Psi(v)} + r).$$

In the sequel, we want to count how many inner iterations are required to return to the situation where $\Psi(v) \leq \tau$. Let Ψ_0 denotes the value of $\Psi(v)$ after the μ-update, the subsequent values in the same outer iteration are denoted as Ψ_k, $k = 1, 2, \ldots, K$, where K denotes the total number of inner iterations in the outer iteration. Hence, we have

$$\Psi_0 \leq \tau + \frac{\theta}{2(1-\theta)}\left(2\tau + 2\sqrt{2r\tau} + r\right).$$

According to the decrease of $f(\tilde{\alpha})$ in Theorem 5.6, we have

$$\Psi_{k+1} \leq \Psi_k - \beta(\Psi_k)^{1-\gamma}, \quad k = 0, 1, \ldots, K-1, \tag{5.3}$$

where $\beta = \frac{1}{\sqrt{2}(1+2\kappa)C(u)p^{\frac{2p+1}{2p}}}$ and $\gamma = \frac{2p+1}{4p}$.

Lemma 5.7 (Lemma 14 in [25]). *Suppose $t_0, t_1, \ldots, t_K$ be a sequence of positive numbers such that*

$$t_{k+1} \leq t_k - \beta t_k^{1-\gamma}, \quad k = 0, 1, \ldots, K-1,$$

where $\beta > 0$ and $0 < \gamma \leq 1$. Then $K \leq \lceil \frac{t_0^\gamma}{\beta\gamma} \rceil$.

The following lemma provides an estimate for the number of inner iterations between two successive barrier parameter updates, in terms of Ψ_0.

Lemma 5.8. *One has*

$$K \leq 2\sqrt{2}(1+2\kappa)C(u)p^{\frac{2p+1}{2p}}\Psi_0^{\frac{2p+1}{4p}}$$

Proof. Combining the results of (5.3) and Lemma 5.7, we can easily obtain the result of the lemma. This completes the proof. □

The number of outer iterations is bounded above by $\frac{1}{\theta}\log\frac{r}{\varepsilon}$ (cf. [26] Π.17, page 116). By multiplying the number of outer iterations and the number of inner iterations, we get an upper bound for the total number of iterations, namely,

$$O\left(\frac{(1+2\kappa)C(u)p^{\frac{2p+1}{2p}}}{\theta}\left(\tau + \frac{\theta}{2(1-\theta)}\left(2\tau + 2\sqrt{2r\tau} + r\right)\right)^{\frac{2p+1}{4p}}\log\frac{r}{\varepsilon}\right).$$

Then, the iteration bounds for large-update methods is established in the following theorem.

Theorem 5.9. *For large-update methods, one takes for θ a constant (independent on r), namely $\theta = \Theta(1)$, and $\tau = O(r)$. The best iteration bound then becomes*

$$O\left((1+2\kappa)p^{\frac{2p+1}{2p}}r^{\frac{2p+1}{4p}}\log\frac{r}{\varepsilon}\right).$$

Corollary 5.10. *Let $p = O(\log r)$. Then the iteration bound for large-update methods reduces to*

$$O\left((1+2\kappa)\sqrt{r}\log r\log\frac{r}{\varepsilon}\right),$$

which matches the currently best known iteration bound for large-update methods.

5.5 Iteration bounds for small-update methods

For the analysis of the iteration bound of a small-update method, we need to estimate the upper bound of Ψ_0 more accurately. This due to the following lemma.

Lemma 5.11 (Corollary 6.1 in [31]). *Let $0 < \theta < 1$ and $v_+ = \dfrac{v}{\sqrt{1-\theta}}$. If $\Psi(v) \leq \tau$, then*

$$\Psi(v_+) \leq r\psi\left(\frac{\varrho(\frac{\tau}{r})}{\sqrt{1-\theta}}\right).$$

From Lemma 5.11, Lemma 3.8, and the fact that $1 - \sqrt{1-\theta} = \frac{\theta}{1+\sqrt{1-\theta}} \leq \theta$, we have

$$\Psi_0 \leq r\psi\left(\frac{\varrho(\frac{\tau}{r})}{\sqrt{1-\theta}}\right) \leq r\left(\frac{\varrho(\frac{\tau}{r})}{\sqrt{1-\theta}} - 1\right)^2 \leq \frac{1}{1-\theta}\left(\theta\sqrt{r} + \sqrt{2\tau}\right)^2.$$

It follows from Theorem 5.8 that the total number of iterations is bounded above by

$$O\left(\frac{(1+2\kappa)C(\mu)p^{\frac{2p+1}{2p}}}{\theta}\left(\frac{1}{1-\theta}\left(\theta\sqrt{r} + \sqrt{2\tau}\right)^2\right)^{\frac{2p+1}{4p}}\log\frac{r}{\varepsilon}\right).$$

The following theorem provides the currently best known iteration bound for small-update methods.

Theorem 5.12. *For small-update methods, one takes* $\theta = \Theta(\frac{1}{\sqrt{r}})$ *and* $\tau = O(1)$. *The best iteration bound then becomes*

$$O\left((1+2\kappa)\sqrt{r}\log\frac{r}{\varepsilon}\right),$$

which matches the currently best known iteration bound for small-update methods.

6 Conclusions and Remarks

In this paper, we have considered a new parametric kernel function with trigonometric barrier term as well as the corresponding barrier function. Based on this parametric kernel function, we designed and analyzed a class of large- and small-update versions of the primal-dual IPMs for the Cartesian $P_*(\kappa)$-SCLCP. The parametric kernel function is not only used for determining the search directions but also for measuring the distance between the given iterate and the corresponding μ-center for the algorithms. By using EJAs, we derived the iteration bounds that match the currently best known iteration bounds for large- and small-update methods, namely $O((1+2\kappa)\sqrt{r}\log r\log\frac{r}{\varepsilon})$ and $O((1+2\kappa)\sqrt{r}\log\frac{r}{\varepsilon})$, respectively.

The generalization of the general nonlinear complementarity problems over symmetric cone deserves to be investigated.

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